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EXPLICIT SOLUTIONS FOR A SYSTEM OF FIRST-ORDER PARTIAL DIFFERENTIAL EQUATIONS-II

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Abstract. In this note we give an explicit formula for the solution of conservative form of a system studied in a previous article [6], in the domain $\{(x,t) : x > 0, t > 0\}$ with initial conditions at $t = 0$ and with Bardos Leroux Nedelec boundary conditions at $x = 0$.

1. Introduction

In this note we consider the conservative form of a system considered in [6], namely

$$\begin{aligned} u_t + f(u)_x &= 0, \\ v_t + (f'(u)v)_x &= 0, \end{aligned} \tag{1.1}$$

with $f''(u) > 0$, in the domain $\Omega = \{(x,t) : x > 0, t > 0\}$. We give an explicit formula for the solution of (1.1) with prescribed initial conditions

$$\begin{pmatrix} u(x,0) \\ v(x,0) \end{pmatrix} = \begin{pmatrix} u_0(x) \\ v_0(x) \end{pmatrix}, \tag{1.2}$$

at $t = 0$, the Bardos Leroux and Nedelec [1, 9] boundary condition for u

$$\begin{gathered} \text{either} \quad u(0+,t) = u_b^+(t) \\ \text{or} \quad f'(u(0+,t)) \le 0 \text{ and } f(u(0+,t)) \ge f(u_b^+(t)), \end{gathered} \tag{1.3}$$

and a weak form of Dirichlet boundary conditions for v

$$\text{if } f'(u(0+,t)) > 0, \text{ then } v(0+,t) = v_b(t). \tag{1.4}$$

Here $u_b^+(t) = \max\{u_b(t), \lambda\}$, where λ is the unique point where $f'(u)$ changes sign.

In [6], explicit solution was constructed for the system where the second equation in (1.1) was replaced by

$$V_t + f'(u)V_x = 0 \tag{1.5}$$

with the weak form of Dirichlet boundary condition $V(0,t) = V_b(t)$. Taking derivative of (1.5) withe respect to x and setting $v = V_x$ we obtain the conservative equation for v. In this note we give Dirichlet boundary condition for v, which is equivalent to giving Neumann Boundary condition for V. Here we explain the

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required modification of the formula in [6] in the construction of solution to (1.1)-(1.4).

LeFloch [8] was the first who studied the system (1.1) when $f(u)$ is strictly convex and constructed explicit formula for the pure initial value problem using Lax formula. One important property of the system is the formation of δ - wave solutions for certain types of initial data which are of bounded variation. Such systems come in applications, for example, the special case $f(u) = u^2/2$ in (1.1), is the one-dimensional model in the large scale structure formation. Initial value problem for this quadratic case was also studied by Joseph [2, 5] by different way, using the vanishing viscosity method and Hopf-Cole transformation.

2. A formula for the solution in the quarter plane

We consider the system (1.1) with initial condition (1.2) and boundary condition (1.3) and (1.4). We assume $u_0(x)$ is bounded measurable and $v_0(x)$ is Lipschitz continuous functions of $x \geq 0$, $u_b(t)$ and $v_b(t)$ are Lipschitz continuous functions of $t > 0$.

We assume the flux $f(u)$ satisfies the conditions

$$f''(u) > 0, \quad \lim_{u\to\infty} \frac{f(u)}{u} = \infty,$$

and let $f^*(u)$ be the convex dual of $f(u)$ namely, $f^*(u) = \max_{\theta\in R^1}\{\theta u - f(\theta)\}$.

As in [6], we introduce some notation and describe the construction of (u, v) and then verify it is a solution. For each fixed $(x, y, t), x \geq 0, y \geq 0, t > 0$, $C(x, y, t)$ denotes the following class of paths β in the quarter plane $\Omega = \{(z, s) : z \geq 0, s \geq 0\}$. Each path is connected from the initial point $(y, 0)$ to (x, t) and is of the form $z = \beta(s)$, where β is a piecewise linear function of maximum three lines and always linear in the interior of Ω. Thus for $x > 0$ and $y > 0$, the curves are either a straight line or have exactly three straight lines with one lying on the boundary $x = 0$. For $y = 0$ the curves are made up of one straight line or two straight lines with one piece lying on the boundary $x = 0$. Associated with the flux $f(u)$ and boundary data $u_b(t)$, we define the functional $J(\beta)$ on $C(x, y, t)$

$$J(\beta) = -\int_{\{s:\beta(s)=0\}} f(u_B(s)^+)ds + \int_{\{s:\beta(s)\neq 0\}} f^*\big(\frac{d\beta(s)}{ds}\big)ds.$$

