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EXISTENCE OF SOLUTIONS FOR A SYSTEM OF ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

ROBERT DALMASSO

Abstract. In this article, we establish the existence of radial solutions for a system of nonlinear elliptic partial differential equations with Dirichlet boundary conditions. Also we discuss the question of uniqueness, and illustrate our results with examples.

1. Introduction

In this article, we study the existence for non-trivial solutions $(u, v) \in (C^2(\overline{B_R}))^2$ of the boundary-value problem

$$\begin{gathered} \Delta u = g(v) \quad \text{in } B_R\,, \\ \Delta v = f(u) \quad \text{in } B_R\,, \\ u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial B_R\,, \end{gathered} \tag{1.1}$$

where B_R denotes the open ball of radius R centered at the origin in $\mathbb{R}^n$ $(n \geq 1)$, $\partial/\partial\nu$ is the outward normal derivative and f, g satisfy the following hypotheses:

- (H1) f, $g : \mathbb{R} \to \mathbb{R}$ are C^1 functions;
- (H2) g is increasing on $(0, \infty)$, $g' > 0$ on $(-\infty, 0)$, $g(0) = 0$ and $\lim_{v\to-\infty} g(v) = -\infty$;
- (H3) $f, f' > 0$ on $(0, \infty)$.

Now we state our main results.

Theorem 1.1. *Let f, g satisfy* (H1)–(H3). *Assume moreover that*

- (H4) *There exist m, $M > 0$ such that $m \leq f(u) \leq M$ for all $u \geq 0$.*

Then (1.1) *has at least one radial solution $(u, v) \in (C^2(\overline{B}_R))^2$.*

Theorem 1.2. *Let f, g satisfy* (H1)–(H3). *Assume moreover that*

- (H5) $f(0) > 0$;
- (H6) *There exist a, b, a', b', $p > 0$ and $q \geq 1$ such that $pq < 1$,*

$$f(u) \geq au^p \quad \forall\, u \geq 0\,, \quad f(u) \leq a'u^p \quad \forall\, u \geq 1\,,$$
$$b|v|^q \leq |g(v)| \leq b'|v|^q \quad \forall\, v \in \mathbb{R}\,.$$

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Then (1.1) *has at least one radial solution* $(u, v) \in (C^2(\overline{B}_R))^2$.

When $n = 1$ we have the following result.

Theorem 1.3. *Assume that* $n = 1$. *Let* f, g *satisfy* (H1)–(H3). *Assume moreover that*

(H7) *There exist* a, a', b, $b' > 0$ *and* p, $q \geq 1$ *such that* $pq > 1$,

$$au^p \leq f(u) \leq a'u^p \quad \forall\, u \geq 0\,,$$
$$b|v|^q \leq |g(v)| \leq b'|v|^q \quad \forall\, v \in \mathbb{R}\,,$$

Then (1.1) *has at least one non-trivial symmetric solution* $(u, v) \in (C^2([-R, R])^2$.

Remark 1.4. Assumptions (H2) and (H4) (resp. (H5)) imply that if a solution $(u, v) \in (C^2(\overline{B}_R))^2$ of (1.1) exists, then $u \neq 0$ and $v \neq 0$.

The particular case of homogeneous nonlinearities $f(u) = |u|^p$, $g(v) = |v|^{q-1}v$ with p, $q > 0$ has been studied in [1].

When $n = 1$ and $g(v) = v$ the uniqueness of a non-trivial solution was proved in [2] for $f : \mathbb{R} \to [0, \infty)$ C^1 and satisfying the following condition:

$$0 < f(u) < uf'(u) \quad \text{for } u > 0\,.$$

An existence result was also given.

Since we are interested in radial solutions the problem reduces to the one-dimensional (singular if $n \geq 2$) boundary-value problem

$$\begin{gathered}\Delta u = g(v) \quad \text{in } [0, R)\,,\\ \Delta v = f(u) \quad \text{in } [0, R)\,,\\ u(R) = u'(R) = u'(0) = v'(0) = 0\,,\end{gathered} \tag{1.2}$$

where Δ denotes the polar form of the Laplacian

$$\Delta = r^{1-n}\frac{d}{dr}(r^{n-1}\frac{d}{dr})\,.$$

In Section 2 we give some preliminary results. Theorems 1.1 and 1.2 are proved in Section 3. We also give two other existence theorems. Theorem 1.3 is proved in Section 4. We examine the uniqueness question in Section 5. Finally in Section 6 we give some examples.

2. Preliminaries

Throughout this section we assume that f and g satisfy (H1)–(H3). When $n \geq 3$ we also assume that f verifies (H5) or that f and g satisfy the following conditions:

(H8) There exist a, b, p, $q > 0$ such that

$$f(u) \geq au^p \quad \forall\, u \geq 0\,, \quad |g(-v)| \geq bv^q \quad \forall\, v \geq 0\,,$$
$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-2}{n}\,.$$

Notice that when $pq \leq 1$ we have

$$\frac{1}{p+1} + \frac{1}{q+1} \geq 1\,.$$

We will use a two-dimensional shooting argument as in [1]. Let α, $\beta > 0$. We introduce the inital value problem

$$\begin{gathered} \Delta u(r) = g(v(r))\,, \quad r \geq 0\,, \\ \Delta v(r) = f(u(r))\,, \quad r \geq 0\,, \\ u(0) = \alpha\,, \quad v(0) = -\beta\,, \quad u'(0) = v'(0) = 0\,. \end{gathered} \tag{2.1}$$

Lemma 2.1. *Let α, $\beta > 0$ be fixed. If $(u, v) \in (C^2([0,\infty))^2$ is a solution of (2.1) such that $uu' < 0$ on $(0,\infty)$, then $v < 0$ on $(0,\infty)$.*

Proof. We have $0 < u \leq \alpha$ on $[0,\infty)$. Therefore, by (H3),

$$r^{n-1}v'(r) = \int_0^r s^{n-1} f(u(s))\,ds \,>\, 0 \quad \text{for } r > 0\,. \tag{2.2}$$

Assume that the conclusion of the lemma is false. Then (2.2) implies that there exist a, $b > 0$ such that

$$v(r) \geq a \quad \text{for } r \geq b\,.$$

With the help of (H2) we deduce that

$$(r^{n-1}u'(r))' \geq g(a) r^{n-1} \quad \text{for } r \geq b\,,$$

hence

$$r^{n-1}u'(r) \geq g(a)\frac{r^n - b^n}{n} + b^{n-1}u'(b) \quad \text{for } r \geq b\,,$$

which implies, using (H2) again, that $u'(r) > 0$ for r large and we reach a contradiction. □

Now we define the functions F, G and G_n by

$$F(u) = \int_0^u f(s)\,ds\,, \quad G(v) = \int_0^v g(s)\,ds \quad u, v \in \mathbb{R}\,,$$

$$G_n(r,s) = \begin{cases} r - s & \text{if } n = 1\,, \\ s\ln(\frac{r}{s}) & \text{if } n = 2\,, \\ \frac{s}{n-2}(1 - (\frac{s}{r})^{n-2}) & \text{if } n \geq 3\,. \end{cases}$$

Lemma 2.2. *Let α, $\beta > 0$ be fixed. Assume that for some $a > 0$, $(u, v) \in (C^2([0,a])^2$ is a solution of (2.1) on $[0,a]$ such that $uu' < 0$ on $(0,a)$. Then*

$$|v(r)| \leq \max(\beta, G^{-1}(F(\alpha) + G(-\beta)))\,, \quad 0 \leq r \leq a\,,$$

where G^{-1} denotes the inverse of $G : [0,\infty) \to [0,\infty)$.

Proof. We have $0 < u \leq \alpha$ on $[0,a)$. As in Lemma 2.1 we deduce that $v' > 0$ on $(0,a]$. We have

$$\int_0^r (v'\Delta u + u'\Delta v)\,ds = \int_0^r (g(v)v' + f(u)u')\,ds\,,$$

for $r \in [0,a]$. Since

$$\begin{aligned} \int_0^r (v'\Delta u + u'\Delta v)\,ds &= \int_0^r (u'v')'\,ds + 2(n-1)\int_0^r \frac{u'(s)v'(s)}{s}\,ds \\ &= u'(r)v'(r) + 2(n-1)\int_0^r \frac{u'(s)v'(s)}{s}\,ds\,, \end{aligned}$$

and

$$\int_0^r (g(v)v' + f(u)u')\,ds = G(v(r)) + F(u(r)) - G(-\beta) - F(\alpha)\,,$$

we obtain

$$F(u(r)) + G(v(r)) = F(\alpha) + G(-\beta(\alpha)) + u'(r)v'(r) + 2(n-1)\int_0^r \frac{u'(s)v'(s)}{s}\,ds\,,$$

for $r \in [0,a]$, which implies that

$$G(v(r)) \le F(\alpha) + G(-\beta) \quad 0 \le r \le a\,,$$

and the lemma follows. $\square$

Lemma 2.3. *For each $\alpha > 0$, $\beta > 0$ there exists $T > 0$ such that problem* (2.1) *on $[0,T]$ has a unique solution $(u,v) \in (C^2[0,T])^2$ such that $u > 0$ (resp. $v < 0$) in $[0,T]$ and $u' < 0$ (resp. $v' > 0$) in $(0,T]$.*

