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OSCILLATION OF SOLUTIONS FOR FORCED NONLINEAR NEUTRAL HYPERBOLIC EQUATIONS WITH FUNCTIONAL ARGUMENTS

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ABSTRACT. This article studies the forced oscillatory behavior of solutions to nonlinear hyperbolic equations with functional arguments. Our main tools are the integral averaging method and a generalized Riccati technique.

1. INTRODUCTION

In this work we consider the oscillatory behavior of solution to the hyperbolic equation

$$\begin{aligned} &\frac{\partial}{\partial t}\Big(r(t)\frac{\partial}{\partial t}\Big(u(x,t)+\sum_{i=1}^{l}h_i(t)u(x,\rho_i(t))\Big)\Big)-a(t)\Delta u(x,t)\\ &-\sum_{i=1}^{k}b_i(t)\Delta u(x,\tau_i(t))+\sum_{i=1}^{m}q_i(x,t)\varphi_i(u(x,\sigma_i(t)))\\ &=f(x,t),\quad (x,t)\in\Omega\equiv G\times(0,\infty), \end{aligned}\tag{1.1}$$

where Δ is the Laplacian in $\mathbb{R}^n$ and G is a bounded domain of $\mathbb{R}^n$ with piecewise smooth boundary ∂G. We consider the boundary conditions

$$u=\psi \quad \text{on } \partial G\times[0,\infty), \tag{1.2}$$

$$\frac{\partial u}{\partial \nu}+\mu u=\tilde{\psi} \quad \text{on } \partial G\times[0,\infty), \tag{1.3}$$

where ν denotes the unit exterior normal vector to ∂G and $\psi,\tilde{\psi}\in C(\partial G\times(0,\infty);\mathbb{R})$, $\mu\in C(\partial G\times(0,\infty);[0,\infty))$.

We use the following assumptions in this article:

(H1) $r(t)\in C^1([0,\infty);(0,\infty))$,
$h_i(t)\in C([0,\infty);[0,\infty))$ $(i=1,2,\dots,l)$,
$a(t),b_i(t)\in C([0,\infty);[0,\infty))$ $(i=1,2,\dots,k)$,
$q_i(x,t)\in C(\overline{\Omega};[0,\infty))$ $(i=1,2,\dots,m)$, $f(x,t)\in C(\overline{\Omega};\mathbb{R})$;

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(H2) $\rho_i(t) \in C([0,\infty);\mathbb{R})$, $\lim_{t\to\infty} \rho_i(t) = \infty$ $(i = 1, 2, \dots, l)$,
$\tau_i(t) \in C([0,\infty);\mathbb{R})$, $\lim_{t\to\infty} \tau_i(t) = \infty$ $(i = 1, 2, \dots, k)$,
$\sigma_i(t) \in C([0,\infty);\mathbb{R})$, $\lim_{t\to\infty} \sigma_i(t) = \infty$ $(i = 1, 2, \dots, m)$;

(H3) $\varphi_i(s) \in C^1(\mathbb{R};\mathbb{R})$ $(i = 1, 2, \dots, m)$ are convex on $[0,\infty)$ and $\varphi_i(-s) = -\varphi_i(s)$ for $s \geq 0$.

By a *solution* of (1.1) we mean a function $u \in C^2(\overline{G}\times [t_{-1},\infty)) \cap C(\overline{G}\times[\tilde{t}_{-1},\infty))$ which satisfies (1.1), where

$$t_{-1} = \min\{0, \min_{1\le i\le l}\{\inf_{t\ge 0} \rho_i(t)\}, \min_{1\le i\le k}\{\inf_{t\ge 0} \tau_i(t)\}\},$$
$$\tilde{t}_{-1} = \min\{0, \min_{1\le i\le m}\{\inf_{t\ge 0} \sigma_i(t)\}\}.$$

A solution u of (1.1) is said to be *oscillatory* in Ω if u has a zero in $G \times (t,\infty)$ for any $t > 0$.

Definition 1.1. We say that the pair of functions (H_1, H_2) belongs to the class $\mathbb{H}$, if $H_1, H_2 \in C(D; [0,\infty))$ and satisfy

$$H_i(t,t) = 0, \quad H_i(t,s) > 0 \quad \text{for } t > s \text{and } i = 1, 2,$$

where $D = \{(t,s) : 0 < s \le t < \infty\}$. Moreover, the partial derivatives $\partial H_1/\partial t$ and $\partial H_2/\partial s$ exist on D and satisfy

$$\frac{\partial H_1}{\partial t}(s,t) = h_1(s,t)H_1(s,t), \quad \frac{\partial H_2}{\partial s}(t,s) = -h_2(t,s)H_2(t,s),$$

where $h_1, h_2 \in C_{\rm loc}(D;\mathbb{R})$.

There are many articles devoted to the study of interval oscillation criteria for nonlinear hyperbolic equations with functional arguments by dealing with Riccati techniques; see for example [1, 2, 3, 4, 5, 6, 7, 10, 11, 12, 14, 15]. There are also some papers which deal with neutral hyperbolic or second order neutral differential equations, [4, 5, 12, 15]. However, it seems that very little is known about interval forced oscillations of the neutral hyperbolic equation (1.1).

On the other hand, oscillation criteria of second order neutral differential equations have been studied by many authors. We make reference to result by Tanaka [8], and extend them.

The aim of this paper is to establish sufficient conditions for every solution of (1.1) to be oscillatory by using Riccati techniques. Equation (1.1) is naturally classified into two classes according to whether

(C1) $\int_{t_0}^{\infty} \frac{1}{r(t)} dt = \infty$; or

(C2) $\int_{t_0}^{\infty} \frac{1}{r(t)} dt < \infty$.

2. Reduction to one-dimensional problems

In this section we reduce the multi-dimensional oscillation problems for (1.1) to one-dimensional oscillation problems. It is known that the first eigenvalue λ_1 of the eigenvalue problem

$$-\Delta w = \lambda w \quad \text{in } G,$$
$$w = 0 \quad \text{on } \partial G$$

is positive, and the corresponding eigenfunction $\Phi(x)$ can be chosen so that $\Phi(x) > 0$ in G. The following notation will be used in this article.

$$\begin{gathered}
U(t)=K_\Phi\int_G u(x,t)\Phi(x)dx,\quad \tilde U(t)=\frac{1}{|G|}\int_G u(x,t)dx,\\
F(t)=K_\Phi\int_G f(x,t)\Phi(x)dx,\quad \tilde F(t)=\frac{1}{|G|}\int_G f(x,t)dx,\\
\Psi(t)=K_\Phi\int_{\partial G}\psi\frac{\partial\Phi}{\partial\nu}(x)dS,\quad \tilde\Psi(t)=\frac{1}{|G|}\int_{\partial G}\tilde\psi dS,\\
q_i(t)=\min_{x\in\overline{G}} q_i(x,t),
\end{gathered}$$

where $K_\Phi=(\int_G\Phi(x)dx)^{-1}$ and $|G|=\int_G dx$.