We call β_0 is straight line path connecting $(y, 0)$ and (x, t) which does not touch the boundary $x = 0$, $\{(0, t), t > 0\}$, then let

$$A(x, y, t) = J(\beta_0) = tf^*\big(\frac{x - y}{t}\big).$$

For any $\beta \in C^*(x, y, t) = C(x, y, t) - \{\beta_0\}$, that is made up of three straight lines connecting $(y, 0)$ to $(0, t_2)$ in the interior and $(0, t_2)$ to $(0, t_1)$ on the boundary and $(0, t_1)$ to (x, t) in the interior, $t_2 < t_1 < t$, it can be easily seen that

$$J(\beta) = J(x, y, t, t_1, t_2) = -\int_{t_2}^{t_1} f(u_B(s)^+)ds + t_2 f^*\big(\frac{y}{-t_2}\big) + (t - t_1)f^*\big(\frac{x}{t - t_1}\big).$$

For the curves made up of two straight lines with one piece lying on the boundary $x = 0$ which connects $(0, 0)$ and $(0, t_1)$ and the other connecting $(0, t_1)$ to (x, t).

$$J(\beta) = J(x, y, t, t_1, t_2 = 0) = -\int_0^{t_1} f(u_B(s)^+)ds + (t - t_1)f^*\big(\frac{x}{t - t_1}\big).$$

In the following, we list some facts which was proved in [3], that are used later in the construction of solution, which follow from some basic convex analysis and arguments of Lax [7].

There exists a $\beta^* \in C^*(x,y,t)$ or correspondingly $t_1(x,y,t)$, $t_2(x,y,t)$ so that

$$B(x,y,t) = J(\beta^*) = J(x,y,t,t_1(x,y,t),t_2(x,y,t)) = \min\{J(\beta) : \beta \in C^*(x,y,t)\}$$

is a locally Lipschitz continuous function of $(x,y,t), x \geq 0, y \geq 0, t \geq 0$.

Secondly, the functions

$$Q(x,y,t) = \min\{J(\beta) : \beta \in C(x,y,t)\} = \min\{A(x,y,t), B(x,y,t)\},$$

and

$$U(x,t) = \min\{Q(x,y,t) + U_0(y),\ 0 \leq y < \infty\} \tag{2.1}$$

are locally Lipschitz continuous functions in their variables, where we have taken $U_0(y) = \int_0^y u_0(z)dz$.

Thirdly minimum in (2.1) is attained at some value of $y \geq 0$ which depends on (x,t), we call it $y(x,t)$. For each fixed $t > 0$, this minimizer is unique except for a countable number of points of $x > 0$.

Finally, for each fixed $t > 0$, except for one point of x, either $A(x,y(x,t),t) < B(x,y(x,t),t)$ or $A(x,y(x,t),t) > B(x,y(x,t),t)$.If $A(x,y(x,t),t) < B(x,y(x,t),t)$,

$$U(x,t) = tf^*(\frac{x - y(x,t)}{t}) + U_0(y),$$

and if $A(x,y(x,t),t) > B(x,y(x,t),t)$

$$U(x,t) = J(x,y(x,t),t,t_1(x,y(x,t),t),t_2(x,y(x,t),t)) + U_0(y).$$

Here and hence forth $y(x,t)$ is a minimizer in (2.1) and we denote $A(x,t) = A(x,y(x,t),t)$, $B(x,t) = B(x,y(x,t),t)$,$t_2(x,t) = t_2(x,y(x,t),t)$ and $t_1(x,t) = t_1(x,y(x,t),t)$.