Proof. Let α, $\beta > 0$ be given. Choose $T = T(\alpha,\beta) > 0$ such that

$$T = \min\Big(\big(\frac{n\alpha}{-g(-\beta)}\big)^{1/2}, \big(\frac{n\beta}{f(\alpha)}\big)^{1/2}\Big)\,,$$

and consider the set of functions

$$Z = \{(u,v) \in (C[0,T])^2;\ \alpha/2 \le u(r) \le \alpha \text{ and } \\ -\beta \le v(r) \le -\beta/2 \text{ for } 0 \le r \le T\}\,.$$

Clearly Z is a bounded closed convex subset of the Banach space $(C[0,T])^2$ endowed with the norm $\|(u,v)\| = \max(\|u\|_\infty, \|v\|_\infty)$. Define

$$L(u,v)(r) = (\alpha + \int_0^r G_n(r,s)g(v(s))\,ds, -\beta + \int_0^r G_n(r,s)f(u(s))\,ds)\,,$$

for $r \in [0,T]$ and $(u,v) \in (C[0,T])^2$. It is easily verified that L is a compact operator mapping Z into itself, and so there exists $(u,v) \in Z$ such that $(u,v) = L(u,v)$ by the Schauder fixed point theorem. Clearly $(u,v) \in (C^2[0,T])^2$ and (u,v) is a solution of (2.1) on $[0,T]$. Since f and g are C^1 the uniqueness follows. Since $u > 0$ and $v < 0$ in $[0,T]$, direct integration of the system (2.1) implies that $u' < 0$ and $v' > 0$ in $(0,T]$.

By Lemma 2.3 for any α, $\beta > 0$ problem (2.1) has a unique local solution: Let $[0, R_{\alpha,\beta})$ denote the maximum interval of existence of that solution ($R_{\alpha,\beta} = \infty$ possibly). Define

$$P_{\alpha,\beta} = \{s \in (0, R_{\alpha,\beta})\,;\, u(\alpha,\beta,r)u'(\alpha,\beta,r) < 0\,\forall\, r \in (0,s]\}$$

where $(u(\alpha,\beta,.), v(\alpha,\beta,.))$ is the solution of (2.1) in $[0,R_{\alpha,\beta})$. $P_{\alpha,\beta} \ne \emptyset$ by Lemma 2.3. Set

$$r_{\alpha,\beta} = \sup P_{\alpha,\beta}\,.$$

$\square$

Lemma 2.4. *We have $u'(\alpha,\beta,r) < 0$ for $r \in (0, r_{\alpha,\beta})$ and $v'(\alpha,\beta,r) > 0$ for $r \in (0, r_{\alpha,\beta}]$.*

Proof. The first assertion follows from the definition of $r_{\alpha,\beta}$. Since $u(\alpha,\beta,r) > 0$ for $r \in [0, r_{\alpha,\beta})$, integrating the second equation in (2.1) from 0 to $r \in (0, r_{\alpha,\beta}]$ we obtain $v'(\alpha,\beta,r) > 0$ for $r \in (0, r_{\alpha,\beta}]$. $\square$

Lemma 2.5. *For any α, $\beta > 0$ we have $r_{\alpha,\beta} < R_{\alpha,\beta}$.*

Proof. If not, there exist α, $\beta > 0$ such that $r_{\alpha,\beta} = R_{\alpha,\beta}$. Suppose first that $R_{\alpha,\beta} < \infty$. Noting $u = u(\alpha, \beta, .)$ and $v = v(\alpha, \beta, .)$ we have $0 < u \le \alpha$ in $[0, R_{\alpha,\beta})$. Then we easily deduce that u, u', v and v' are bounded on $[0, R_{\alpha,\beta})$ and we obtain a contradiction with the definition of $R_{\alpha,\beta}$. Now assume that $R_{\alpha,\beta} = \infty$. We have $0 < u \le \alpha$ in $[0, \infty)$. By Lemma 2.1 $v < 0$ in $[0, \infty)$. When $n = 1$ (H2) implies that $u'' < 0$ in $[0, \infty)$ and we deduce that $u'(r) \le u'(1) < 0$ for all $r \ge 1$, from which we obtain $u(r) \le u(1) + u'(1)(r - 1)$ for all $r \ge 1$. Thus we can find $r > 1$ such that $u(r) < 0$ and we have a contradiction. If $n = 2$, (H2) implies that $(ru'(r))' < 0$ on $(0, \infty)$. We deduce that $ru'(r) \le u'(1) < 0$ for all $r \ge 1$, from which we obtain $u(r) \le u(1) + u'(1) \ln r$ for all $r \ge 1$. Thus we can find $r \ge 1$ such that $u(r) < 0$ and we obtain a contradiction. Now let $n \ge 3$. Suppose first that f satisfies (H5). From the second equation in (2.1), using (H3), we obtain

$$v(r) \ge -\beta + \frac{f(0)}{2n} r^2 \quad \forall\, r \ge 0\,,$$

which implies, with the help of (H5), that $v(r) \to \infty$ as $r \to \infty$ and we have a contradiction. Now suppose that f verifies (H8). Let $z = -v$. We have u, $z > 0$ on $[0, \infty)$ and, by (H8),

$$-\Delta u \ge b z^q\,, \quad -\Delta z \ge a u^p \quad \text{on } [0, \infty)\,.$$

Since $pq < 1$ we have

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-2}{n}\,.$$

Then we obtain a contradiction with the help of the nonexistence results established in [3]-[5].

Proposition 2.6. *For each $\alpha > 0$, there exists a unique $\beta > 0$ such that*

$$u(\alpha, \beta, r_{\alpha,\beta}) = u'(\alpha, \beta, r_{\alpha,\beta}) = 0\,.$$

Proof. We first prove the uniqueness. Let $\alpha > 0$ be fixed. Suppose that there exist $\beta > \gamma > 0$ such that $u(\alpha, \beta, r_{\alpha,\beta}) = u'(\alpha, \beta, r_{\alpha,\beta}) = u(\alpha, \gamma, r_{\alpha,\gamma}) = u'(\alpha, \gamma, r_{\alpha,\gamma}) = 0$. In order to simplify our notations we denote $u(\alpha, \beta, r)$, $u(\alpha, \gamma, r)$, $v(\alpha, \beta, r)$ and $v(\alpha, \gamma, r)$ by $u(r)$, $w(r)$, $v(r)$ and $z(r)$. Define $b = \min(r_{\alpha,\beta}, r_{\alpha,\gamma})$. Suppose that there exists $a \in (0, b]$ such that $v - z < 0$ in $[0, a)$ and $(v - z)(a) = 0$. Using (H2) we obtain $\Delta(u - w) = g(v) - g(z) < 0$ in $[0, a)$. Since $(u - w)(0) = (u - w)'(0) = 0$, we deduce that $u - w < 0$ in $(0, a]$. Using (H3) we obtain $\Delta(v - z) = f(u) - f(w) < 0$ in $(0, a]$. We have $(v - z)(0) < 0$, $(v - z)'(0) = 0$ and $(v - z)(a) = 0$. Therefore we reach a contradiction. Thus $v - z < 0$ in $[0, b]$. As before we show that $u - w < 0$ in $(0, b]$. Since $(u - w)'(0) = 0$ we deduce that $(u - w)'(b) < 0$. By Lemma 2.4 we have

$$(u - w)'(b) = \begin{cases} u'(r_{\alpha,\gamma}) < 0 & \text{if } r_{\alpha,\beta} > r_{\alpha,\gamma}\,, \\ 0 & \text{if } r_{\alpha,\beta} = r_{\alpha,\gamma}\,, \\ -w'(r_{\alpha,\beta}) > 0 & \text{if } r_{\alpha,\beta} < r_{\alpha,\gamma}\,. \end{cases}$$

Therefore, $b = r_{\alpha,\gamma} < r_{\alpha,\beta}$. Now $(u - w)(b) = u(r_{\alpha,\gamma}) > 0$ and we obtain a contradiction. The case $0 < \beta < \gamma$ can be handled in the same way.

Now we prove the existence. Suppose that there exists $\alpha > 0$ such that for any $\beta > 0$ $u(\alpha, \beta, r_{\alpha,\beta}) > 0$ or $u'(\alpha, \beta, r_{\alpha,\beta}) < 0$. Define the sets

$$B = \{\beta > 0;\ u(\alpha, \beta, r_{\alpha,\beta}) = 0 \text{ and } u'(\alpha, \beta, r_{\alpha,\beta}) < 0\}\,,$$
$$C = \{\beta > 0;\ u(\alpha, \beta, r_{\alpha,\beta}) > 0 \text{ and } u'(\alpha, \beta, r_{\alpha,\beta}) = 0\}\,.$$

The proof of the proposition is completed by using the next two lemmas which contradict the fact that $(0,\infty)=B\cup C$. $\square$

Lemma 2.7. (i) *If $B\neq\emptyset$, then* $\inf B>0$.
(ii) *If $C\neq\emptyset$, then* $\sup C<\infty$.