Theorem 2.1. *If the functional differential inequality*

$$\frac{d}{dt}\Big(r(t)\frac{d}{dt}\Big(y(t)+\sum_{i=1}^{l}h_i(t)y(\rho_i(t))\Big)\Big)+\sum_{i=1}^{m}q_i(t)\varphi_i(y(\sigma_i(t)))\le\pm G(t)\tag{2.1}$$

has no eventually positive solution, then every solution of (1.1), (1.2) *is oscillatory in Ω, where*

$$G(t)=F(t)-a(t)\Psi(t)-\sum_{i=1}^{k}b_i(\tau_i(t))\Psi(\tau_i(t)).$$

Proof. Suppose to the contrary that there is a non-oscillatory solution u of (1.1), (1.2). Without loss of generality we may assume that $u(x,t)>0$ in $G\times[t_0,\infty)$ for some $t_0>0$ because the case $u(x,t)<0$ can be treated similarly. Since (H2) holds, we see that $u(x,\rho_i(t))>0$ $(i=1,2,\dots,l)$, $u(x,\tau_i(t))>0$ $(i=1,2,\dots,k)$ and $u(x,\sigma_i(t))>0$ $(i=1,2,\dots,m)$ in $G\times[t_1,\infty)$ for some $t_1\ge t_0$. Multiplying (1.1) by $K_\Phi\Phi(x)$ and integrating over G, we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big(r(t)\frac{d}{dt}\Big(U(t)+\sum_{i=1}^{l}h_i(t)U(\rho_i(t))\Big)\Big)-a(t)K_\Phi\int_G\Delta u(x,t)\Phi(x)dx\\
&-\sum_{i=1}^{k}b_i(t)K_\Phi\int_G\Delta u(x,\tau_i(t))\Phi(x)dx+\sum_{i=1}^{m}K_\Phi\int_G q_i(x,t)\varphi_i(u(x,\sigma_i(t)))\Phi(x)dx\\
&=F(t),\quad t\ge t_1.
\end{aligned}\tag{2.2}$$

Using Green's formula, it is obvious that

$$K_\Phi\int_G\Delta u(x,t)\Phi(x)dx\le-\Psi(t),\quad t\ge t_1,\tag{2.3}$$

$$K_\Phi\int_G\Delta u(x,\tau_i(t))\Phi(x)dx\le-\Psi(\tau_i(t)),\quad t\ge t_1.\tag{2.4}$$

An application of Jensen's inequality shows that

$$\sum_{i=1}^{m}K_\Phi\int_G q_i(x,t)\varphi_i(u(x,\sigma_i(t)))\Phi(x)dx\ge\sum_{i=1}^{m}q_i(t)\varphi_i(U(\sigma_i(t)))\tag{2.5}$$

for $t \ge t_1$. Combining (2.2)–(2.5) yields

$$\frac{d}{dt}\Big(r(t)\frac{d}{dt}\Big(U(t)+\sum_{i=1}^{l}h_i(t)U(\rho_i(t))\Big)\Big)+\sum_{i=1}^{m}q_i(t)\varphi_i(U(\sigma_i(t)))\le G(t)$$

for $t \ge t_1$. Therefore, $U(t)$ is an eventually positive solution of (2.1). This contradicts the hypothesis and completes the proof. □

Theorem 2.2. *If the functional differential inequality*

$$\frac{d}{dt}\Big(r(t)\frac{d}{dt}\Big(y(t)+\sum_{i=1}^{l}h_i(t)y(\rho_i(t))\Big)\Big)+\sum_{i=1}^{m}q_i(t)\varphi_i(y(\sigma_i(t)))\le \pm\tilde{G}(t) \tag{2.6}$$

has no eventually positive solution, then every solution of (1.1), (1.3) *is oscillatory in* Ω, *where*

$$\tilde{G}(t)=\tilde{F}(t)+a(t)\tilde{\Psi}(t)+\sum_{i=1}^{k}b_i(\tau_i(t))\tilde{\Psi}(\tau_i(t)).$$

Proof. Suppose to the contrary that there is a non-oscillatory solution u of (1.1), (1.3). Without loss of generality we may assume that $u(x,t)>0$ in $G\times[t_0,\infty)$ for some $t_0>0$. Since (H2) holds, we see that $u(x,\rho_i(t))>0$ $(i=1,2,\dots,l)$, $u(x,\tau_i(t))>0$ $(i=1,2,\dots,k)$ and $u(x,\sigma_i(t))>0$ $(i=1,2,\dots,m)$ in $G\times[t_1,\infty)$ for some $t_1\ge t_0$. Dividing (1.1) by $|G|$ and integrating over G, we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big(r(t)\frac{d}{dt}\Big(\tilde{U}(t)+\sum_{i=1}^{l}h_i(t)\tilde{U}(\rho_i(t))\Big)\Big)-\frac{a(t)}{|G|}\int_G\Delta u(x,t)dx\\
&-\sum_{i=1}^{k}\frac{b_i(t)}{|G|}\int_G\Delta u(x,\tau_i(t))dx+\frac{1}{|G|}\sum_{i=1}^{m}\int_G q_i(x,t)\varphi_i(u(x,\sigma_i(t)))dx\\
&=\tilde{F}(t),\quad t\ge t_1.
\end{aligned}\tag{2.7}$$

It follows from Green's formula that

$$\frac{1}{|G|}\int_G\Delta u(x,t)dx\le\tilde{\Psi}(t),\quad t\ge t_1, \tag{2.8}$$

$$\frac{1}{|G|}\int_G\Delta u(x,\tau_i(t))dx\le\tilde{\Psi}(\tau_i(t)),\quad t\ge t_1. \tag{2.9}$$

Applying Jensen's inequality, we observe that

$$\frac{1}{|G|}\sum_{i=1}^{m}\int_G q_i(x,t)\varphi_i(u(x,\sigma_i(t)))dx\ge\sum_{i=1}^{m}q_i(t)\varphi_i(\tilde{U}(\sigma_i(t))),\quad t\ge t_1. \tag{2.10}$$

This together with (2.7)–(2.10) yield

$$\frac{d}{dt}\Big(r(t)\frac{d}{dt}\Big(\tilde{U}(t)+\sum_{i=1}^{l}h_i(t)\tilde{U}(\rho_i(t))\Big)\Big)+\sum_{i=1}^{m}q_i(t)\varphi_i(\tilde{U}(\sigma_i(t)))\le\tilde{G}(t)$$

for $t\ge t_1$. Hence $\tilde{U}(t)$ is an eventually positive solution of (2.6). This contradicts the hypothesis and completes the proof. □

3. Second-order functional differential inequalities

We look for sufficient conditions so that the functional differential inequality

$$\frac{d}{dt}\Big(r(t)\frac{d}{dt}\Big(y(t)+\sum_{i=1}^{l}h_i(t)y(\rho_i(t))\Big)\Big)+\sum_{i=1}^{m}q_i(t)\varphi_i(y(\sigma_i(t)))\le f(t) \tag{3.1}$$

has no eventually positive solution, where $f(t)\in C([0,\infty);\mathbb{R})$.

3.1. **Case: (C1) is satisfied.** We assume the following hypotheses:

(H4) For some $j\in\{1,2,\dots,m\}$, there exists a positive constant σ such that $\sigma_j'(t)\ge\sigma$, $t\ge\sigma_j(t)$, $\varphi_j'(s)>0$ and $\varphi_j'(s)$ is nondecreasing for $s>0$;
(H5) $\rho_i(t)\le t$ $(i=1,2,\dots,l)$;
(H6) $\sum_{i=1}^{l}h_i(t)\le h<1$ for some $h>0$;
(H7) there exists $T\ge 0$ such that $T\le a<b$ and $f(t)\le 0$ for all $t\in[a,b]$.

