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SCATTERING FOR WAVE EQUATIONS WITH DISSIPATIVE TERMS IN LAYERED MEDIA

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Abstract. In this article, we show the existence of scattering solutions to wave equations with dissipative terms in layered media. To analyze the wave propagation in layered media, it is necessary to handle singular points called thresholds in the spectrum. Our main tools are Kato's smooth perturbation theory and some approximate operators.

1. Introduction

In this article, we study the wave propagation in Ω (layered media), expressed as
$$\Omega = \{(x, y) : x \in \mathbb{R}^N,\, 0 < y < \pi\},$$
where $N \in \mathbb{N}$ is a fixed number.

We consider wave equations with dissipative terms:
$$\begin{gathered} \partial_t^2 u(x,y,t) + b(x,y)\partial_t u(x,y,t) - \Delta u(x,y,t) = 0, \quad (x,y,t) \in \Omega \times [0,\infty), \\ u(x,0,t) = u(x,\pi,t) = 0, \quad (x,t) \in \mathbb{R}^N \times [0,\infty), \end{gathered} \tag{1.1}$$
where $\partial_t = \partial/\partial t$, $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \dots + \partial^2/\partial x_N^2 + \partial^2/\partial y^2$, and $b(x, y)$ is a measurable non-negative function that decays as $|x| \to \infty$.

We consider (1.1) as a perturbed system for
$$\begin{gathered} \partial_t^2 u(x,y,t) - \Delta u(x,y,t) = 0, \quad (x,y,t) \in \Omega \times (-\infty,\infty), \\ u(x,0,t) = u(x,\pi,t) = 0, \quad (x,t) \in \mathbb{R}^N \times (-\infty,\infty). \end{gathered} \tag{1.2}$$

The primary purpose of the present paper is to show the existence of scattering solutions for $b(x, y)$ under the following conditions (cf. Mochizuki-Nakazawa [10]): For $b_0 > 0$, $\delta \in (0, 1]$, and $m \in \mathbb{N} \cup \{0\}$,
$$0 \le b(x,y) \le b_0 \Big(\prod_{k=0}^{m} \log^{[k]}(e_m + r) \Big)^{-1} \big(\log^{[m]}(e_m + r) \big)^{-\delta}, \tag{1.3}$$

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where $r = |x|$ and

$$e_0 = 1, \quad e_m = e^{e_{m-1}}, \quad \log^{[0]} s = s, \quad \log^{[m]} s = \log\log^{[m-1]} s \quad (m \geq 1).$$

For instance, if $m = 0$, then (1.3) is expressed as

$$0 \leq b(x,y) \leq b_0(1+r)^{-1-\delta}. \tag{1.4}$$

Moreover, it can be easily observed that

$$\int_0^\infty \Big(\prod_{k=0}^m \log^{[k]}(e_m + r)\Big)^{-1}\big(\log^{[m]}(e_m+r)\big)^{-\delta} dr < \infty.$$

Hence, (1.3) represents the short-range condition.

To explain the thresholds, we define a self-adjoint operator, L_0, in $L^2(\Omega)$ by

$$L_0 u = -\varDelta u, \quad D(L_0) = \{u \in H_0^1(\Omega) : \varDelta u \in L^2(\Omega)\}.$$

For $z \notin \mathbb{R}$, we put $R(z) = (L_0 - z^2)^{-1}$. Then we have

$$(R(z)\varphi)(x,y) = \frac{2}{\pi}\sum_{n=1}^\infty \sin ny \int_0^\pi \sin ny'(r_n(z)\varphi)(x,y')dy', \tag{1.5}$$

for $\varphi \in C_0^\infty(\Omega)$, where $r_n(z) = (-\varDelta_x - (z^2-n^2))^{-1}$ and $\varDelta_x = \sum_{j=1}^N \partial^2/\partial x_j^2$.

Therefore, $\sigma(L_0) = \sigma_{ac}(L_0) = \cup_{n=1}^\infty [n^2, \infty) = [1^2, \infty)$. The operator $r_n(z)$ (and accordingly $R(z)$) has singularity at $z^2 = n^2$. $\{n^2\}_{n\in\mathbb{N}}$ are called the *thresholds* of L_0.