Lemma 2.8. *Sets B and C are open.*

Proof of Lemma 2.7. We have

$$u(\alpha,\beta,r)=\alpha+\int_0^r G_n(r,s)g(v(\alpha,\beta,s))\,ds\,,\quad 0\le r<R_{\alpha,\beta}\,, \tag{2.3}$$

$$v(\alpha,\beta,r)=-\beta+\int_0^r G_n(r,s)f(u(\alpha,\beta,s))\,ds\,,\quad 0\le r<R_{\alpha,\beta}\,. \tag{2.4}$$

(i) Let $\beta\in B$. Assume first that $v(\alpha,\beta,.)<0$ on $[0,r_{\alpha,\beta})$. Then Lemma 2.4, Equation (2.3) and assumption (H2) imply

$$r_{\alpha,\beta}\ge\big(\frac{2n\alpha}{-g(-\beta)}\big)^{1/2}. \tag{2.5}$$

Now, if there exists $s_{\alpha,\beta}\in[0,r_{\alpha,\beta})$ such that $v(\alpha,\beta,s_{\alpha,\beta})=0$, Lemma 2.4 implies that $-\beta\le v(\alpha,\beta,.)<0$ on $[0,s_{\alpha,\beta})$ and $v(\alpha,\beta,.)>0$ on $(s_{\alpha,\beta},r_{\alpha,\beta}]$. Then from (2.3) and (H2) we obtain

$$\begin{aligned}
\alpha&=-\int_0^{r_{\alpha,\beta}} G_n(r_{\alpha,\beta},s)g(v(\alpha,\beta,s))\,ds\\
&\le-\int_0^{s_{\alpha,\beta}} G_n(r_{\alpha,\beta},s)g(v(\alpha,\beta,s))\,ds\\
&\le-g(-\beta)\int_0^{s_{\alpha,\beta}} G_n(r_{\alpha,\beta},s)\,ds\\
&\le-g(-\beta)\frac{r_{\alpha,\beta}^2}{2n}
\end{aligned}$$

and (2.5) still holds.

Suppose that $\inf B=0$ and let (β_j) be a sequence in B decreasing to zero. Then $r_{\alpha,\beta_j}\to+\infty$ by (2.5) and (H2). Let $r>0$ be fixed. We can assume that $r_{\alpha,\beta_j}>r$ for all j. If $v(\alpha,\beta_j,s)<0$ for $s\in[0,r]$, using (H2) we have

$$u(\alpha,\beta_j,r)=\alpha+\int_0^r G_n(r,s)g(v(\alpha,\beta_j,s))\,ds\ge\alpha+\frac{r^2g(-\beta_j)}{2n}\,.$$

If $v(\alpha,\beta_j,s_{\alpha,\beta_j})=0$ with $s_{\alpha,\beta_j}<r$ we write

$$\begin{aligned}
u(\alpha,\beta_j,r)&=\alpha+\int_0^{s_{\alpha,\beta_j}} G_n(r,s)g(v(\alpha,\beta_j,s))\,ds+\int_{s_{\alpha,\beta_j}}^r G_n(r,s)g(v(\alpha,\beta_j,s))\,ds\\
&\ge\alpha+\int_0^{s_{\alpha,\beta_j}} G_n(r,s)g(v(\alpha,\beta_j,s))\,ds\\
&\ge\alpha+g(-\beta_j)\int_0^{s_{\alpha,\beta_j}} G_n(r,s)\,ds\\
&\ge\alpha+\frac{g(-\beta_j)r^2}{2n}
\end{aligned}$$

Therefore, using Lemma 2.4 we obtain

$$u(\alpha,\beta_j,s)\geq \alpha+\frac{g(-\beta_j)r^2}{2n}\quad \text{for } s\in[0,r]\,,$$

from which we deduce with the help of (H2) that $u(\alpha,\beta_j,s)\geq \alpha/2$ for $s\in[0,r]$ and j large. From (2.4), using (H3) we obtain

$$v(\alpha,\beta_j,r)\geq -\beta_j+\frac{r^2}{2n}f(\frac{\alpha}{2})$$

for j large. Thus if we choose r such that

$$-\beta_j+\frac{r^2}{2n}f(\frac{\alpha}{2})\geq 1\,,$$

using Lemma 2.4 we obtain $v(\alpha,\beta_j,s)\geq 1$ for $r\leq s\leq r_{\alpha,\beta_j}$ and j large. There exists $k>0$ such that

$$\int_r^{r_{\alpha,\beta_j}} G_n(r_{\alpha,\beta_j},s)\,ds\geq kr^2_{\alpha,\beta_j}$$

for j large. Now we write

$$\begin{aligned}
\alpha&=-\int_0^{r_{\alpha,\beta_j}} G_n(r_{\alpha,\beta_j},s)g(v(\alpha,\beta_j,s))\,ds\\
&=-\int_0^{r} G_n(r_{\alpha,\beta_j},s)g(v(\alpha,\beta_j,s))\,ds-\int_r^{r_{\alpha,\beta_j}} G_n(r_{\alpha,\beta_j},s)g(v(\alpha,\beta_j,s))\,ds\\
&\leq -g(-\beta_j)\int_0^{r} G_n(r_{\alpha,\beta_j},s)\,ds-g(1)\int_r^{r_{\alpha,\beta_j}} G_n(r_{\alpha,\beta_j},s)\,ds\\
&\leq -g(-\beta_j)rr_{\alpha,\beta_j}-g(1)kr^2_{\alpha,\beta_j}
\end{aligned}$$

for j large, where we have used the fact that $G_n(r_{\alpha,\beta_j},s)\leq r_{\alpha,\beta_j}$ for $0\leq s\leq r_{\alpha,\beta_j}$. Since the last term above tends to $-\infty$ as $j\to\infty$ we obtain a contradiction.

(ii) Let $\beta\in C$. We claim that $v(\alpha,\beta,r_{\alpha,\beta})>0$. If not, by Lemma 2.4 and (H2) we have $\Delta u(\alpha,\beta,.)<0$ on $[0,r_{\alpha,\beta})$ for some $\beta\in C$. Since $u'(\alpha,\beta,0)=0$, we obtain $u'(\alpha,\beta,r_{\alpha,\beta})<0$, a contradiction. Therefore (2.4) implies

$$\beta<\int_0^{r_{\alpha,\beta}} G_n(r_{\alpha,\beta},s)f(u(\alpha,\beta,s))\,ds \tag{2.6}$$

for $\beta\in C$. Suppose that $\sup C=\infty$ and let (β_j) be a sequence in C increasing to ∞. Since $0<u(\alpha,\beta_j,r)\leq\alpha$ for $r\in[0,r_{\alpha,\beta_j}]$, (2.6) implies that $r_{\alpha,\beta_j}\to\infty$ as $j\to\infty$. Then we can assume that $r_{\alpha,\beta_j}\geq 1$ and that $f(\alpha)\leq\beta_j$ for all j. From (2.4) we obtain

$$-\beta_j\leq v(\alpha,\beta_j,r)\leq -\frac{2n-1}{2n}\beta_j\leq -\frac{\beta_j}{2}\,,\quad \text{for } r\in[0,1]\,,$$

and using (2.3) we deduce that $u(\alpha,\beta_j,1)\leq\alpha+g(-\beta_j/2)/2n$. However by (H2), $u(\alpha,\beta_j,1)<0$ for j large; thus we reach a contradiction. □

Remark 2.9. The proof above shows that, when $\beta\in C$, there exists $s_{\alpha,\beta}\in(0,r_{\alpha,\beta})$ such that $v(\alpha,\beta,.)<0$ on $[0,s_{\alpha,\beta})$ and $v(\alpha,\beta,.)>0$ on $(s_{\alpha,\beta},r_{\alpha,\beta}]$. When $\beta\in B$, $s_{\alpha,\beta}$ may not exist.

Proof of Lemma 2.8. Let $\beta \in B$. We have $u(\alpha,\beta,r_{\alpha,\beta}) = 0$ and $u'(\alpha,\beta,r_{\alpha,\beta}) < 0$. Therefore, we can find $\varepsilon > 0$ such that

$$u(\alpha,\beta,r_{\alpha,\beta}+\varepsilon) < 0 \quad\text{and}\quad u'(\alpha,\beta,r_{\alpha,\beta}+\varepsilon) < 0\,.$$