Theorem 3.1. *Assume that* (C1), (H4)–(H7) *hold. If the Riccati inequality*

$$z'(t)+\frac{1}{2}\frac{1}{P_K(t)}z^2(t)\le -q_j(t) \tag{3.2}$$

has no solution on $[T,\infty)$ *for all large* T*, then* (3.1) *has no eventually positive solution, where*

$$P_K(t)=\frac{r(\sigma_j(t))}{2K(1-h)\sigma}.$$

Proof. Suppose that $y(t)$ is a positive solution of (3.1) on $[t_0,\infty)$ for some $t_0>0$. From (3.1) there exist $j\in\{1,2,\dots,m\}$ and $a,b\ge t_0$ such that $f(t)\le 0$ on the interval $I\in[a,b]$, and so,

$$\frac{d}{dt}\Big(r(t)\frac{d}{dt}\Big(y(t)+\sum_{i=1}^{l}h_i(t)y(\rho_i(t))\Big)\Big)+q_j(t)\varphi_j(y(\sigma_j(t)))\le 0,\quad t\in I$$

for $t\ge t_0$. If we set the function

$$z(t)=y(t)+\sum_{i=1}^{l}h_i(t)y(\rho_i(t)),$$

then we see that

$$(r(t)z'(t))'\le -q_j(t)\varphi_j(y(\sigma_j(t)))\le 0,\quad t\ge t_0. \tag{3.3}$$

Then we conclude that $z'(t)\ge 0$ or $z'(t)<0$, $t\ge t_1$ for some $t_1\ge t_0$. From the well known argument (cf. Yoshida [13]), we see that $z'(t)\ge 0$, $z(t)\ge 0$ and

$$y(\sigma_j(t))\ge(1-h)z(\sigma_j(t)),\ t\ge t_2$$

for some $t_2\ge t_1$. Setting

$$w(t)=\frac{r(t)z'(t)}{\varphi_j((1-h)z(\sigma_j(t)))},$$

we show that

$$\begin{aligned} w'(t) &= \frac{(r(t)z'(t))'}{\varphi_j((1-h)z(\sigma_j(t)))} - (1-h)r(t)z'(t)\frac{\varphi_j'((1-h)z(\sigma_j(t)))z'(\sigma_j(t))\sigma_j'(t)}{\varphi_j^2((1-h)z(\sigma_j(t)))} \\ &\le -q_j(t)\frac{\varphi_j(y(\sigma_j(t)))}{\varphi_j((1-h)z(\sigma_j(t)))} - \frac{(1-h)\sigma\varphi_j'((1-h)z(\sigma_j(t)))}{r(\sigma_j(t))}w^2(t), \quad t \ge t_2. \end{aligned} \tag{3.4}$$

It follows from (H4) that

$$\varphi_j'((1-h)z(\sigma_j(t))) \ge \varphi_j'((1-h)k) \equiv K, \quad t \ge t_2. \tag{3.5}$$

Combining (3.5) and (3.4), we have

$$w'(t) + \frac{1}{2}\frac{1}{P_K(t)}w^2(t) \le -q_j(t), \quad t \ge t_2. \tag{3.6}$$

That is, $w(t)$ is a solution of (3.1) on $[t_2, \infty)$. This is a contradiction and the proof is complete. □

(H8) There exists an oscillatory function $\theta(t)$ such that

$$\big(r(t)\theta'(t)\big)' = f(t) \quad \text{and} \quad \lim_{t\to\infty} \tilde{\theta}(t) = 0,$$

where

$$\tilde{\theta}(t) = \theta(t) - \sum_{i=1}^{l} h_i(t)\theta(\rho_i(t)).$$

Theorem 3.2. *Assume that* (C1), (H4)–(H6), (H8) *hold. If the Riccati inequality* (3.2) *has no solution on* $[T, \infty)$ *for all large* T, *then* (3.1) *has no eventually positive solutions.*

Proof. Suppose that $y(t)$ is a positive solution of (3.1) on $[t_0, \infty)$ for some $t_0 > 0$. From (3.1) there exists $j \in \{1, 2, \dots, m\}$ such that

$$\frac{d}{dt}\Big(r(t)\frac{d}{dt}\Big(y(t) + \sum_{i=1}^{l} h_i(t)y(\rho_i(t))\Big)\Big) + q_j(t)\varphi_j(y(\sigma_j(t))) \le f(t), \quad t \ge t_0.$$

Define the function $\tilde{z}(t)$ by

$$\tilde{z}(t) = y(t) + \sum_{i=1}^{l} h_i(t)y(\rho_i(t)) - \theta(t),$$

then it obvious that

$$(r(t)\tilde{z}'(t))' \le -q_j(t)\varphi_j(y(\sigma_j(t))) \le 0, \quad t \ge t_0, \tag{3.7}$$

so that $\tilde{z}'(t) \ge 0$ or $\tilde{z}'(t) < 0$, $t \ge t_1$ for some $t_1 \ge t_0$. By standard arguments (cf. Yoshida [13]), we see that $\tilde{z}'(t) \ge 0$, $\tilde{z}(t) \ge 0$ and

$$y(t) \ge (1-h)\tilde{z}(t) + \tilde{\theta}(t), \quad t \ge t_2$$

for some $t_2 \ge t_1$. Since (H8) holds, there exists a number $t_3 \ge t_2$ such that

$$|\tilde{\theta}(t)| \le \frac{(1-h)k}{2}, \quad t \ge t_3.$$

In view of $\tilde{z}(t) \ge k$, we observe that

$$y(t) \ge (1-h)\tilde{z}(t) - \frac{(1-h)k}{2} \ge \frac{(1-h)k}{2} \equiv \tilde{k} > 0, \quad t \ge t_3. \tag{3.8}$$

Setting

$$\tilde{w}(t) = \frac{r(t)\tilde{z}'(t)}{\varphi_j\big((1-h)\tilde{z}(\sigma_j(t)) - \tilde{k}\big)},$$

for $t \geq t_3$, we have

$$\begin{aligned}
\tilde{w}'(t) &= \frac{(r(t)\tilde{z}'(t))'}{\varphi_j\big((1-h)\tilde{z}(\sigma_j(t)) - \tilde{k}\big)} \\
&\quad - r(t)\tilde{z}'(t)\frac{\varphi_j'\big((1-h)\tilde{z}(\sigma_j(t)) - \tilde{k}\big)(1-h)\tilde{z}'(\sigma_j(t))\sigma_j'(t)}{\varphi_j^2\big((1-h)\tilde{z}(\sigma_j(t)) - \tilde{k}\big)} \\
&\leq -q_j(t)\frac{\varphi_j(y(\sigma_j(t)))}{\varphi_j\big((1-h)\tilde{z}(\sigma_j(t)) - \tilde{k}\big)} - \frac{(1-h)\sigma\varphi_j'\big((1-h)\tilde{z}(\sigma_j(t)) - \tilde{k}\big)}{r(\sigma_j(t))}\tilde{w}^2(t).
\end{aligned} \tag{3.9}$$

It follow from (3.8) and (H4) that

$$\varphi_j'\Big((1-h)\tilde{z}(\sigma_j(t)) - \tilde{k}\Big) \geq \varphi_j'(\tilde{k}) \equiv K, \quad t \geq t_3. \tag{3.10}$$

Combining (3.9) with (3.10) yields

$$\tilde{w}'(t) + \frac{1}{2}\frac{1}{P_K(t)}\tilde{w}^2(t) \leq -q_j(t), \quad t \geq t_3. \tag{3.11}$$