The solution of (1.2) is represented by the superposition of several modes; that is, the solution u of (1.2) is represented as

$$u(x,y,t) = \sum_{n=1}^\infty u_n(x,t)\sin ny,$$

where $u_n(x,t)$ is the solution of

$$\partial_t^2 u_n(x,t) - \varDelta_x u_n(x,t) + n^2 u_n(x,t) = 0, \quad (x,t) \in \mathbb{R}^N \times (-\infty,\infty).$$

To explain the main results, we put $f(t) = {}^t(u(t), \partial_t u(t))$. Then (1.1) and (1.2) can be expressed as

$$\partial_t f(t) = -iAf(t) \quad \text{and} \quad \partial_t f(t) = -iA_0 f(t),$$

where

$$A_0 = i\begin{pmatrix} 0 & 1 \\ \varDelta & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & b(x,y) \end{pmatrix}, \tag{1.6}$$

and $A = A_0 - iB$.

Let $\nabla = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3, \dots, \partial/\partial x_N, \partial/\partial y)$ and $\dot{H}_0^1(\Omega)$ be the completion of $C_0^\infty(\Omega)$ with respect to $\|\nabla f\|_{L^2(\Omega)}$. Let $\mathcal{H} = \dot{H}_0^1(\Omega) \times L^2(\Omega)$ be the Hilbert space with the inner product

$$\langle f, g\rangle = \int_\Omega (\nabla f_1(x,y)\cdot\overline{\nabla g_1(x,y)} + f_2(x,y)\overline{g_2(x,y)})\,dx\,dy,$$

where $f = {}^t(f_1, f_2)$ and $g = {}^t(g_1, g_2)$. The norm of $\mathcal{H}$ is denoted by $\|\cdot\|$.

We define the domains of A and A_0 as

$$D(A) = D(A_0) = \big\{f = {}^t(f_1,f_2) \in \mathcal{H} : \varDelta f_1 \in L^2(\Omega), f_2 \in H_0^1(\Omega)\big\}.$$

A_0 is self-adjoint, and hence, $U(t) = e^{-itA_0}$ ($t \in \mathbb{R}$) is unitary. Moreover, $-iA$ generates a contraction semi-group $V(t)$, $t \geq 0$; see Reed-Simon [14, Theorem X-50].

We have $\sigma(A_0) = \sigma_{ac}(A_0) = \cup_{n=1}^{\infty}(-\infty, -n] \cup [n, \infty) = (-\infty, -1] \cup [1, \infty)$ (cf. Proposition 3.1). $\{\pm n\}_{n\in\mathbb{N}}$ are called the *thresholds* of A_0.

The main result of this paper can be stated as follows.

Theorem 1.1. *Let us assume* (1.3)*. Then for the above defined A_0 and A, it holds that*

1. *A has no real eigenvalues.*
2. *The wave operator*
$$W = s - \lim_{t\to\infty} U(-t)V(t)$$
exists. Moreover, W is not zero as an operator in $\mathcal{H}$.

Corollary 1.2. *There exist non-trivial initial data $f \in D(A)$ and $f_+ \in D(A_0)$ such that*
$$\lim_{t\to\infty} \|V(t)f - U(t)f_+\| = 0. \tag{1.7}$$

If $V(t)f$ satisfies (1.7), then $V(t)f$ is called the scattering solution to $\partial_t f(t) = -iAf(t)$, $f(0) \in D(A)$. The proof of Corollary 1.2 is obtained in the same manner as in Kadowaki [5, Corollary 2]; hence, it is omitted here.

Remark 1.3. When we assume the Neumann conditions instead of the Dirichlet conditions, we can obtain the same results as in Theorem 1.1.

Spectral analysis near the thresholds on layered media has been performed by several authors (e.g., Sveshnikov [19], [20], Werner [21], and Ramm-Werner [13]). In [20], it has been proved that the limiting absorption principle does not hold at the thresholds in the case of $N = 1$. In [13], it has been shown that the limiting amplitude principle does not hold at the thresholds for $N = 1, 2$ but holds $N \geq 3$.

In the cases of other media, the existence of the thresholds is known. For example, Ben-Artzi [3] and Weder [22], [23] have derived the limiting absorption principle at the thresholds on inhomogeneous layered media in $\mathbb{R}^2$ and stratified media, respectively.