But then by continuous dependence on initial data there exists $\eta > 0$ such that

$$u(\alpha,\gamma,r_{\alpha,\beta}+\varepsilon) < 0 \quad\text{and}\quad u'(\alpha,\gamma,r_{\alpha,\beta}+\varepsilon) < 0\,, \tag{2.7}$$

for $|\gamma-\beta| < \eta$. The first inequality in (2.7) implies that there exists $x \in (0, r_{\alpha,\beta}+\varepsilon)$ such that $u(\alpha,\gamma,x) = 0$ and $u(\alpha,\gamma,r) > 0$ for $r \in [0,x)$. We claim that $x = r_{\alpha,\gamma}$. Then $(\beta-\eta,\beta+\eta) \subset B$. Thus B is open. (H3) implies that $\Delta v(\alpha,\beta,r) > 0$ for $r \in [0,x)$. Then $v'(\alpha,\gamma,r) > 0$ for $r \in (0,x]$. Suppose first that $v'(\alpha,\gamma,r) > 0$ for $r \in (0,r_{\alpha,\beta}+\varepsilon)$. Then $v(\alpha,\gamma,.)$ is increasing on $[0,r_{\alpha,\beta}+\varepsilon]$. We deduce that $\Delta u(\alpha,\gamma,.)$ is increasing on $[0,r_{\alpha,\beta}+\varepsilon]$. If $\Delta u(\alpha,\gamma,r_{\alpha,\beta}+\varepsilon) \le 0$, then $u'(\alpha,\gamma,r) < 0$ for $r \in (0,r_{\alpha,\beta}+\varepsilon]$. If $\Delta u(\alpha,\gamma,r_{\alpha,\beta}+\varepsilon) > 0$, then there exists $y \in (0,r_{\alpha,\beta}+\varepsilon)$ such that $\Delta u(\alpha,\gamma,.) < 0$ in $[0,y)$ and $\Delta u(\alpha,\gamma,.) > 0$ in $(y,r_{\alpha,\beta}+\varepsilon]$. We deduce that $r \to r^{n-1}u'(\alpha,\gamma,r)$ is decreasing (resp. increasing) in $[0,y]$ (resp. $[y,r_{\alpha,\beta}+\varepsilon]$). Then the second inequality in (2.7) implies that $u'(\alpha,\gamma,r) < 0$ for $r \in (0,r_{\alpha,\beta}+\varepsilon]$. Therefore $x = r_{\alpha,\gamma}$ for $|\gamma-\beta| < \eta$. Suppose now that there exists $r \in (x,r_{\alpha,\beta}+\varepsilon)$ such that $v'(\alpha,\gamma,r) \le 0$. Let $t \in (x,r_{\alpha,\beta}+\varepsilon)$ be the first zero of $v'(\alpha,\gamma,.)$. Then $v(\alpha,\gamma,.)$ is increasing on $[0,t]$. We deduce that $\Delta u(\alpha,\gamma,.)$ is increasing on $[0,t]$. If $\Delta u(\alpha,\gamma,t) \le 0$, then $u'(\alpha,\gamma,r) < 0$ for $r \in (0,t]$ and we conclude that $x = r_{\alpha,\gamma}$. If $\Delta u(\alpha,\gamma,t) > 0$, then there exists $y \in (0,t)$ such that $\Delta u(\alpha,\gamma,.) < 0$ in $[0,y)$ and $\Delta u(\alpha,\gamma,.) > 0$ in $(y,t]$. We deduce that $r \to r^{n-1}u'(\alpha,\gamma,r)$ is decreasing (resp. increasing) in $[0,y]$ (resp. $[y,t]$). If $u'(\alpha,\gamma,t) \le 0$, then $u'(\alpha,\gamma,x) < 0$ and $x = r_{\alpha,\gamma}$. If $u'(\alpha,\gamma,t) > 0$, let $s \in (y,t)$ be such that $u'(\alpha,\gamma,s) = 0$. If $s > x$ then $x = r_{\alpha,\gamma}$. If $s < x$, then $u(\alpha,\gamma,r) > 0$ for $r \in [0,x]$ and we reach a contradiction. Finally if $t = x$, then $u(\alpha,\gamma,r) > 0$ for $r \in [0,x) \cup (x,t]$. We deduce that $\Delta v(\alpha,\gamma,r) > 0$ for $r \in [0,x) \cup (x,t]$ which implies that $v'(\alpha,\gamma,r) > 0$ for $r \in (0,t]$ and we reach again a contradiction.

Now let $\beta \in C$. We have $u(\alpha,\beta,r_{\alpha,\beta}) > 0$ and $u'(\alpha,\beta,r_{\alpha,\beta}) = 0$. By Remark 2.9 we have $v(\alpha,\beta,r_{\alpha,\beta}) > 0$, hence $\Delta u(\alpha,\beta,r_{\alpha,\beta}) = u''(\alpha,\beta,r_{\alpha,\beta}) > 0$. Therefore we can find $\varepsilon > 0$ such that

$$u(\alpha,\beta,r) > 0\,, \quad r \in [0,r_{\alpha,\beta}+\varepsilon], \quad u'(\alpha,\beta,r_{\alpha,\beta}+\varepsilon) > 0\,.$$

Then by continuous dependence on initial data there exists $\eta > 0$ such that

$$u(\alpha,\gamma,r) > 0\,, \quad r \in [0,r_{\alpha,\beta}+\varepsilon], \quad u'(\alpha,\gamma,r_{\alpha,\beta}+\varepsilon) > 0\,, \tag{2.8}$$

for $|\gamma-\beta| < \eta$. The second inequality in (2.8) implies that there exists $x \in (0,r_{\alpha,\beta}+\varepsilon)$ such that $u'(\alpha,\gamma,x) = 0$ and $u'(\alpha,\gamma,r) < 0$ for $r \in (0,x)$. Therefore, $x = r_{\alpha,\gamma}$ for $|\gamma-\beta| < \eta$ and $(\beta-\eta,\beta+\eta) \subset C$. Thus C is open. □

3. Proof of Theorems 1.1 and 1.2

We use the notation introduced in Section 2. The following result clearly implies Theorems 1.1 and 1.2.

Proposition 3.1. *Let f and g satisfy* (H1)–(H3)*, and* (H4) *or* (H5), (H6)*. Then for any $\alpha > 0$ there exists a unique $(\beta(\alpha),r(\alpha)) \in (0,\infty)\times(0,\infty)$ such that $u(\alpha,\beta(\alpha),r(\alpha)) = u'(\alpha,\beta(\alpha),r(\alpha)) = 0$, $u(\alpha,\beta(\alpha),r) > 0$ for $r \in [0,r(\alpha))$ and $u'(\alpha,\beta(\alpha),r) < 0$ for $r \in (0,r(\alpha))$. Moreover, β, $r \in C^1(0,\infty)$, $\beta'(\alpha) > 0$ for $\alpha > 0$, $\lim_{\alpha\to 0} r(\alpha) = 0$ and $\lim_{\alpha\to\infty} r(\alpha) = \infty$.*

Proof. Let $\alpha > 0$ be fixed. The existence and uniqueness of $(\beta(\alpha), r(\alpha))$ satisfying the first part of the proposition are given by Proposition 2.6. In order to simplify our notations we denote $u(\alpha, \beta(\alpha), r)$ and $v(\alpha, \beta(\alpha), r)$ by $u_\alpha(r)$ and $v_\alpha(r)$. We begin with the following lemma. □

Lemma 3.2. *For any $\alpha > 0$ there exists $s(\alpha) \in (0, r(\alpha))$ such that $v_\alpha(r) < 0$ for $r \in [0, s(\alpha))$ and $v_\alpha(r) > 0$ for $r \in (s(\alpha), r(\alpha)]$*

The proof of the above lemma follows the same arguments as in the proof of Lemma 2.7 (ii).

Now for α, $\beta > 0$ define

$$\varphi(\alpha,\beta,r) = \frac{\partial u}{\partial \alpha}(\alpha,\beta,r)\,, \quad \psi(\alpha,\beta,r) = \frac{\partial v}{\partial \alpha}(\alpha,\beta,r),$$
$$\rho(\alpha,\beta,r) = \frac{\partial u}{\partial \beta}(\alpha,\beta,r)\,, \quad \chi(\alpha,\beta,r) = \frac{\partial v}{\partial \beta}(\alpha,\beta,r)$$

for $r \in [0, R_{\alpha,\beta})$. Then φ, ψ, ρ and χ satisfy the linearized equations

$$\begin{gathered}
\Delta\varphi(\alpha,\beta,r) = g'(v(\alpha,\beta,r))\psi(\alpha,\beta,r)\,, \quad 0 \le r < R_{\alpha,\beta}\,,\\
\Delta\psi(\alpha,\beta,r) = f'(u(\alpha,\beta,r))\varphi(\alpha,\beta,r)\,, \quad 0 \le r < R_{\alpha,\beta}\,,\\
\varphi(\alpha,\beta,0) = 1\,,\ \psi(\alpha,\beta,0) = \varphi'(\alpha,\beta,0) = \psi'(\alpha,\beta,0) = 0\,,
\end{gathered}$$

and

$$\begin{gathered}
\Delta\rho(\alpha,\beta,r) = g'(v(\alpha,\beta,r))\chi(\alpha,\beta,r)\,, \quad 0 \le r < R_{\alpha,\beta}\,,\\
\Delta\chi(\alpha,\beta,r) = f'(u(\alpha,\beta,r))\rho(\alpha,\beta,r)\,, \quad 0 \le r < R_{\alpha,\beta}\,,\\
\chi(\alpha,\beta,0) = -1\,,\ \rho(\alpha,\beta,0) = \rho'(\alpha,\beta,0) = \chi'(\alpha,\beta,0) = 0\,.
\end{gathered}$$

Lemma 3.3. *We have φ, ψ, φ', $\psi' > 0$ on $(0, r_{\alpha,\beta}]$ and χ, ρ, χ', $\rho' < 0$ on $(0, r_{\alpha,\beta}]$.*