Therefore, $\tilde{w}(t)$ is a solution of (3.2). This contradicts the hypothesis and completes the proof. $\square$

Theorem 3.3. *Assume that* (C1) (H4)–(H7) *(or that* (H4)–(H6), (H8)*) hold. If for each $T > 0$ and some $K > 0$, there exist $(H_1, H_2) \in \mathbb{H}$, $\phi(t) \in C^1((0,\infty);(0,\infty))$ and $a, b, c \in \mathbb{R}$ such that $T \leq a < c < b$ and*

$$\begin{aligned}
&\frac{1}{H_1(c,a)}\int_a^c H_1(s,a)\{q_j(s) - \frac{1}{2}P_K(s)\lambda_1^2(s,a)\}\phi(s)ds \\
&+ \frac{1}{H_2(b,c)}\int_c^b H_2(b,s)\{q_j(s) - \frac{1}{2}P_K(s)\lambda_2^2(b,s)\}\phi(s)ds > 0,
\end{aligned} \tag{3.12}$$

where

$$\lambda_1(s,t) = \frac{\phi'(s)}{\phi(s)} + h_1(s,t), \quad \lambda_2(t,s) = \frac{\phi'(s)}{\phi(s)} - h_2(t,s).$$

Then (3.1) *has no eventually positive solutions.*

Proof. Suppose that $y(t)$ is a positive solution of (3.1) on $[t_0, \infty)$ for some $t_0 > 0$. Proceeding as in the proof of Theorem 3.1, multiplying (3.6) or (3.11) by $H_2(t,s)$ and integrating over $[c,t]$ for $t \in [c,b)$, we have

$$\begin{aligned}
&\int_c^t H_2(t,s)q_j(s)\phi(s)ds \\
&\leq -\int_c^t H_2(t,s)w'(s)\phi(s)ds - \frac{1}{2}\int_c^t H_2(t,s)\frac{1}{P_K(s)}w^2(s)\phi(s)ds \\
&\leq H_2(t,c)w(c)\phi(c) + \frac{1}{2}\int_c^t H_2(t,s)P_K(s)\lambda_2^2(t,s)\phi(s)ds \\
&\quad - \frac{1}{2}\int_c^t H_2(t,s)\{w(s)/\sqrt{P_K(s)} - \lambda_2(t,s)\sqrt{P_K(s)}\}^2\phi(s)ds,
\end{aligned}$$

and so

$$\frac{1}{H_2(t,c)}\int_c^t H_2(t,s)\{q_j(s)-\frac{1}{2}P_K(s)\lambda_2^2(t,s)\}\phi(s)ds \le w(c)\phi(c).$$

Letting $t \to b^-$ in the last inequality, we obtain

$$\frac{1}{H_2(b,c)}\int_c^b H_2(b,s)\{q_j(s)-\frac{1}{2}P_K(s)\lambda_2^2(b,s)\}\phi(s)ds \le w(c)\phi(c). \tag{3.13}$$

On the other hand, multiplying (3.6) by $H_1(s,t)$, integrating over $[t,c]$ for $t \in (a,c]$ and letting $t \to a^+$, we obtain

$$\frac{1}{H_1(c,a)}\int_a^c H_1(s,a)\{q_j(s)-\frac{1}{2}P_K(s)\lambda_1^2(s,a)\}\phi(s)ds \le -w(c)\phi(c). \tag{3.14}$$

Adding (2.1) and (2.6), we obtain

$$\begin{aligned}&\frac{1}{H_1(c,a)}\int_a^c H_1(s,a)\{q_j(s)-\frac{1}{2}P_K(s)\lambda_1^2(s,a)\}\phi(s)ds\\&+\frac{1}{H_2(b,c)}\int_c^b H_2(b,s)\{q_j(s)-\frac{1}{2}P_K(s)\lambda_2^2(b,s)\}\phi(s)ds \le 0,\end{aligned}$$

which is contrary to (3.12). Pick up a sequence $\{T_i\} \subset [t_0,\infty)$ such that $T_i \to \infty$ as $i \to \infty$. By the assumptions, for each $i \in \mathbb{N}$, there exists $a_i,\ b_i,\ c_i \in [0,\infty)$ such that $T_i \le a_i < c_i < b_i$, and (3.12) holds with $a,\ b,\ c$ replaced by $a_i,\ b_i,\ c_i$, respectively. Therefore, every solution $y(t)$ of (3.1) has at least one zero $t_i \in (a_i,b_i)$. The case when (3.11) follows by a similar arguments. This is a contradiction and the proof is complete. □

Theorem 3.4. *Assume* (C1), (H4)–(H7) *(or* (H4)–(H6), (H8)*). If for each $T > 0$ and some $K > 0$, there exist functions $(H_1,H_2) \in \mathbb{H}$, $\phi(t) \in C^1((0,\infty);(0,\infty))$, such that*

$$\limsup_{t\to\infty}\int_T^t H_1(s,T)\{q_j(s)-\frac{1}{2}P_K(s)\lambda_1^2(s,T)\}\phi(s)ds > 0 \tag{3.15}$$

and

$$\limsup_{t\to\infty}\int_T^t H_2(t,s)\{q_j(s)-\frac{1}{2}P_K(s)\lambda_2^2(t,s)\}\phi(s)ds > 0, \tag{3.16}$$

then (3.1) *has no eventually positive solutions.*

Proof. For any $T \ge t_0$, let $a = T$ and choose $T = a$ in (3.12). Then there exists $c > a$ such that

$$\int_a^c H_1(s,a)\{q_j(s)-\frac{1}{2}P_K(s)\lambda_1^2(s,a)\}\phi(s)ds > 0. \tag{3.17}$$

Next, choose $T = c$ in (3.16). Then there exists $b > c$ such that

$$\int_c^b H_2(b,s)\{q_j(s)-\frac{1}{2}P_K(s)\lambda_2^2(b,s)\}\phi(s)ds > 0. \tag{3.18}$$

Combining (3.17) and (3.18), we obtain (3.12). By Theorem 3.3, the proof is complete. □

3.2. **Case: (C2) is satisfied.** We use the following notation:

$$\rho_*(t) = \min_{1\le i\le l} \rho_i(t), \quad \pi(t) = \int_t^\infty \frac{1}{r(s)} ds,$$
$$A(t) = 1 - \sum_{i=1}^{l} h_i(t) - \log \frac{\pi(\rho_*(t))}{\pi(t)}, \quad [\delta(t)]_\pm = \max\{0, \pm\delta(t)\}.$$

Theorem 3.5. *Assume that*(C2), (H4)–(H7) *hold. If the Riccati inequality*

$$z_i'(t) + \frac{1}{2}\frac{1}{P_i(t)} z_i^2(t) \le -Q_i(t) \ (i = 1, 2) \tag{3.19}$$

has no solution on $[T, \infty)$ *for all large* T, *then* (3.1) *has no eventually positive solutions, where*

$$P_1(t) = P_K(t), \quad P_2(t) = \frac{r(t)}{2\varphi_j'(c_1\pi(t))},$$
$$Q_1(t) = q_j(t), \ Q_2(t) = q_j(t)\frac{\varphi_j\big([c_1 A(\sigma_j(t))\pi(\rho_*(\sigma_j(t)))]_+\big)}{\tilde{K}}.$$