Wave equations with dissipative terms have been studied by Mochizuki [8] and Kadowaki [5]. In [8], the existence of scattering solutions has been shown for wave equations in $\mathbb{R}^N$, $N \neq 2$ (for $N = 2$, see Nakazawa [11]). The above proof was based on Kato's smooth perturbation theory (Kato [6]). In [5], the same problem was dealt with for stratified media. In that proof, in addition to the concept employed in [8], an approximate operator employed by Simon [18] and the well-known properties of compact operators have been used.

In [18], $(H - i)^{-2}H$ has been used as an approximate operator, where H is the Schrödinger operator with absorption (non-self-adjoint operator). Concretely, in that study, the set
$$\{(H - i)^{-2}Hv : v \in D(H) \cap (L^2(\mathbb{R}^N))_b^{\perp}\}$$
was proven to be dense in $(L^2(\mathbb{R}^N))_b^{\perp}$, where $(L^2(\mathbb{R}^N))_b$ is the space generated by the eigenvector of H with real eigenvalues. The reason for using the approximate operator is as follows. For the spectral analysis of non-self-adjoint operators, it is difficult to use localized method for the spectrum because the spectral resolution

theory for non-self-adjoint operators has not been established yet. Even if $\Psi(\lambda)$ belongs to $C_0^\infty(\mathbb{R})$, it is difficult to define $\Psi(H)$. Hence, an approximate operator was used instead of $\Psi(H)$.

We will prove Theorem 1.1 using the concept employed in [8] and [5]. The existence of the thresholds makes the proof difficult. To eliminate the difficulty, we use $\sqrt{B}(A_0^2 - n^2)(A_0 - i)^{-2}$. This operator plays an important role in the proof (see section 2 (A3) and section 3). In addition, we use approximate operators of Simon's type: $\prod_{k=q}^{p}(A^2 - k^2)(A - i)^{-l}$, where $l = 1, 2$ (see Lemma 2.1 and the proofs for Lemma 2.5 and Theorem 2.3).

There are several other results on scattering problems for dissipative wave (hyperbolic) equations (e.g., Mochizuki-Nakazawa [10], Petkov [12], etc.). However, there are no results for dissipative wave equations in layered media.

Before concluding this section, we will briefly explain the contents of the present paper. In section 2, we will describe an abstract result (Theorem 2.3) and provide its proof. In section 3, we will prove Theorem 1.1 by applying Theorem 2.3. In section 4, we will provide a resolvent estimate. In section 5, we will consider the case where $b(x, y)$ satisfies (5.1). Hence, we will be able to show that the total energy of all solutions of (1.1) decays (i.e., (1.1) has only dissipative solutions).

2. Abstract result

To prove Theorem 1.1, we prepare an abstract theorem (Theorem 2.3).

Let $\mathcal{H}$ be a separable Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. Let A be a linear operator in $\mathcal{H}$, and let $\mathcal{H}_b$ be the space generated by the eigenvector of A with real eigenvalues. We assume that $-iA$ generates a contraction semi-group $V(t)$ $(t \geq 0)$.

In order to conduct a density argument, we prepare subspaces of Simon's type.

Lemma 2.1. *Let $l, p \in \mathbb{N}$. Let $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_p \in \mathbb{R}$ be a finite sequence. Then the set*
$$\Phi = \{\prod_{k=1}^{p}(A - \alpha_k)\{(A - i)^{-1}\}^l f : f \in \mathcal{H}_b^\perp\}$$
is dense in $\mathcal{H}_b^\perp$.

Proof. Let $\alpha \in \mathbb{R}$. Using the same approach as that used by Petkov [12, Lemma 1.1.6], we can prove that the set
$$\tilde{\Phi}_0 = \{(A - \alpha)\{(A - i)^{-1}\}^l f : f \in D(A) \cap \mathcal{H}_b^\perp\}$$
is dense in $\mathcal{H}_b^\perp$. We use
$$\Phi_0 = \{(A - \alpha)\{(A - i)^{-1}\}^l f : f \in \mathcal{H}_b^\perp\}.$$
Then since $\tilde{\Phi}_0 \subset \Phi_0$, Φ_0 is also dense in $\mathcal{H}_b^\perp$. Thus, repeating the density argument for Φ_0, we observe that Φ is also dense in $\mathcal{H}_b^\perp$. □

Remark 2.2. If A is a self-adjoint operator, then the same assertion as that used in Lemma 2.1 remains true when $(A - i)^{-1}$ is replaced by $(A + i)^{-1}$.