Proof. By (H2) and (H3) we have $\Delta\psi(\alpha,\beta,0) = f'(\alpha) > 0$ and $\Delta\rho(\alpha,\beta,0) = -g'(-\beta) < 0$. Then $\psi' > 0$ and $\rho' < 0$ on $(0, \eta]$ for some $\eta > 0$ and we can define

$$r_0 = \sup\{r \in (0, r_{\alpha,\beta}];\ \psi'\rho' < 0 \text{ on } (0, r]\}\,.$$

We have $\psi > 0$ and $\rho < 0$ on $(0, r_0]$. Since

$$r^{n-1}\varphi'(\alpha,\beta,r) = \int_0^r s^{n-1} g'(v(\alpha,\beta,s))\psi(\alpha,\beta,s)\,ds,$$
$$r^{n-1}\chi'(\alpha,\beta,r) = \int_0^r s^{n-1} f'(u(\alpha,\beta,s))\rho(\alpha,\beta,s)\,ds\,,$$

using (H2) and (H3) we deduce that $\varphi' > 0$ and $\chi' < 0$ on $(0, r_0]$. Therefore, $\varphi > 0$ and $\chi < 0$ on $(0, r_0]$. Since

$$r^{n-1}\psi'(\alpha,\beta,r) = \int_0^r s^{n-1} f'(u(\alpha,\beta,s))\varphi(\alpha,\beta,s)\,ds,$$
$$r^{n-1}\rho'(\alpha,\beta,r) = \int_0^r s^{n-1} g'(v(\alpha,\beta,s))\chi(\alpha,\beta,s)\,ds$$

using (H2) and (H3) we deduce that $\psi' > 0$ and $\rho' < 0$ on $(0, r_0]$. Therefore, $r_0 = r_{\alpha,\beta}$ and the lemma follows. □

Now let $D = \{(\alpha, \beta, r);\, \alpha\,,\, \beta > 0 \text{ and } r \in [0, R_{\alpha,\beta})\}$. D is open in $(0, \infty) \times (0, \infty) \times [0, \infty)$. Consider the map $H : D \to \mathbb{R}^2$ defined by

$$H(\alpha, \beta, r) = (u(\alpha, \beta, r), u'(\alpha, \beta, r))\,.$$

Then $H \in C^1(D, \mathbb{R}^2)$ and

$$H(\alpha, \beta(\alpha), r(\alpha)) = 0 \quad \text{for } \alpha > 0\,. \tag{3.1}$$

Using Lemmas 3.2, 3.3 and (H2) we obtain

$$|D_{(\beta,t)} H(\alpha, \beta(\alpha), r(\alpha))| = \rho(\alpha, \beta(\alpha), r(\alpha)) u''_\alpha(r(\alpha)) \;< 0\,.$$

Therefore, by the implicit function theorem $\alpha \to (\beta(\alpha), r(\alpha))$ is a C^1 map for $\alpha > 0$. Differentiating (3.1) with respect to α we obtain

$$\varphi(\alpha, \beta(\alpha), r(\alpha)) + \rho(\alpha, \beta(\alpha), r(\alpha))\beta'(\alpha) = 0 \quad \text{for } \alpha > 0\,. \tag{3.2}$$

From (3.2) and Lemma 3.3 we deduce that $\beta'(\alpha) > 0$ for $\alpha > 0$. Now we have two cases to consider.

Case 1: f satisfies (H1), (H3) and (H4). Assume that $\beta(\alpha) \to \infty$ as $\alpha \to \infty$. Using (H4) we have

$$v_\alpha(r(\alpha)) = -\beta(\alpha) + \int_0^{r(\alpha)} G_n(r(\alpha), s) f(u_\alpha(s))\, ds \le -\beta(\alpha) + M \frac{r(\alpha)^2}{2n}\,.$$

Then Lemma 3.2 implies that $r(\alpha) \to \infty$ as $\alpha \to \infty$. Now suppose that $\beta(\alpha) \to c < \infty$ as $\alpha \to \infty$. Using Lemma 3.2 and (H2) we can write

$$\begin{aligned}
0 = u_\alpha(r(\alpha)) &= \alpha + \int_0^{r(\alpha)} G_n(r(\alpha), s) g(v_\alpha(s))\, ds \\
&= \alpha + \int_0^{s(\alpha)} G_n(r(\alpha), s) g(v_\alpha(s))\, ds + \int_{s(\alpha)}^{r(\alpha)} G_n(r(\alpha), s) g(v_\alpha(s))\, ds \\
&\ge \alpha + g(-\beta(\alpha)) \frac{r(\alpha)^2}{2n}\,,
\end{aligned}$$

and again we deduce that $r(\alpha) \to \infty$ as $\alpha \to \infty$.

Assume that $\beta(\alpha) \to 0$ as $\alpha \to 0$. Using (H4) we have

$$v_\alpha(r(\alpha)) = -\beta(\alpha) + \int_0^{r(\alpha)} G_n(r(\alpha), s) f(u_\alpha(s))\, ds \ge -\beta(\alpha) + m \frac{r(\alpha)^2}{2n}\,. \tag{3.3}$$

Then Lemma 2.2 implies that $r(\alpha) \to 0$ as $\alpha \to 0$. Now suppose that $\beta(\alpha) \to c > 0$ as $\alpha \to 0$. We claim that $r(\alpha) \to 0$ as $\alpha \to 0$. If not there exist $r_0 > 0$ and a sequence $(\alpha_k)_{k\in\mathbb{N}}$ such that $\alpha_k \to 0$ as $k \to \infty$ and $r(\alpha_k) \ge r_0$ for all $k \in \mathbb{N}$. Let $r_1 \in (0, r_0)$ be such that

$$-c + f(0) \frac{r_1^2}{2n} \le -\frac{c}{2}\,.$$

For $s \in [0, r_1]$ we have

$$\begin{aligned}
\lim_{k\to\infty} v_{\alpha_k}(s) &= \lim_{k\to\infty} (-\beta(\alpha_k) + \int_0^s G_n(s, x) f(u_{\alpha_k}(x))\, dx) \\
&= -c + f(0) \frac{s^2}{2n} \le -\frac{c}{2}\,.
\end{aligned} \tag{3.4}$$

Clearly $(v_k)_{k\in\mathbb{N}}$ converges uniformly on $[0,r_1]$. Then, for $s\in(0,r_1]$, using (3.4) and (H2) we have

$$\begin{aligned}\lim_{k\to\infty} u_{\alpha_k}(s) &= \lim_{k\to\infty}(\alpha_k+\int_0^s G_n(s,x)g(v_{\alpha_k}(x))\,dx)\\ &= \int_0^s G_n(s,x)g(-c+f(0)\frac{x^2}{2n})\,dx<0\,,\end{aligned}$$

and we reach a contradiction. Therefore our claim is proved and the proof of Proposition 3.1 is complete in Case 1.

Case 2: f satisfies (H1), (H3), (H5) and (H6). From the proof of Lemma 2.2 we obtain

$$G(v_\alpha(r(\alpha)))=F(\alpha)+G(-\beta(\alpha))+2(n-1)\int_0^{r(\alpha)}\frac{u'_\alpha(s)v'_\alpha(s)}{s}\,ds\,. \tag{3.5}$$

Let $\alpha\geq 1$. Using Lemma 3.2, (2.4), (H3) and (H6) we have

$$0<v_\alpha(r(\alpha))\leq a'\alpha^p\frac{r(\alpha)^2}{2n}\,, \tag{3.6}$$

and

$$\beta(\alpha)=\int_0^{s(\alpha)}G_n(s(\alpha),s)f(u_\alpha(s))\,ds\leq a'\alpha^p\frac{s(\alpha)^2}{2n}\leq a'\alpha^p\frac{r(\alpha)^2}{2n}\,. \tag{3.7}$$

Then, with the help of (H2), (H3), (H6), (3.6) and (3.7), we can write

$$\begin{aligned}|r^{n-1}u'_\alpha(r)| &= |\int_0^r s^{n-1}g(v_\alpha(s))\,ds|\\ &\leq \frac{r^n}{n}\max(|g(-\beta(\alpha))|,g(v_\alpha(r(\alpha))))\\ &\leq \frac{r^n}{2^q n^{q+1}}b'a'^q\alpha^{pq}r(\alpha)^{2q}\,,\end{aligned}$$

and

$$|r^{n-1}v'_\alpha(r)|=|\int_0^r s^{n-1}f(u_\alpha(s))\,ds|\leq\frac{r^n}{n}f(\alpha)\leq\frac{r^n}{n}a'\alpha^p\,,$$

for $r\in[0,r(\alpha)]$. Therefore,

$$|\int_0^{r(\alpha)}\frac{u'_\alpha(s)v'_\alpha(s)}{s}\,ds|\leq\frac{b'a'^{q+1}}{2^{q+1}n^{q+2}}\alpha^{p(q+1)}r(\alpha)^{2(q+1)}\,. \tag{3.8}$$

Now using (3.5), (3.6), (3.8) an (H6) we obtain

$$\begin{aligned}&\frac{a'^{q+1}b'}{(q+1)2^{q+1}n^{q+1}}\alpha^{p(q+1)}r(\alpha)^{2(q+1)}\\ &\geq G(v_\alpha(r(\alpha)))\\ &\geq\frac{a}{p+1}\alpha^{p+1}+\frac{b}{q+1}\beta(\alpha)^{q+1}-\frac{b'a'^{q+1}}{2^q n^{q+1}}\alpha^{p(q+1)}r(\alpha)^{2(q+1)}\,.\end{aligned} \tag{3.9}$$

Since $pq<1$, we deduce that $r(\alpha)\to\infty$ as $\alpha\to\infty$.