Proof. Suppose that $y(t)$ is a positive solution of (3.1) on $[t_0, \infty)$ for some $t_0 > 0$. Proceeding as in the proof of Theorem 3.1, we obtain the inequality (3.3). Thus we see that $z'(t) \ge 0$, $z(t) \ge 0$ or $z'(t) < 0$, $z(t) \ge 0$, $t \ge t_1$ for some $t_1 \ge t_0$.
Case 1. $z'(t) \ge 0$, $z(t) \ge 0$ for $t \ge t_1$. The proof of this case is similar as Theorem 3.1, and so we omit it.
Case 2. $z'(t) < 0$, $z(t) \ge 0$ for $t \ge t_1$. Then there exists a constant $k_1 > 0$ such that $z(t) \le k_1$, $t \ge t_2$ for some $t_2 \ge t_1$. Consequently we have

$$\varphi_j(z(t)) \le \varphi_j(k_1) \equiv \tilde{K}, \quad t \ge t_2. \tag{3.20}$$

If we define

$$w_2(t) = \frac{r(t)z'(t)}{\varphi_j(z(t))},$$

then

$$\begin{aligned} w_2'(t) &= \frac{(r(t)z'(t))'}{\varphi_j(z(t))} - r(t)z'(t)\frac{\varphi_j'(z(t))z'(t)}{\varphi_j^2(z(t))} \\ &\le -q_j(t)\frac{\varphi_j(y(\sigma_j(t)))}{\varphi_j(z(t))} - \frac{\varphi_j'(z(t))}{r(t)} w_2^2(t), \quad t \ge t_2. \end{aligned} \tag{3.21}$$

Using [8, Lemma 5.2], we see that $z(t) \ge c_1\pi(t)$, $t \ge t_3$ for some $t_3 \ge t_2$, and that

$$\varphi_j'(z(t)) \ge \varphi_j'(c_1\pi(t)), \quad t \ge t_3. \tag{3.22}$$

By [8, Theorem 3.2], we show that

$$y(t) \ge c_1 A(t)\pi(\rho_*(t)), \quad t \ge t_3,$$

and that

$$\varphi_j(y(\sigma_j(t))) \ge \varphi_j([c_1 A(\sigma_j(t))\pi(\rho_*(\sigma_j(t)))]_+), \quad t \ge t_3. \tag{3.23}$$

Combining (3.20)–(3.23), we can derive the inequality

$$w_2'(t) + \frac{1}{2}\frac{1}{P_2(t)} w_2^2(t) \le -Q_2(t), \quad t \ge t_3.$$

Therefore, $w_2(t)$ is a solution of (3.19). This contradicts the hypothesis and completes the proof. □

Theorem 3.6. *Assume that* (C2), (H4)–(H6), (H8) *hold. If the Riccati inequality*
$$z_i'(t) + \frac{1}{2}\frac{1}{P_i(t)} z_i^2(t) \le -\tilde{Q}_i(t) \quad (i = 1, 2) \tag{3.24}$$
has no solution on $[T, \infty)$ *for all large* T, *then* (3.1) *has no eventually positive solutions, where*
$$\tilde{Q}_1(t) = q_j(t), \quad \tilde{Q}_2(t) = q_j(t) \frac{\varphi_j\Big([c_1 A(\sigma_j(t))\pi(\rho_*(\sigma_j(t))) + \tilde{\theta}(\sigma_j(t))]_+\Big)}{\tilde{K}}.$$

Proof. Suppose that $y(t)$ is a positive solution of (3.1) on $[t_0, \infty)$ for some $t_0 > 0$. Proceeding as in the proof of Theorem 3.2, we see that $\tilde{z}'(t) \ge 0$, $\tilde{z}(t) \ge 0$ or $\tilde{z}'(t) < 0$, $\tilde{z}(t) \ge 0$, $t \ge t_1$ for some $t_1 \ge t_0$.

Case 1. $\tilde{z}'(t) \ge 0$, $\tilde{z} \ge 0$. Then it can be treated similarly as in the proof of Theorem 3.2.

Case 2. $\tilde{z}'(t) < 0$, $\tilde{z}(t) \ge 0$. By Tanaka [8, Theorem 3.2], we obtain
$$y(\sigma_j(t)) \ge [c_1 A(\sigma_j(t))\pi(\rho_*(\sigma_j(t))) + \tilde{\theta}(\sigma_j(t))]_+, \quad t \ge t_2.$$
Setting $\tilde{w}_2(t) = w_2(t)$, it obvious that
$$\tilde{w}_2'(t) \le -q_j(t)\frac{\varphi_j(y(\sigma_j(t)))}{\varphi_j(z(t))} - \frac{\varphi_j'(z(t))}{r(t)}\tilde{w}_2^2(t), \quad t \ge t_2.$$
Substituting (3.20) and (3.22) into this inequality yields
$$\tilde{w}_2'(t) + \frac{1}{2}\frac{1}{P_2(t)}\tilde{w}_2^2(t) \le -q_j(t)\frac{\varphi_j(y(\sigma_j(t)))}{\tilde{K}}.$$
It is clear that $\tilde{w}_2(t)$ is a solution of (3.24). This contradicts the hypothesis and completes the proof. $\square$

Theorem 3.7. *Assume that* (C2), (H4)–(H7) *hold. If for each* $T > 0$ *and some* $K > 0$, $\tilde{K} > 0$ *there exist* $(H_1, H_2) \in \mathbb{H}$, $\phi(t) \in C^1((0,\infty);(0,\infty))$ *and* $a, b, c \in \mathbb{R}$ *such that* $T \le a < c < b$ *and* (3.12) *and*
$$\begin{aligned} &\frac{1}{H_1(c,a)}\int_a^c H_1(s,a)\{Q_2(s) - \frac{1}{2}P_2(s)\lambda_1^2(s,a)\}\phi(s)ds \\ &+ \frac{1}{H_2(b,c)}\int_c^b H_2(b,s)\{Q_2(s) - \frac{1}{2}P_2(s)\lambda_2^2(b,s)\}\phi(s)ds > 0 \end{aligned} \tag{3.25}$$
hold, then (3.1) *has no eventually positive solutions.*

Theorem 3.8. *Assume that* (C2), (H4)–(H7) *hold. If for each* $T > 0$ *and some* $K > 0$, $\tilde{K} > 0$, *there exist functions* $(H_1, H_2) \in \mathbb{H}$, $\phi(t) \in C^1((0,\infty);(0,\infty))$, *such that* (3.15), (3.16) *and*
$$\limsup_{t\to\infty}\int_T^t H_1(s,T)\{Q_2(s) - \frac{1}{2}P_2(s)\lambda_1^2(s,T)\}\phi(s)ds > 0 \tag{3.26}$$
and
$$\limsup_{t\to\infty}\int_T^t H_2(t,s)\{Q_2(s) - \frac{1}{2}P_2(s)\lambda_2^2(t,s)\}\phi(s)ds > 0, \tag{3.27}$$
then (3.1) *has no eventually positive solutions.*