Theorem 2.1. *Assume u_0 is bounded measurable and locally Lipschitz continuous, v_0 is Lipschitz continuous in $x \geq 0$ and $u_b(t)$ ans $v_b(t)$ are Lipschitz continuous functions. Then for every $\{(x,t), x \geq 0, t > 0$, $U(x,t)$ defined by the minimization problem (2.1) is a locally Lipschitz continuous function. For almost every (x,t) there is only one minimizer $y(x,t)$ and let $t_1(x,t)$ and $t_2(x,t)$ as described before. Define*

$$u(x,t) = \begin{cases} (f^*)'(\frac{x-y(x,t)}{t}), & \text{if } A(x,t) < B(x,t), \\ (f^*)'(\frac{x}{t-t_1(x,t)}), & \text{if } A(x,t) > B(x,t), \end{cases} \tag{2.2}$$

and

$$V(x,t) = \begin{cases} \int_0^{y(x,t)} v_0(z)dz, & \text{if } A(x,t) < B(x,t), \\ -\int_{t_2(x,t)}^{t_1(x,t)} f'(u_b^+(s)v_b(s)ds, & \text{if } A(x,t) > B(x,t), \end{cases} \tag{2.3}$$

and set

$$v(x,t) = \partial_x(V(x,t)). \tag{2.4}$$

Then the function $(u(x,t), v(x,t))$ is a weak solution of (1.1), satisfying the initial condition (1.2) and boundary conditions (1.3) and (1.4). Further u satisfies the entropy condition $u(x-,t) \geq u(x+,t)$ for $x > 0$, $t > 0$.

Proof. The proof is by direct verification and most part is identical to [6] and so that part is omitted. We give here only the verification of the boundary condition (1.4).

Suppose $f'(u(0+,t)) > 0$ then $f'(u(x,t)) > 0$ for $0 < x \le \epsilon$ for some sufficiently small ϵ and u and v are given by (2.2)-(2.4). Then $t_2(x,t)$ is constant for $x \in [0,\epsilon)$ and

$$u(x,t) = (f^*)'(\frac{x}{t - t_1(x,t}),$$

so that $t - t_1(x,t) = x/f'(u(x,t))$. It follows that $\lim_{x\to 0} t_1(x,t) = t$, since we assumed that

$$\lim_{x\to 0} f'(u(x,t)) = f'(u(0+,t)) = f'(u_b(t)) > 0. \tag{2.5}$$

Now

$$\begin{aligned} v(x,t) &= -\partial_x \int_{t_2(x,t)}^{t_1(x,t)} f'(u_b^+(s)v_b(s)ds. \\ &= -f'(u_b(t_1(x,t)))v_b(t_1(x,t))\partial_x t_1(x,t) \end{aligned} \tag{2.6}$$

Again differentiating the relation $t - t_1(x,t) = x/f'(u(x,t))$ with respect to x, we have

$$\partial_x t_1(x,t) = \frac{xf''(u(x,t))u_x - f'(u(x,t))}{(f'(u(x,t)))^2} \tag{2.7}$$

By (2.5)-(2.7) and using the fact $\lim_{x\to 0} t_1(x,t) = t$, we get the weak boundary condition (1.4). □

Explicit formula for Riemann initial boundary value problem. It is illustrative to compute the solution constructed in the above theorem for the Riemann type initial boundary data, namely u_0, v_0, u_b and v_b are all constants.

Theorem 2.2. *For Riemann initial boundary value problems, the formulae* (2.2) - (2.4) *takes the form*

Case 1: $f'(u_0) = f'(u_b) > 0$,

$$(u(x,t), v(x,t)) = \begin{cases} (u_0, v_b), & \text{if } x < f'(u_0)t, \\ (u_0, v_0), & \text{if } x > f'(u_0)t. \end{cases}$$

Case 2: $f'(u_0) = f'(u_b) < 0$,

$$(u(x,t), v(x,t) = (u_0, v_0)$$

Case 3: $0 < f'(u_b) < f'(u_0)$,

$$(u(x,t), v(x,t)) = \begin{cases} (u_b, v_b), & \text{if } x < f'(u_b)t, \\ (x/t, 0), & \text{if } f'(u_b)t < x < f'(u_0)t \\ (u_0, v_0), & \text{if } x > f'(u_0)t \end{cases}$$

Case 4: $f'(u_b) < 0 < f'(u_0)$,

$$(u(x,t), v(x,t)) = \begin{cases} (x/t, 0), & \text{if } 0 < x < f'(u_0)t \\ (u_0, v_0), & \text{if } x > f'(u_0)t \end{cases}$$