Now assume that $\alpha\leq 1$. (H3) and (H5) imply that there exist m, $M>0$ such that

$$m\leq f(u_\alpha(r))\leq M\quad\forall\, r\in[0,r(\alpha)]\,.$$

Then we show that $\lim_{\alpha\to 0}r(\alpha)=0$ as in Case 1. ☐

We conclude this section with the following theorems.

Theorem 3.4. *Let f, g satisfy* (H1)–(H3). *Assume moreover that*

(H9) *There exist a, b, $p > 0$ and $q \geq 1$ such that*
$$f(u) \geq au^p \quad \forall\, u \geq 0\,, \quad |g(-v)| \geq bv^q \quad \forall\, v \geq 0\,,$$
and
$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-2}{n} \quad \text{if} \quad n \geq 3\,.$$

Then there exists $R > 0$ such that (1.1) *has at least one non-trivial radial solution $(u, v) \in (C^2(\overline{B}_R))^2$.*

Since Proposition 2.6 holds, with the help of the first part of Proposition 3.1, we conclude the statement of the above theorem.

Remark 3.5. Notice that , when $f(0) > 0$, p may be less than 1. If $f(0) = 0$, necessarily $p \geq 1$ since f is C^1.

Theorem 3.6. *Let f, g satisfy* (H1)–(H3). *Moreover assume that*

(H10) *There exist a, a', b, $b' > 0$ and p, $q \geq 1$ such that $pq > 1$,*
$$f(u) \geq au^p \quad \forall\, u \geq 0\,, \quad f(u) \leq a'u^p \quad \forall\, u \in [0,1]\,,$$
$$b|v|^q \leq |g(v)| \leq b'|v|^q \quad \forall\, v \in \mathbb{R}\,,$$
$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-2}{n} \quad \text{if}\; n \geq 3\,.$$

Then there exists $R_0 \geq 0$ such that for all $R > R_0$ problem (1.1) *has at least one non-trivial radial solution $(u, v) \in (C^2(\overline{B}_R))^2$.*

Proof. Since Proposition 2.6 holds we have the first part of Proposition 3.1. We also have β, $r \in C^1(0, \infty)$ and $\beta'(\alpha) > 0$ for $\alpha > 0$. Now let $\alpha \in (0, 1]$. (3.5)-(3.9) hold and, since $pq > 1$, we conclude that $\lim_{\alpha\to 0} r(\alpha) = \infty$. Then we take
$$R_0 = \inf_{\alpha>0} r(\alpha) \geq 0\,.$$
□

4. Proof of Theorem 1.3

We use again the notation introduced in Section 2. The following result implies Theorem 1.3.

Proposition 4.1. *Assume that $n = 1$. Let f and g satisfy* (H1)–(H3), (H7). *Then for any $\alpha > 0$ there exists a unique $(\beta(\alpha), r(\alpha)) \in (0, \infty) \times (0, \infty)$ such that $u(\alpha, \beta(\alpha), r(\alpha)) = u'(\alpha, \beta(\alpha), r(\alpha)) = 0$, $u(\alpha, \beta(\alpha), r) > 0$ for $r \in [0, r(\alpha))$ and $u'(\alpha, \beta(\alpha), r) < 0$ for $r \in (0, r(\alpha))$. Moreover, β, $r \in C^1(0, \infty)$, $\beta'(\alpha) > 0$ for $\alpha > 0$, $\lim_{\alpha\to 0} r(\alpha) = \infty$ and $\lim_{\alpha\to\infty} r(\alpha) = 0$.*

Proof. Let $\alpha > 0$ be fixed. The existence and uniqueness of $(\beta(\alpha), r(\alpha))$ satisfying the first part of the proposition are given by Proposition 2.6. Clearly Lemmas 3.2 and 3.3 also hold. Then we have β, $r \in C^1(0, \infty)$, $\beta'(\alpha) > 0$ for $\alpha > 0$. We show that $\lim_{\alpha\to 0} r(\alpha) = \infty$ as in the proof of Theorem 3.6. As in the preceding section we denote $u(\alpha, \beta(\alpha), r)$ and $v(\alpha, \beta(\alpha), r)$ by $u_\alpha(r)$ and $v_\alpha(r)$. □

Now we give some lemmas where $s(\alpha)$ is defined in Lemma 3.2.

Lemma 4.2. *There exists a constant $C > 0$ such that*

$$u'_\alpha(s(\alpha))^2 \le C u_\alpha(s(\alpha))(\alpha^{p+1} + \beta(\alpha)^{q+1})^{\frac{q}{q+1}},$$
$$v'_\alpha(s(\alpha))^2 \le C\alpha^p(\alpha^{p+1} + \beta(\alpha)^{q+1})^{\frac{1}{q+1}},$$

for all $\alpha > 0$.

Proof. Since $n = 1$ we have

$$F(u_\alpha(r)) + G(v_\alpha(r)) = F(\alpha) + G(-\beta(\alpha)) + u'_\alpha(r)v'_\alpha(r) \tag{4.1}$$

for $r \in [0, r(\alpha)]$. Then (4.1) and (H7) imply that there exist two constants C_1, $C_2 > 0$ such that

$$C_1(\alpha^{p+1} + \beta(\alpha)^{q+1})^{\frac{1}{q+1}} \le v_\alpha(r(\alpha)) \le C_2(\alpha^{p+1} + \beta(\alpha)^{q+1})^{\frac{1}{q+1}}. \tag{4.2}$$

Using (H1), (H2), Lemma 2.4 and Lemma 3.2 we have

$$\begin{aligned}\frac{u'_\alpha(s(\alpha))^2}{2} &= -\int_{s(\alpha)}^{r(\alpha)} g(v_\alpha(s))u'_\alpha(s)\,ds = \int_{s(\alpha)}^{r(\alpha)} g'(v_\alpha(s))v'_\alpha(s)u_\alpha(s)\,ds \\ &\le u_\alpha(s(\alpha))\int_{s(\alpha)}^{r(\alpha)} g'(v_\alpha(s))v'_\alpha(s)\,ds = u_\alpha(s(\alpha))g(v_\alpha(r(\alpha)))\end{aligned}$$

Then with the help of (4.2) and (H7) we obtain

$$u'_\alpha(s(\alpha))^2 \le C u_\alpha(s(\alpha))(\alpha^{p+1} + \beta(\alpha)^{q+1})^{\frac{q}{q+1}},$$

for another positive constant C. Now, using (H1), (H3) and Lemma 3.2, we write

$$\begin{aligned}\frac{v'_\alpha(s(\alpha))^2}{2} &= \int_0^{s(\alpha)} f(u_\alpha(s))v'_\alpha(s)\,ds \\ &= f(\alpha)\beta(\alpha) - \int_0^{s(\alpha)} f'(u_\alpha(s))u'_\alpha(s)v_\alpha(s)\,ds \\ &\le f(\alpha)\beta(\alpha),\end{aligned}$$

from which we obtain, using (H7),

$$v'_\alpha(s(\alpha))^2 \le C\alpha^p(\alpha^{p+1} + \beta(\alpha)^{q+1})^{\frac{1}{q+1}},$$

for some positive constant C. $\square$

Lemma 4.3. *There exist two constants $c \in (0,1)$ and $M > 0$ such that*

$$u_\alpha(s(\alpha)) \ge c\max(\alpha, \beta(\alpha)^{\frac{q+1}{p+1}}) \quad \forall\, \alpha \ge M. \tag{4.3}$$

Moreover,

$$\frac{2}{a'}\frac{\beta(\alpha)}{\alpha^p} \le s(\alpha)^2 \le \frac{2}{ac^p}\frac{\beta(\alpha)}{\alpha^p} \quad \forall\, \alpha \ge M. \tag{4.4}$$

Proof. We argue by contradiction. Suppose first that there exists a sequence $(\alpha_k)_{k\in\mathbb{N}}$ increasing to ∞ such that

$$u_{\alpha_k}(s(\alpha_k)) \le \frac{1}{k}\alpha_k \quad \forall\, k \ge 2. \tag{4.5}$$

Using Lemma 4.2 and (4.5) we have

$$\begin{aligned}|u'_{\alpha_k}(s(\alpha_k))v'_{\alpha_k}s(\alpha_k))| &\le C\frac{1}{\sqrt{k}}\alpha_k^{\frac{p+1}{2}}(\alpha_k^{p+1}+\beta(\alpha_k)^{q+1})^{1/2}\\ &\le C\frac{1}{\sqrt{k}}(\alpha_k^{p+1}+\beta(\alpha_k)^{q+1}).\end{aligned}$$

From the above inequality, (H7), Lemma 3.2 and (4.1), we obtain

$$u_{\alpha_k}(s(\alpha_k))^{p+1} \ge d(\alpha_k^{p+1}+\beta(\alpha_k)^{q+1})$$

for some positive constant d when k is large and we obtain a contradiction with (4.5).