Theorem 3.9. *Assume that* (C2), (H4)–(H6), (H8) *hold. If for each* $T > 0$ *and some* $K > 0$, $\tilde{K} > 0$, *there exist* $(H_1, H_2) \in \mathbb{H}$, $\phi(t) \in C^1((0,\infty);(0,\infty))$ *and* $a, b, c \in \mathbb{R}$ *such that* $T \le a < c < b$ *and* (3.12) *and*

$$\begin{aligned} &\frac{1}{H_1(c,a)}\int_a^c H_1(s,a)\{\tilde{Q}_2(s) - \frac{1}{2}P_2(s)\lambda_1^2(s,a)\}\phi(s)ds \\ &+ \frac{1}{H_2(b,c)}\int_c^b H_2(b,s)\{\tilde{Q}_2(s) - \frac{1}{2}P_2(s)\lambda_2^2(b,s)\}\phi(s)ds > 0 \end{aligned} \tag{3.28}$$

hold, then (3.1) *has no eventually positive solutions.*

Theorem 3.10. *Assume that* (C2), (H4)–(H6), (H8) *hold. If for each* $T > 0$ *and some* $K > 0$, $\tilde{K} > 0$, *there exist functions* $(H_1, H_2) \in \mathbb{H}$, $\phi(t) \in C^1((0,\infty);(0,\infty))$, *such that* (3.15), (3.16) *and*

$$\limsup_{t\to\infty}\int_T^t H_1(s,T)\{\tilde{Q}_2(s) - \frac{1}{2}P_2(s)\lambda_1^2(s,T)\}\phi(s)ds > 0 \tag{3.29}$$

and

$$\limsup_{t\to\infty}\int_T^t H_2(t,s)\{\tilde{Q}_2(s) - \frac{1}{2}P_2(s)\lambda_2^2(t,s)\}\phi(s)ds > 0, \tag{3.30}$$

then (3.1) *has no eventually positive solutions.*

4. Oscillation criteria for (1.1)

In this section, by combining the results of Sections 2 and 3, we establish sufficient conditions for oscillation of solutions to (1.1).

(H9) There exists $T \le a < b \le \tilde{a} < \tilde{b}$ such that

$$G(t)\ [\text{resp. } \tilde{G}(t)] = \begin{cases} \le 0, & t \in [a,b], \\ \ge 0, & t \in [\tilde{a},\tilde{b}] \end{cases}$$

for each $T \ge 0$;

(H10) there exists an oscillatory function $\Theta(t)$ such that

$$\Big(r(t)\Theta'(t)\Big)' = G(t)\ [\text{resp.}\tilde{G}(t)], \quad \lim_{t\to\infty}\tilde{\Theta}(t) = 0,$$

where

$$\tilde{\Theta}(t) = \Theta(t) - \sum_{i=1}^{l} h_i(t)\Theta(\rho_i(t)).$$

Using the Riccati inequality, we derive sufficient conditions for every solution of hyperbolic equation (1.1) to be oscillatory. We are going to use the following lemma which is due to Usami [9].

Lemma 4.1. *If there exists a function* $\phi(t) \in C^1([T_0,\infty);(0,\infty))$ *such that*

$$\int_{T_1}^\infty \Big(\frac{\bar{p}(t)|\phi'(t)|^\beta}{\phi(t)}\Big)^{1/(\beta-1)} dt < \infty, \quad \int_{T_1}^\infty \frac{1}{\bar{p}(t)(\phi(t))^{\beta-1}} dt = \infty,$$

$$\int_{T_1}^\infty \phi(t)\bar{q}(t)dt = \infty$$

for some $T_1 \ge T_0$, *then the Riccati inequality*

$$x'(t) + \frac{1}{\beta}\frac{1}{\bar{p}(t)}|x(t)|^\beta \le -\bar{q}(t)$$

has no solution on $[T, \infty)$ *for all large* T*, where* $\beta > 1$*,* $\bar{p}(t) \in C([T_0, \infty); (0, \infty))$ *and* $\bar{q}(t) \in C([T_0, \infty); \mathbb{R})$*,*

4.1. **Oscillation results by Riccati inequality for case (C1).** Combining Theorems 2.1–3.2 and Lemma 4.1, we obtain the following theorem.

Theorem 4.2. *Assume that* (C1), (H1)–(H6), (H9) *(or* (H1)–(H6), (H10)*) and that if*

$$\int_{T_1}^{\infty} \Big(\frac{P_K(t)\phi'(t)^2}{\phi(t)}\Big)dt < \infty, \quad \int_{T_1}^{\infty} \frac{1}{P_K(t)\phi(t)}dt = \infty, \quad \int_{T_1}^{\infty} \phi(t)q_j(t)dt = \infty,$$

then every solution $u(x,t)$ *of* (1.1), (1.2) *(or* (1.1), (1.3)*) is oscillatory in* Ω*.*

Example 4.3. Consider the equation

$$\begin{aligned} &\frac{\partial}{\partial t}\Big(e^{-2t}\frac{\partial}{\partial t}\Big(u(x,t) + \frac{1}{2}u(x,t-\pi)\Big)\Big) - e^{-3t}\Delta u(x,t) \\ &- \frac{1}{2}e^{-2t}\Delta u(x,t-2\pi) - \big(e^{-t} + e^{-2t}\big)\,\Delta u\big(x,t-\frac{3}{2}\pi\big) + e^{-t}u\big(x,t-\frac{\pi}{2}\big) \\ &= e^{-3t}\sin x \sin t,\ (0,\pi)\times(0,\infty), \end{aligned} \tag{4.1}$$

$$u(0,t) = u(\pi,t) = 0, \quad t > 0. \tag{4.2}$$

Here $l = m = 1$, $k = 2$, $r(t) = e^{-2t}$, $h_1(t) = 1/2$, $\rho_1(t) = t - \pi$, $q_1(x,t) = e^{-t}$, $\sigma_1(t) = t - \pi/2$ and $f(x,t) = e^{-3t}\sin x \sin t$. It is easy to see that $\Phi(x) = \sin x$ and

$$G(t) = F(t) = \frac{\pi}{4}e^{-3t}\sin t, \quad \tilde{\Theta}(t) = \frac{\pi}{16}\big(1 + \frac{1}{2}e^{\pi}\big)e^{-t}\cos t.$$

Then $\int^{\infty} e^{-t}dt < \infty$; hence, [8, Corollary 2.1] is not applicable to this problem. Taking $\phi(t) = e^t$, we find

$$\begin{gathered} \int^{\infty} \Big(\frac{P_K(t)\phi'(t)^2}{\phi(t)}\Big)dt = \int^{\infty} \Big(\frac{e^{-2t+\pi}\cdot e^{2t}}{e^t}\Big)dt < \infty, \\ \int^{\infty} \Big(\frac{1}{P_K(t)\phi(t)}\Big)dt = \int^{\infty} \Big(\frac{1}{e^{-2t+\pi}\cdot e^{t}}\Big)dt = \infty, \\ \int^{\infty} \phi(t)q_1(t)dt = \int^{\infty} \big(e^t \cdot e^{-t}\big)\,dt = \infty. \end{gathered}$$

It follows from Theorem 4.2 that every solution u of (4.1), (4.2) is oscillatory in $(0,\pi)\times(0,\infty)$. For example, $u = \sin x \sin t$ is such a solution.

4.2. **Interval oscillation results for case (C1).** Combining Theorems 2.1, 2.2, 3.3, and 3.4, we have the following theorems.