Let A_0 and B be self-adjoint operators in $\mathcal{H}$. Let $E(\lambda)$ be the spectral family of A_0 and $\{U(t)\}_{t\in\mathbb{R}}$ be the unitary group $\{e^{-itA_0}\}_{t\in\mathbb{R}}$. We assume the following three conditions:

(A1) $\sigma(A_0) = \sigma_{ac}(A_0) = (-\infty, -m] \cup [m, \infty)$ for some $m \geq 0$;

(A2) B is nonnegative and A_0-compact;
(A3) There exists a sequence: $m = a_1 < a_2 < a_3 < \dots < a_n < \dots$ such that $\lim_{n\to\infty} a_n = \infty$, and

$$\sqrt{B} F_n(A_0) E_{a_n, a_{n+1}}(A_0)$$

is A_0-smooth, where $E_{\alpha,\beta}(A_0) = E((-\beta,-\alpha) \cup (\alpha,\beta))$ for $0 < \alpha < \beta$ and

$$F_n(\lambda) = \{(\lambda - a_n)(\lambda - i)^{-1}\}\{(\lambda + a_n)(\lambda - i)^{-1}\} = (\lambda^2 - a_n^2)(\lambda - i)^{-2}.$$

In this article, we define a bounded operator K to be A_0-smooth (Kato [6]) if there exists a positive constant C such that

$$\int_{-\infty}^{\infty} \|KU(t)f\|^2 dt \le C\|f\|^2$$

for any $f \in \mathcal{H}$ (cf. Reed -Simon [15, p. 144, Lemma 2]). By (A2), $-i(A_0 - iB)$ generates a contraction semigroup (see [14, Theorem X-50]).

Theorem 2.3. *Assume* (A1), (A2), (A3). *Let $A = A_0 - iB$. Then the assertion is in Theorem 1.1 holds.*

To prove the above theorem, we first find the following relations.

$$\|V(t)f\|^2 + 2\int_0^t \|\sqrt{B}V(\tau)f\|^2 d\tau = \|f\|^2 \tag{2.1}$$

for $f \in D(A)$. Next, (2.1) implies

$$\int_0^{\infty} \|\sqrt{B}V(\tau)f\|^2 d\tau \le \frac{1}{2}\|f\|^2. \tag{2.2}$$

for $f \in D(A)$. We use

$$W(t) = U(-t)V(t), \quad \tilde{F}_n(A_0) = \prod_{j=1}^{n} F_j(A_0), \quad \tilde{E}_n(A_0) = \sum_{j=1}^{n} E_{a_j, a_{j+1}}(A_0).$$

Next, we prepare the following two lemmas.

Lemma 2.4. *Let $f \in \mathcal{H}$. Then for every $n \in \mathbb{N}$,*

$$\lim_{s,t\to\infty} \|\tilde{F}_n(A_0)\tilde{E}_n(A_0)(W(t) - W(s))f\| = 0.$$

Proof. Put $M_k = \sup_{\lambda\in\mathbb{R}} |F_k(\lambda)|$. For any $\varepsilon > 0$, there exists $h \in D(A)$ such that

$$\|f - h\| < \frac{\varepsilon}{2\prod_{j=1}^{n} M_j}. \tag{2.3}$$

Since

$$\begin{aligned} &\|\tilde{F}_n(A_0)\tilde{E}_n(A_0)(W(t) - W(s))h\| \\ &\le \sum_{k=1}^{n} \prod_{j=1, j\neq k}^{n} M_j \|F_k(A_0)E_{a_k,a_{k+1}}(A_0)(W(t) - W(s))h\|, \end{aligned}$$

it is sufficient to show that

$$\lim_{s,t\to\infty} \|F_k(A_0)E_{a_k,a_{k+1}}(A_0)(W(t) - W(s))h\| = 0 \tag{2.4}$$

for $k = 1, 2, 3 \dots, n$. Indeed, it follows from (2.3) and (2.4) that

$$\limsup_{s,t\to\infty} \|\tilde{F}_n(A_0)\tilde{E}_n(A_0)(W(t) - W(s))f\| \le 2\varepsilon.$$

Thus, we obtain the desired result.