Case 5: $f'(u_b) < 0$ *and* $f'(u_0) \le 0$,

$$(u(x,t), v(x,t)) = (u_0, v_0)$$

Case 6: $f'(u_0) < f'(u_b)$ *and* $s = \frac{f(u_b)-f(u_0)}{u_b-u_0} > 0$ *:*

$$(u(x,t), v(x,t)) = \begin{cases} (u_b, v_b), & \text{if } x < st, \\ (1/2(u_b+u_0), (1/2)(u_b-u_0)(v_0+v_b)t\delta_{x=st}) & \text{if } x = st \\ (u_0, v_0), & \text{if } x > st. \end{cases}$$

3. Solution in a strip

The solution we have obtained for the quarter plane problem can be easily generalized to the ship $\Omega = \{(x,t) : 0 < x < 1, t > 0\}$. Here we prescribe

$$(u(x,0+), v(x,0+) = (u_0(x), v_0(x)), \quad 0 \le x \le 1. \tag{3.1}$$

As before for u component we prescribe a weak form of Dirichlet boundary conditions at $x = 0$ and at $x = 1$:

$$\begin{gathered} \text{either} \quad u(0+,t) = u_l^+(t) \\ \text{or} \quad f'(u(0+,t)) \le 0 \text{ and } f(u(0+,t)) \ge f(u_b^+(t)), \end{gathered} \tag{3.2}$$

$$\begin{gathered} \text{either} \quad u(1-,t) = u_r^+(t) \\ \text{or} \quad f'(u(1-,t)) \ge 0 \text{ and } f(u(1-,t)) \ge f(u_r^+(t)). \end{gathered} \tag{3.3}$$

Here $u_l^+(t) = \max\{u_l(t), \lambda\}$, $u_r^-(t) = \min\{u_r(t), \lambda\}$ where as before λ is the point of minimum of f. We get explicit formula for the entropy weak solution of the first component u of (1.1) with initial condition $u(x,0) = u_0(x)$ and the boundary conditions (3.2) and (3.3) by Joseph and Gowda [4]. Once u is obtained, the boundary conditions for $v(0+,t) = v_l(t)$ is prescribed only if the characteristics at $(0,t)$ has positive speed, ie $f'(u(0+,t)) > 0$. So the weak form of boundary conditions for v component at $x = 0$ is

$$\text{if } f'(u(0+,t)) > 0, \text{ then } v(0+,t) = v_l(t). \tag{3.4}$$

Similarly the weak form of the boundary condition at $x = 1$ is

$$\text{if } f'(u(1-,t)) < 0, \text{ then } v(1-,t) = v_r(t). \tag{3.5}$$

We assume the initial conditions $u_0(x)$ is bounded measurable, and locally Lipschitz, and $v_0(x)$ is Lipschitz continuous on $0 \le x \le 1$ and boundary datas $u_l(t), v_b(t)$ are Lipschitz continuous $[0,T]$, for each $T > 0$.

For the statement of the theorem, we introduce some notations. For each fixed (x,y,t), $0 \le x \le 1$, $0 \le y \le 1$, $t > 0$, $|i-j| \le 1$,$i,j = 0,1,2,3,\dots$, $C_{ij}(x,y,t)$ denotes the following class of paths β in the strip

$$\Omega = \{(z,s) : 0 \le z \le 1, s \ge 0\}$$

Each path connects $(y,0)$ to (x,t) and is of the form $z = \beta(s)$ where $\beta(s)$ is piecewise linear function which are straight lines in the interior of D , and having i straight line pieces lie on $x = 0$ and j of them lie on $x = 1$. The points of intersection of the straight line pieces of the curve lying in Ω with the boundaries $x = 0$ and $x = 1$ are called corners of the curve β.

Denote

$$C(x,y,t) = \cup_{i\ge 0, j\ge 0, |i-j|\le 1} C_{i,j}(x,y,t)$$

For fixed (x,y,t), we define

$$J(\beta)=-\int_{\{s:\beta(s)=0\}} f(u_l^+(s))ds-\int_{\{s:\beta(s)=1\}} f(u_r^-(s))ds+\int_{\{s:0<\beta(s)<1\}} f^*(\frac{d\beta}{ds})ds. \tag{3.6}$$

Denote $C^*(x,y,t)=C(x,y,t)-\{\beta_0\}$, where β_0 is the straight line path joining (x,t) to $(y,0)$.