Theorem 4.4. *Assume that* (C1), (H1)–(H6), (H9) *hold. If for each* $T > 0$ *and some* $K > 0$*, there exist functions* $(H_1, H_2) \in \mathbb{H}$*,* $\phi(t) \in C^1((0,\infty); (0,\infty))$ *and* $a, b, c, \tilde{a}, \tilde{b}, \tilde{c} \in \mathbb{R}$ *such that* $T \le a < c < b < \tilde{a} < \tilde{c} < \tilde{b}$*,* (3.12) *and*

$$\begin{aligned} &\frac{1}{H_1(\tilde{c},\tilde{a})}\int_{\tilde{a}}^{\tilde{c}} H_1(s,\tilde{a})\{q_j(s) - \frac{1}{2}P_K(s)\lambda_1^2(s,\tilde{a})\}\phi(s)ds \\ &+ \frac{1}{H_2(\tilde{b},\tilde{c})}\int_{\tilde{c}}^{\tilde{b}} H_2(\tilde{b},s)\{q_j(s) - \frac{1}{2}P_K(s)\lambda_2^2(\tilde{b},s)\}\phi(s)ds > 0 \end{aligned}$$

hold, then every solution $u(x,t)$ *of* (1.1), (1.2) *(or* (1.1), (1.3)*) is oscillatory in* Ω*.*

Theorem 4.5. *Assume that* (C1), (H1), (H6), (H10) *hold. If for each* $T > 0$ *and some* $K > 0$*, there exist functions* $(H_1, H_2) \in \mathbb{H}$*,* $\phi(t) \in C^1((0,\infty);(0,\infty))$ *and* $a, b, c \in \mathbb{R}$ *such that* $T \le a < c < b$ *and* (3.12) *hold, then every solution of* (1.1), (1.2) *(or* (1.1), (1.3)*) is oscillatory in* Ω*.*

Theorem 4.6. *Assume that* (C1), (H1)–(H6), (H9) *(or* (H1)–(H6), (H10)*) hold. If for some functions* $(H_1, H_2) \in \mathbb{H}$*, each* $T \ge 0$ *and some* $K > 0$*, the conditions* (3.15) *and* (3.16) *hold, then every solution of* (1.1), (1.2) *(or* (1.1), (1.3)*) is oscillatory in* Ω*.*

Example 4.7. Consider the problem

$$\begin{gathered}\frac{\partial^2}{\partial t^2}\Big(u(x,t) + \frac{1}{2}u(x,t-\pi)\Big) - \Delta u(x,t) - 5t^{-2}\Delta u(x,t-2\pi) + 5t^{-2}u(x,t-\pi) \\ = \frac{1}{2}\sin x \sin t, \quad (0,\pi)\times(0,\infty),\end{gathered} \tag{4.3}$$

$$u(0,t) = u(\pi,t) = 0, \quad t > 0. \tag{4.4}$$

Here $l = k = m = 1$, $r(t) = 1$, $h_1(t) = 1/2$, $\rho_1(t) = t - \pi$, $q_1(x,t) = 5t^{-2}$, $\sigma_1(t) = t - \pi$ and $f(x,t) = \frac{1}{2}\sin x \sin t$.

It is easy to verify that $\Phi(x) = \sin x$ and

$$G(t) = F(t) = \frac{\pi}{8}\sin t \quad \text{and} \quad \tilde{\Theta}(t) = -\frac{3}{16}\pi \sin t.$$

Since

$$\int^{\infty} 5t^{-2}[\frac{1}{2} \pm \frac{3}{16}\pi \sin t]_{+} dt < \infty,$$

Then [8, Theorem 2.1] does not apply; however, by choosing $\phi(t) = t^2$ and $H_1(s,t) = H_2(t,s) = (t-s)^2$,

$$\limsup_{t\to\infty} \int_T^t (s-T)^2\{5s^{-2} - \frac{1}{2}\frac{1}{2}\frac{4T^2}{s^2(s-T)^2}\}s^2 ds > 0$$

and

$$\limsup_{t\to\infty} \int_T^t (t-s)^2\{5s^{-2} - \frac{1}{2}\frac{1}{2}\frac{4(t-2s)^2}{s^2(t-s)^2}\}s^2 ds > 0$$

hold. Therefore, Theorem 4.6 implies that every solution u of the problem (4.3), (4.4) is oscillatory in $(0,\pi)\times(0,\infty)$. In fact, one such solution is $u = \sin x \sin t$.

4.3. **Oscillation results by Riccati inequality for case (C2).** Combining Theorems 2.1, 2.2, and 3.5, we have the following theorem.

Theorem 4.8. *Assume that* (C2), (H1)–(H6), (H9) *hold. If for* $i = 1, 2$*,*

$$\int_{T_1}^{\infty} \Big(\frac{P_i(t)\phi'(t)^2}{\phi(t)}\Big)dt < \infty, \quad \int_{T_1}^{\infty} \frac{1}{P_i(t)\phi(t)}dt = \infty, \quad \int_{T_1}^{\infty} \phi(t)Q_i(t)dt = \infty, \tag{4.5}$$

then every solution of (1.1), (1.2) *(or* (1.1), (1.3)*) is oscillatory in* Ω*.*

Example 4.9. Consider the equation

$$
\begin{aligned}
&\frac{\partial}{\partial t}\Big(e^{1/8}\frac{\partial}{\partial t}\Big(u(x,t)+\frac{1}{2}u(x,t-\pi)\Big)\Big)\\
&-\frac{1}{2}e^{1/8}\Delta u(x,t)-\frac{1}{16}e^{1/8}\Delta u\big(x,t-\frac{\pi}{2}\big)+e^{2t}u(x,t-2\pi)\\
&=e^{2t}\sin x\sin t,\quad (0,\pi)\times(0,\infty),
\end{aligned}
\tag{4.6}
$$

$$
u(0,t)=u(\pi,t)=0,\quad t>0. \tag{4.7}
$$

Here $l=k=m=1$, $r(t)=e^{t/8}$, $h_1(t)=1/2$, $\rho_1(t)=t-\pi$, $q_1(x,t)=e^{2t}$, $\sigma_1(t)=t-2\pi$ and $f(x,t)=e^{2t}\sin x\sin t$. It is easy to see that $\Phi(x)=\sin x$ and

$$
\begin{gathered}
\int^{\infty}\Big(\frac{P_1(t)\phi'(t)^2}{\phi(t)}\Big)dt=\int^{\infty}\Big(\frac{e^{\frac{1}{8}(t-2\pi)}\cdot e^{-2t}}{e^{-t}}\Big)dt<\infty,\\
\int^{\infty}\Big(\frac{P_2(t)\phi'(t)^2}{\phi(t)}\Big)dt=\int^{\infty}\Big(\frac{\frac{1}{2}e^{\frac{1}{8}t}\cdot e^{-2t}}{e^{-t}}\Big)dt<\infty,\\
\int^{\infty}\frac{1}{P_1(t)\phi(t)}dt=\int^{\infty}\frac{1}{\big(e^{\frac{1}{8}(t-2\pi)}\cdot e^{-t}\big)}dt=\infty,\\
\int^{\infty}\frac{1}{P_2(t)\phi(t)}dt=\int^{\infty}\frac{1}{\big(\frac{1}{2}e^{\frac{1}{8}t}\cdot e^{-t}\big)}dt=\infty,\\
\int^{\infty}\phi(t)Q_1(t)dt=\int^{\infty}(e^{-t}\cdot e^{2t})=\infty,\\
\int^{\infty}\phi(t)Q_2(t)dt=\int^{\infty}e^{-t}\cdot e^{2t}\big[c\big(\frac{1}{2}-\frac{\pi}{8}\big)\cdot 8e^{-\frac{1}{8}(t-3\pi)}\big]_+dt=\infty,
\end{gathered}
$$

where $\phi(t)=e^{-t}$. Therefore it follows from Theorem 4.8 that every solution u of problem (4.6), (4.7) is oscillatory in $(0,\pi)\times(0,\infty)$. For example $u=\sin x\sin t$ is such a solution.