Now, we prove (2.4). This is proven using the same approach as that used in Mochizuki [8]. Hence, we provide a brief overview of the proof. Let $g \in \mathcal{H}$. Using the equality

$$\begin{aligned}&\langle F_k(A_0)E_{a_k,a_{k+1}}(A_0)(W(t) - W(s))h, g\rangle\\ &= -\int_s^t \langle \sqrt{B}V(\tau)h, \sqrt{B}U(\tau)F_k(A_0)E_{a_k,a_{k+1}}(A_0)g\rangle d\tau\end{aligned}$$

together with (A3) and (2.2), we obtain (2.4). □

Lemma 2.5. *Let $f \in \mathcal{H}$. Then $w - \lim_{t\to\infty} V(t)f = 0$.*

Proof. Let $g \in \mathcal{H}$. For any $\varepsilon > 0$, there exists $h \in \mathcal{H}$ such that

$$\|g - \overline{F}_1(A_0)(A_0 + i)^{-2}h\| < \frac{\varepsilon}{\|f\|} \tag{2.5}$$

by Lemma 2.1 (and Remark 2.2) for $l = 2$ and $\alpha_1 = a_1$ and $\alpha_2 = -a_1$. For ε and h, there exists $n \in \mathbb{N}$ such that

$$\sup_{|\lambda|>a_{n+1}} |\overline{F}_1(\lambda)(\lambda^2 + i)^{-2}| < \frac{\varepsilon}{\|f\|\|h\|}. \tag{2.6}$$

Moreover, for ε, h, and n, there exists $\varphi \in \mathcal{H}$ such that

$$\|h - \prod_{k=2}^{n} \overline{F}_k(A_0)\varphi\| < \frac{\varepsilon}{M_1\|f\|} \tag{2.7}$$

by Lemma 2.1 (and Remark 2.2) for $l = 1$ and $\alpha_1 = a_2$, $\alpha_2 = -a_2$, $\alpha_3 = a_3$, $\alpha_4 = -a_3 \dots$, $M_1 = \sup_{\lambda\in\mathbb{R}} |F_1(\lambda)|$.

Let $f \in \mathcal{H}$ and $\tau \ge 0$. Using the values of h, φ, and n, we decompose $\langle (W(t) - W(s))f, U(-\tau)g\rangle$ into four parts:

$$\langle (W(t) - W(s))f, U(-\tau)g\rangle = \sum_{j=1}^{4} I_j(s,t),$$

where

$$\begin{gathered}I_1(s,t) = \langle (W(t) - W(s))f, U(-\tau)(g - \overline{F}_1(A_0)(A_0 + i)^{-2}h)\rangle,\\ I_2(s,t) = \langle (W(t) - W(s))f, U(-\tau)(I_d - \tilde{E}_n(A_0))\overline{F}_1(A_0)(A_0 + i)^{-2}h)\rangle,\\ I_3(s,t) = \langle (A_0 - i)^{-2}\tilde{E}_n(A_0)F_1(A_0)(W(t) - W(s))f, U(-\tau)(h - \prod_{k=2}^{n} \overline{F}_k(A_0)\varphi)\rangle,\\ I_4(s,t) = \langle (A_0 - i)^{-2}\tilde{E}_n(A_0)\tilde{F}_n(A_0)(W(t) - W(s))f, U(-\tau)\varphi\rangle.\end{gathered}$$

By (2.5), (2.6), and (2.7), we have

$$|I_1(s,t)| + |I_2(s,t)| + |I_3(s,t)| \le 6\varepsilon.$$

Hence, we have

$$|\langle (W(t) - W(s))f, U(-\tau)g\rangle| \le 6\varepsilon + \|(\tilde{E}_n\tilde{F}_n)(A_0)(W(t) - W(s))f\|\|\varphi\| \tag{2.8}$$

uniformly for $\tau \geq 0$. Put $\tau = 0$ in (2.8). Then Lemma 2.4 implies that

$$\limsup_{s,t\to\infty} |\langle (W(t) - W(s))f, g\rangle| \leq 6\varepsilon.$$

Since $g \in \mathcal{H}$ is arbitrary, there exists $f_+ \in \mathcal{H}$ such that

$$\mathrm{w} - \lim_{t\to\infty} W(t)f = f_+.$$

Now, we return to (2.8). It is noted that $f_+^{(n)} := s - \lim_{s\to\infty}(\tilde{E}_n\tilde{F}_n)(A_0)W(s)f$ exists by Lemma 2.4. Thus, we have

$$|\langle W(t)f - f_+, U(-\tau)g\rangle| \leq 6\varepsilon + \|(\tilde{E}_n\tilde{F}_n)(A_0)W(t)f - f_+^{(n)}\|\|\varphi\|$$

in (2.8) as $s \to \infty$.