Now suppose that there exists a sequence $(\alpha_k)_{k\in\mathbb{N}}$ increasing to ∞ such that

$$u_{\alpha_k}(s(\alpha_k)) \le \frac{1}{k}\beta(\alpha_k)^{\frac{q+1}{p+1}} \quad \forall\, k\ge 2\,. \tag{4.6}$$

Using Lemma 4.2 and (4.6) we have

$$\begin{aligned}|u'_{\alpha_k}(s(\alpha_k))v'_{\alpha_k}s(\alpha_k))| &\le C\frac{1}{\sqrt{k}}\beta(\alpha_k)^{\frac{q+1}{2(p+1)}}\alpha_k^{p/2}(\alpha_k^{p+1}+\beta(\alpha_k)^{q+1})^{1/2}\\ &\le \frac{C}{2\sqrt{k}}(\alpha_k^{p}\beta(\alpha_k)^{\frac{q+1}{p+1}}+\alpha_k^{p+1}+\beta(\alpha_k)^{q+1})\\ &\le C'\frac{1}{\sqrt{k}}(\alpha_k^{p+1}+\beta(\alpha_k)^{q+1}),\end{aligned}$$

for another constant $C'>0$, where we have used Young's inequality. Then we obtain a contradiction as before.

Now we prove (4.4). Using (H7), we have

$$\beta(\alpha)=\int_0^{s(\alpha)}(s(\alpha)-r)f(u_\alpha(r))\,dr \le \frac{a'}{2}\alpha^p s(\alpha)^2\,,$$

and, with the help of (4.3),

$$\beta(\alpha)=\int_0^{s(\alpha)}(s(\alpha)-r)f(u_\alpha(r))\,dr \ge \frac{ac^p}{2}\alpha^p s(\alpha)^2\,.$$

The proof of the lemma is complete. □

Lemma 4.4. *We have $s(\alpha)\to 0$ as $\alpha\to\infty$.*

Proof. Since $\beta'(\alpha)>0$ for $\alpha>0$ we have $\lim_{\alpha\to\infty}\beta(\alpha)=d\le\infty$. If $d<\infty$ (4.4) implies that $s(\alpha)\to 0$ as $\alpha\to\infty$. If $d=\infty$, using (H7) and (4.3), we can write

$$\begin{aligned}\beta(\alpha)&=\int_0^{s(\alpha)}(s(\alpha)-r)f(u_\alpha(r))\,dr\\ &\ge \frac{a}{2}u_\alpha(s(\alpha))^p s(\alpha)^2 \ge \frac{ac^p}{2}\beta(\alpha)^{p\frac{q+1}{p+1}}s(\alpha)^2\,,\end{aligned}$$

from which we obtain

$$s(\alpha)^2 \le \frac{2}{ac^p}\beta(\alpha)^{\frac{1-pq}{p+1}}\,,$$

and the result follows since $pq>1$. □

Now we show that $r(\alpha) \to 0$ as $\alpha \to \infty$. If not, there exist $r_0 > 0$ and a sequence $(\alpha_k)_{k\in\mathbb{N}}$ increasing to ∞ such that

$$r(\alpha_k) \geq \frac{3r_0}{2} \quad \forall\, k \in \mathbb{N}\,. \tag{4.7}$$

By Lemma 4.4, we can assume that

$$s(\alpha_k) \leq \frac{r_0}{2} \quad \forall\, k \in \mathbb{N}\,. \tag{4.8}$$

Lemma 4.5. *There exists a constant $C > 0$ such that*

$$\begin{gathered} u'_{\alpha_k}(r_0)^2 \leq C u_{\alpha_k}(r_0)(\alpha_k^{p+1} + \beta(\alpha_k)^{q+1})^{\frac{q}{q+1}}, \\ v'_{\alpha_k}(r_0)^2 \leq C \alpha_k^p(\alpha_k^{p+1} + \beta(\alpha_k)^{q+1})^{\frac{1}{q+1}} \end{gathered}$$

for all $k \in \mathbb{N}$.

Proof. Using (H1), (H2), Lemma 2.4, (4.7) and (4.8) we have

$$\begin{aligned} \frac{u'_{\alpha_k}(r_0)^2}{2} &= -\int_{r_0}^{r(\alpha_k)} g(v_{\alpha_k}(s))u'_{\alpha_k}(s)\,ds \\ &= g(v_{\alpha_k}(r_0))u_{\alpha_k}(r_0) + \int_{r_0}^{r(\alpha_k)} g'(v_{\alpha_k}(s))v'_{\alpha_k}(s)u_{\alpha_k}(s)\,ds \\ &\leq g(v_{\alpha_k}(r_0))u_{\alpha_k}(r_0) + u_{\alpha_k}(r_0)\int_{r_0}^{r(\alpha_k)} g'(v_{\alpha_k}(s))v'_{\alpha_k}(s)\,ds \\ &= u_{\alpha_k}(r_0)g(v_{\alpha_k}(r(\alpha_k)))\,. \end{aligned}$$

Then with the help of (4.2) with $\alpha = \alpha_k$ and (H7) we obtain

$$u'_{\alpha_k}(r_0)^2 \leq C u_{\alpha_k}(r_0)(\alpha_k^{p+1} + \beta(\alpha_k)^{q+1})^{\frac{q}{q+1}}\,,$$

for some positive constant C. Now we write

$$\begin{aligned} \frac{v'_{\alpha_k}(r_0)^2}{2} &= \int_0^{r_0} f(u_{\alpha_k}(s))v'_{\alpha_k}(s)\,ds \\ &= f(u_{\alpha_k}(r_0))v_{\alpha_k}(r_0) + f(\alpha_k)\beta(\alpha_k) - \int_0^{r_0} f'(u_{\alpha_k}(s))u'_{\alpha_k}(s)v_{\alpha_k}(s)\,ds \end{aligned}$$

from which, using Lemma 2.4, Lemma 3.2, (H3), and (4.8), we obtain

$$\begin{aligned} &\frac{v'_{\alpha_k}(r_0)^2}{2} \\ &\leq f(u_{\alpha_k}(r_0))v_{\alpha_k}(r_0) + f(\alpha_k)\beta(\alpha_k) - \int_{s(\alpha_k)}^{r_0} f'(u_{\alpha_k}(s))u'_{\alpha_k}(s)v_{\alpha_k}(s)\,ds \\ &\leq f(u_{\alpha_k}(r_0))v_{\alpha_k}(r_0) + f(\alpha_k)\beta(\alpha_k) - v_{\alpha_k}(r_0)\int_{s(\alpha_k)}^{r_0} f'(u_{\alpha_k}(s))u'_{\alpha_k}(s)\,ds \\ &= f(\alpha_k)\beta(\alpha_k) + v_{\alpha_k}(r_0)f(u_{\alpha_k}(s(\alpha_k))) \\ &\leq f(\alpha_k)\beta(\alpha_k) + v_{\alpha_k}(r(\alpha_k))f(\alpha_k)\,. \end{aligned}$$

Then, with the help of (4.2) with $\alpha = \alpha_k$ and (H7), we obtain

$$v'_{\alpha_k}(r_0)^2 \leq C\alpha_k^p(\alpha_k^{p+1} + \beta(\alpha_k)^{q+1})^{\frac{1}{q+1}}\,,$$

for some positive constant C. ◻

Lemma 4.6. *There exists a constant $C > 0$ such that*
$$0 < v_{\alpha_k}(r_0) \le C(\alpha_k^{p+1} + \beta(\alpha_k)^{q+1})^{\frac{1}{q(p+1)}} \quad \forall\, k \in \mathbb{N}\,.$$

Proof. The left hand side inequality follows from (4.8). Now, using (4.7), (4.8), Lemma 2.4, Lemma 3.2, (H2) and (H7), we have
$$\begin{aligned} u_{\alpha_k}(r_0) &= \int_{r_0}^{r(\alpha_k)} (s - r_0) g(v_{\alpha_k}(s))\, ds \\ &\ge \frac{(r(\alpha_k) - r_0)^2}{2} g(v_{\alpha_k}(r_0)) \\ &\ge C v_{\alpha_k}(r_0)^q\,, \end{aligned} \tag{4.9}$$
for some positive constant C. With the help of (H7), (4.1) with $r = r_0$, $\alpha = \alpha_k$, lemma 2.4 and (4.9) we deduce that
$$v_{\alpha_k}(r_0) \le C(\alpha_k^{p+1} + \beta(\alpha_k)^{q+1})^{\frac{1}{q(p+1)}} \quad \forall\, k \in \mathbb{N}\,,$$
for another positive constant C. ☐

Lemma 4.7. *There exist a constant $c \in (0, 1)$ and an integer k_0 such that*
$$u_{\alpha_k}(r_0) \ge c \max(\alpha_k, \beta(\alpha_k)^{\frac{q+1}{p+1}}) \quad \forall\, k \ge k_0\,.$$

Proof. Lemma 4.6 and (H7) imply that
$$\lim_{k\to\infty} G(v_{\alpha_k}(r_0))(\alpha_k^{p+1} + \beta(\alpha_k)^{q+1})^{-1} = 0\,,$$
since $pq > 1$. Now the arguments are the same as in the proof of Lemma 4.3 with r_0 in place of $s(\alpha)$, using Lemma 4.5 instead of Lemma 4.2. ☐

Proof of Proposition 4.1 completed. Using Lemma 4.7 and (H7) we have
$$\begin{aligned} v_{\alpha_k}(r_0) + \beta(\alpha_k) &= \int_0^{r_0} (r_0 - s) f(u_{\alpha_k}(s))\, ds \\ &\ge \frac{ac^p}{2} r_0^2 \max(\alpha_k^p, \beta(\alpha_k)^{p\frac{q+1}{p+1}}) \end{aligned}$$
for all $k \in \mathbb{N}$. Then Lemma 4.6 implies that $r_0 = 0$ since $pq > 1$ and we reach a contradiction. ☐

5. The uniqueness question

Define
$$A = \{\alpha \in (0, \infty);\, r'(\alpha) \ne 0\}\,.$$
Assume that $A \ne \emptyset$. Since A is open there exists $J \subset \mathbb{N}$ such that $A = \cup_{n\in J} I_n$, where $I_n = (a_n, b_n)$.