Combining Theorems 2.1, 2.2, and 3.6, we have the following result.

Theorem 4.10. *Assume* (C1), (H1)–(H6), (H10). *If* (4.5) *and*

$$
\int_{T_1}^{\infty}\phi(t)\tilde{Q}_i(t)dt=\infty\quad (i=1,2)
$$

hold, then every solution $u(x,t)$ of (1.1), (1.2) *(or* (1.1), (1.3)*) is oscillatory in* Ω, *where*

$$
\tilde{Q}_2(t)=q_j(t)\frac{1}{\tilde{K}}\varphi_j\Big(\big[c_1A(\sigma_j(t))\pi(\rho_*(\sigma_j(t)))+\tilde{\Theta}(\sigma_j(t))\big]_+\Big).
$$

4.4. **Interval oscillation results for case (C2).** Combining Theorems 2.1, 2.2, 3.7, and 3.8, we have the following result.

Theorem 4.11. *Assume that* (C2), (H1)–(H6), (H9) *hold. If for each $T>0$ and some $K>0$, $\tilde{K}>0$, there exist functions $(H_1,H_2)\in\mathbb{H}$, $\phi(t)\in C^1((0,\infty);(0,\infty))$ and $a,b,c,\tilde{a},\tilde{b},\tilde{c}\in\mathbb{R}$ such that $T\le a<c<b<\tilde{a}<\tilde{c}<\tilde{b}$, and* (3.12), (3.25),

$$
\begin{aligned}
&\frac{1}{H_1(\tilde{c},\tilde{a})}\int_{\tilde{a}}^{\tilde{c}}H_1(s,\tilde{a})\{q_j(s)-\frac{1}{2}P_K(s)\lambda_1^2(s,\tilde{a})\}\phi(s)ds\\
&+\frac{1}{H_2(\tilde{b},\tilde{c})}\int_{\tilde{c}}^{\tilde{b}}H_2(\tilde{b},s)\{q_j(s)-\frac{1}{2}P_K(s)\lambda_2^2(\tilde{b},s)\}\phi(s)ds>0
\end{aligned}
$$

and

$$\begin{aligned}
&\frac{1}{H_1(\tilde{c},\tilde{a})}\int_{\tilde{a}}^{\tilde{c}} H_1(s,\tilde{a})\{Q_2(s)-\frac{1}{2}P_2(s)\lambda_1^2(s,\tilde{a})\}\phi(s)ds\\
&+\frac{1}{H_2(\tilde{b},\tilde{c})}\int_{\tilde{c}}^{\tilde{b}} H_2(\tilde{b},s)\{Q_2(s)-\frac{1}{2}P_2(s)\lambda_2^2(\tilde{b},s)\}\phi(s)ds>0
\end{aligned}$$

hold, then every solution of (1.1), (1.2) *(or* (1.1), (1.3)*) is oscillatory in* Ω.

Theorem 4.12. *Assume* (C2), (H1)–(H4), (H9). *Also assume that for some functions* $(H_1,H_2)\in\mathbb{H}$*, each* $T\geq 0$ *and some* $K>0$, $\tilde{K}>0$. *If* (3.15), (3.16), (3.26), *and* (3.27) *hold, then every solution of* (1.1), (1.2) *(or* (1.1), (1.3)*) is oscillatory in* Ω.

Combining Theorems 2.1, 2.2, 3.9, and 3.10, we have the following result.

Theorem 4.13. *Assume that* (C2), (H1)–(H6), (H10) *hold. If for each* $T>0$ *and some* $K>0$, $\tilde{K}>0$*, there exist functions* $(H_1,H_2)\in\mathbb{H}$, $\phi(t)\in C^1((0,\infty);(0,\infty))$ *such that* (3.12) *and* (3.28) *hold, then every solution of* (1.1), (1.2) *(or* (1.1), (1.3)*) is oscillatory in* Ω.

Theorem 4.14. *Assume* (C2), (H1)–(H4), (H10). *Also assume that some functions* $(H_1,H_2)\in\mathbb{H}$ *for each* $T\geq 0$ *and some* $K>0$, $\tilde{K}>0$. *If* (3.15), (3.16), (3.29), (3.30) *hold, then every solution of* (1.1), (1.2) *(or* (1.1), (1.3)*) is oscillatory in* Ω.

Example 4.15. Consider the equation

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big(t^3\frac{\partial}{\partial t}\Big(u(x,t)+\frac{1}{2}u(x,t-\pi)\Big)\Big)\\
&-\frac{t^3}{2}\Delta u(x,t)-\big(t+\frac{3}{2}t^2\big)\Delta u\big(x,t-\frac{\pi}{2}\big)+u(x,t-2\pi)\\
&=(\sin t-t\cos t)\sin x,\quad (0,\pi)\times(T_0,\infty),
\end{aligned}\tag{4.8}$$

$$u(0,t)=u(\pi,t)=0,\quad t>T_0=\pi/(1-e^{-1/4}).\tag{4.9}$$

Here $l=k=m=1$, $r(t)=t^3$, $h_1(t)=1/2$, $\rho_1(t)=t-\pi$, $q_1(x,t)=1$, $\sigma_1(t)=t-2\pi$ and $f(x,t)=(\sin t-t\cos t)\sin x$. An easy computation shows that $\Phi(x)=\sin x$ and

$$\pi(t)=\frac{1}{2}t^{-2},\quad \tilde{\Theta}(t)=\frac{\pi}{4}\Big(t^{-2}+\frac{1}{2}(t-\pi)^{-2}\Big)\cos t,\quad A(t)=\frac{1}{2}+2\log\big(\frac{t-\pi}{t}\big)>0.$$

Since

$$\int^{\infty}\big(\frac{1}{2}t^{-2}\big)[cA(t-2\pi)\pi(t-3\pi)\pm\Theta(t-2\pi)]_+dt<\infty,$$

Note that [8, Theorem 3.2] is not applicable to this problem. However, we see from $\phi(t)=t^3$ and $H_1(s,t)=H_2(t,s)=(t-s)^3$ that

$$\begin{gathered}
\limsup_{t\to\infty}\int_T^t(s-T)^3\{1-\frac{1}{2}(s-2\pi)^3\frac{9T^2}{s^2(s-T)^2}\}s^3ds>0,\\
\limsup_{t\to\infty}\int_T^t(s-t)^3\{1-\frac{1}{2}(s-2\pi)^3\frac{9(t-2s)^2}{s^2(s-t)^2}\}s^3ds>0,\\
\limsup_{t\to\infty}\int_T^t(s-T)^3\{1-\frac{1}{2}\frac{s^3}{2}\frac{9T^2}{s^2(s-T)^2}\}s^3ds>0,
\end{gathered}$$

$$\limsup_{t\to\infty}\int_T^t (t-s)^3\{[cA(t-2\pi)\pi(t-3\pi)\pm\tilde{\Theta}(t-2\pi)]_+ - \frac{1}{2}\frac{s^3}{2}\frac{9(t-2s)^2}{s^2(s-t)^2}\}s^3ds > 0.$$

Therefore, Theorem 4.14 implies that every solution u of the problem (4.8), (4.9) is oscillatory in $(0,\pi)\times(T_0,\infty)$. In fact, one such solution is $u=\sin x\sin t$.

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