Let us define $A(x,y,t)$ and $B(x,y,t)$ by

$$A(x,y,t)=J(\beta_0),\quad B(x,y,t)=\min_{\beta\in C^*(x,y,t)} J(\beta) \tag{3.7}$$

where $J(\beta)$ be defined by (3.6).

We recall a few facts from [4]. For each $(x,t)\in\Omega$ and $0\le y\le 1$, the minimum in (3.7) is attained for a path β^* over $C^*(x,y,t)$. Let the corner points of the minimizer β be

$$(\beta^*(t_1(x,y,t)),t_1(x,y,t)),\quad (\beta^*(t_2(x,y,t)),t_2(x,y,t)),\\ \dots,\quad (\beta^*(t_k(x,y,t)),t_k(x,y,t)),$$

$t>t_1(x,y,t)>t_2(x,y,t)\dots>t_k(x,y,t)>0$. For a given $T>0$, there exits positive integer $N(T)$ such that for any $t\le T$, and $k<N(T)$. This is due to the bound of $u_l(t)$ on $[0,T]$ and due to the conditions on $f(u)$. Then the function B is expressed in terms of $x,y,t,t_1(x,y,t),\dots t_k(x,y,t)$ which we denote by $B(x,y,t)=J(x,y,t,t_1(x,y,t)\dots t_k(x,y,t)=J(\beta^*)$. Similarly define $A(x,y,t)=tf^*(\frac{x-t}{t})=J(\beta_0)$, β_0 is the straight line path connecting (x,t) and $(y,0)$. Define the function

$$Q(x,y,t)=\min\{A(x,y,t),B(x,y,t)\}. \tag{3.8}$$

The function

$$U(x,t)=\min_{0\le y\le 1}\big[\int_0^y u_0(z)dz+Q(x,y,t)\big] \tag{3.9}$$

is Lipschitz continuous function of (x,t) in Ω. For almost every (x,t) in Ω, there exists a unique minimizer $y(x,t)$ and either

$A(x,y(x,t),t)<B(x,y(x,t)t)$ and $U(x,t)=\int_0^y(x,t)u_0(z)dz+A(x,y,t)$

or

$A(x,y(x,t),t)>B(x,y(x,t)t)$ in which case $U(x,t)=\int_0^y u_0(z)dz+B(x,y,t)$.

In the second case, let $t_j(x,y,t), j=1,2,\dots k$ corresponds to the corner points of the curve β^* in the evaluation of $B(x,y,t)$. Denote $t_j(x,t)=t_j(x,y(x,t),t)$, $A(x,t)=A(x,y(x,t),t)$ and $B(x,t)=B(x,y(x,t),t)$.

With these notations we have the following theorem.

Theorem 3.1. *Let U be defined by the minimization problem* (3.9) *and $y(x,t)$ be a minimizer (which is unique for a.e points of Ω). Let $u=U_x(x,t)$ exists for a.e. points of Ω and has the form*

$$u(x,t)=\begin{cases}(f^*)'(\frac{x-y(x,t)}{t}), & \textit{if } A(x,t)<B(x,t),\\ (f^*)'(\frac{x}{t-t_1(x,t)}), & \textit{if } A(x,t)>B(x,t),\end{cases}$$

and

$$V(x,t)=\begin{cases}\int_0^{y(x,t)} v_0(z)dz, & \textit{if } A(x,t)<B(x,t),\\ -\int_{t_2(x,t)}^{t_1(x,t)} f'(u_l^+(s)v_l(s)ds, & \textit{if } A(x,t)>B(x,t) \textit{ and } \beta^*(t_1(x,t))=0,\\ -\int_{t_2(x,t)}^{t_1(x,t)} f'(u_r^-(s)v_r(s)ds, & \textit{if } A(x,t)>B(x,t) \textit{ and } \beta^*(t_1(x,t))=1,\end{cases}$$

and set

$$v(x,t) = \partial_x(V(x,t)).$$

Then (u,v) *is a solution to* (1.1) *with initial conditions* (3.1) *and boundary conditions* (3.2)-(3.5). *Further* u *satisfies the entropy condition* $u(x-,t) \geq u(x+,t)$ *for* $0 < x < 1$, $t > 0$.

Proof. The assertions on u is proved in [4]. Once we have that, the verification that v solves the equation and the initial and boundary conditions follows exactly as in section 2 and is omitted. □

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