In the setting of Theorem 1.1 or Theorem 1.2, Proposition 3.1 implies that $A \ne \emptyset$ and that $\inf\{a_n;\, n \in J\} = 0$ and $\sup\{b_n;\, n \in J\} = \infty$.

Case 1: $|J| = 1$. Then $A = (0, \infty)$ and for all $R > 0$, problem (1.2) has a unique solution $(u, v) \in (C^2([0, R])^2$.

Case 2: $|J| \ge 2$. Suppose first that there exist j, $k \in J$ such that $a_j = 0$ and $b_k = \infty$. Let $\gamma = \min\{r(\alpha);\, \alpha \in [b_j, a_k]\}$ and $\delta = \max\{r(\alpha);\, \alpha \in [b_j, a_k]\}$. Then for all $R \in (0, \gamma) \cup (\delta, \infty)$ problem (1.2) has a unique solution $(u, v) \in (C^2([0, R])^2$. Suppose that there exists $j \in J$ such that $a_j = 0$ and that $b_k \ne \infty$ for all $k \in J$. Let $\gamma = \inf\{r(\alpha);\, \alpha \ge b_j\}$. Proposition 3.1 implies that $\gamma > 0$. Then for all $R \in (0, \gamma)$

problem (1.2) has a unique solution $(u,v) \in (C^2([0,R])^2$. Now, if there exists $k \in J$ such that $b_k = \infty$ and that $a_j \neq 0$ for all $j \in J$, we define $\delta = \sup\{r(\alpha);\, \alpha \leq a_k\}$. Proposition 3.1 implies that $\delta < \infty$. Then for all $R \in (\delta, \infty)$ problem (1.2) has a unique solution $(u,v) \in (C^2([0,R])^2$. Otherwise we cannot give any uniqueness result.

In the setting of Theorem 1.3, Proposition 4.1 implies that $A \neq \emptyset$ and that $\inf\{a_n;\, n \in J\} = 0$ and $\sup\{b_n;\, n \in J\} = \infty$.

Case 1: $|J| = 1$. Then $A = (0, \infty)$ and for all $R > 0$ problem (1.2) has a unique solution $(u,v) \in (C^2([0,R])^2$.

Case 2: $|J| \geq 2$. Suppose first that there exist j, $k \in J$ such that $a_j = 0$ and $b_k = \infty$. Let $\gamma = \min\{r(\alpha);\, \alpha \in [b_j, a_k]\}$ and $\delta = \max\{r(\alpha);\, \alpha \in [b_j, a_k]\}$. Then for all $R \in (0, \gamma) \cup (\delta, \infty)$ problem (1.2) has a unique solution $(u,v) \in (C^2([0,R])^2$. Suppose that there exists $j \in J$ such that $a_j = 0$ and that $b_k \neq \infty$ for all $k \in J$. Let $\delta = \sup\{r(\alpha);\, \alpha \geq b_j\}$. Proposition 4.1 implies that $\delta < \infty$. Then for all $R \in (\delta, \infty)$ problem (1.2) has a unique solution $(u,v) \in (C^2([0,R])^2$. Now, if there exists $k \in J$ such that $b_k = \infty$ and that $a_j \neq 0$ for all $j \in J$, we define $\gamma = \inf\{r(\alpha);\, \alpha \leq a_k\}$. Proposition 4.1 implies that $\gamma > 0$. Then for all $R \in (0, \gamma)$ problem (1.2) has a unique solution $(u,v) \in (C^2([0,R])^2$. Otherwise we cannot give any uniqueness result.

In the setting of Theorem 3.6, the proof shows that $A \neq \emptyset$ and that $\inf\{a_n;\, n \in J\} = 0$

Case 1: $|J| = 1$. Then $A = (0, c)$ where $c \leq \infty$. If $c = \infty$, then for all $R > 0$ problem (1.2) has a unique solution $(u,v) \in (C^2([0,R])^2$. If $c < \infty$, then for all $R > r(c)$ problem (1.2) has a unique solution $(u,v) \in (C^2([0,R])^2$.

Case 2: $|J| \geq 2$. Suppose that there exists $j \in J$ such that $a_j = 0$. Let $\delta = \sup\{r(\alpha);\, \alpha \geq b_j\}$. If $\delta < \infty$, then for all $R \in (\delta, \infty)$ problem (1.2) has a unique solution $(u,v) \in (C^2([0,R])^2$. If $\delta = \infty$ we cannot give any uniqueness result. If $a_j \neq 0$ for all $j \in J$ we cannot give any uniqueness result.

6. Examples

In this section we give some examples that illustrate our results.

Example 6.1. Theorem 1.1 applies when f and g are defined in the following six cases:

(1) Let $c > \pi/2$, $m \in \mathbb{N}\backslash\{0\}$ and $f(u) = c + \arctan u^m$, $u \in \mathbb{R}$.
(2) Let $f(u) = 2 - \frac{1}{1+u^2}$, $u \in \mathbb{R}$.
(3) Let $q \geq 1$ and $g(v) = |v|^{q-1}v$, $v \in \mathbb{R}$.
(4) Let r, $q > 1$ and
$$g(v) = \begin{cases} v^r & v \geq 0\,, \\ |v|^{q-1}v & v \leq 0\,. \end{cases}$$
(5) Let p, $q > 1$ and
$$g(v) = \begin{cases} \ln(1 + v^p) & v \geq 0\,, \\ 1 - \exp|v|^q & v \leq 0\,. \end{cases}$$
(6) Let
$$g(v) = \begin{cases} \arctan v^2 & v \geq 0\,, \\ v^2 \arctan v & v \leq 0\,. \end{cases}$$

Example 6.2. Let $p > 0$ and $q \geq 1$. For $m \in \mathbb{N}\backslash\{0\}$ define

$$f(u) = (1 + u^{2m})^{p/(2m)}, \quad u \in \mathbb{R}.$$

Let $h \in C^1(\mathbb{R})$ be such that $h' < 0$ on $(-\infty, 0)$, $h' > 0$ on $(0, \infty)$ and $b \leq h \leq b'$ for some constants b, $b' > 0$. Define

$$g(v) = h(v)|v|^{q-1}v, \quad v \in \mathbb{R}.$$

Then Theorem 1.2 applies. If p, q satisfy the condition in (H9), then we can use Theorem 3.4.

Example 6.3. Let $k \in C^1(\mathbb{R})$ be such that $k' > 0$ on $(0, \infty)$ and $a \leq k \leq a'$ for some constants a, $a' > 0$. Define

$$f(u) = k(u)|u|^{p-1}u, \quad \forall u \in \mathbb{R},$$

where $p \geq 1$. If g is as in Example 6.2 and if p, q satisfy the condition in (H9) (resp. (H10), then Theorem 3.4 (resp. Theorem 3.6) applies. We can also use Theorem 1.3.

References

[1] R. Dalmasso; *Existence and uniqueness of solutions for a semilinear elliptic system*, Int. J. Math. Math. Sci. 10 (2005), 1507-1523.

[2] R. Dalmasso; *Uniqueness of positive solutions for some nonlinear fourth-order equations*, J. Math. Anal. Appl. 201 (1996), 152-168.

[3] E. Mitidieri; *A Rellich type identity and applications*, Commun. in Partial Differential Equations, (18) (1&2),125-151 (1993).

[4] J. Serrin, H. Zou; *Non-existence of positive solutions of semilinear elliptic systems*, Discourses in Mathematics and its Applications, no. 3, Department of Mathematics, Texas A&M University, College Station, Texas, (1994), 55-68.

[5] J. Serrin, H. Zou; *Non-existence of positive solutions of the Lane-Emden system*, Differential Integral Eqs 9 (1996), 635-653.

Robert Dalmasso

Laboratoire Jean Kuntzmann, Equipe EDP, 51 rue des Mathématiques, Domaine universitaire, BP 53, 38041 Grenoble Cedex 9, France

E-mail address: robert.dalmasso@imag.fr