Substituting $\tau = t$, we have

$$|\langle V(t)f - U(t)f_+, g\rangle| \leq 6\varepsilon + \|(\tilde{E}_n\tilde{F}_n)(A_0)W(t)f - f_+^{(n)}\|\|\varphi\|.$$

Further, as $t \to \infty$, we have

$$\limsup_{t\to\infty} |\langle V(t)f - U(t)f_+, g\rangle| \leq 6\varepsilon.$$

Since $w - \lim_{t\to\infty} U(t)f_+ = 0$ by (A1), we have

$$\limsup_{t\to\infty} |\langle V(t)f, g\rangle| \leq 6\varepsilon.$$

Thus, the proof is complete. □

As an immediate consequence of Lemma 2.5, we see that

$$\sigma_p(A) \cap \mathbb{R} = \emptyset, \tag{2.9}$$

whose proof can be found in Kadowaki [5].

Proof of Theorem 2.3. By Lemma 2.1 and (2.9), the set

$$\{(A - \alpha)\{(A - i)^{-1}\}^l f : f \in \mathcal{H}\}$$

is dense in $\mathcal{H}$. We use

$$F_k(A) = \{(A - a_k)(A - i)^{-1}\}\{(A + a_k)(A - i)^{-1}\}.$$

Let $f \in \mathcal{H}$. Then for any $\varepsilon > 0$, there exists $g \in \mathcal{H}$ such that

$$\|f - F_1(A)\{(A - i)^{-1}\}^2 g\| < \varepsilon \tag{2.10}$$

by Lemma 2.1, for $l = 2$ and $\alpha_1 = a_1$ and $\alpha_2 = -a_1$.

For ε and g, there exists $n \in \mathbb{N}$ such that

$$\sup_{|\lambda|>a_{n+1}} |F_1(\lambda)(\lambda^2 - i)^{-2}| < \varepsilon/\|g\|. \tag{2.11}$$

Moreover, for ε, g and n, there exists $\varphi \in \mathcal{H}$ such that

$$\|g - \prod_{k=2}^{n} F_k(A)\varphi\| < \frac{\varepsilon}{M_1} \tag{2.12}$$

by Lemma 2.1, for $l = 1$ and $\alpha_1 = a_2$, $\alpha_2 = -a_2$, $\alpha_3 = a_3$, $\alpha_4 = -a_3$, $\dots$, $M_1 = \sup_{\lambda\in\mathbb{R}} |F_1(\lambda)|$.

Using g, φ and n, we decompose $(W(t) - W(s))f$ into eight parts:

$$(W(t) - W(s))f = \sum_{j=1}^{8} J_j(s, t),$$

where

$$
\begin{gathered}
J_1(s,t)=(W(t)-W(s))(f-F_1(A)\{(A-i)^{-1}\}^2g),\\
J_2(s,t)=U(-t)\left[F_1(A)\{(A-i)^{-1}\}^2-F_1(A_0)(A_0-i)^{-2}\right]V(t)g,\\
J_3(s,t)=-U(-s)\left[F_1(A)\{(A-i)^{-1}\}^2-F_1(A_0)(A_0-i)^{-2}\right]V(s)g,\\
J_4(s,t)=(I_d-\tilde{E}_n(A_0))F_1(A_0)(A_0-i)^{-2}(W(t)-W(s))g,\\
J_5(s,t)=(A_0-i)^{-2}\tilde{E}_n(A_0)F_1(A_0)(W(t)-W(s))\Big(g-\prod_{k=2}^n F_k(A)\varphi\Big),\\
J_6(s,t)=(A_0-i)^{-2}\tilde{E}_n(A_0)F_1(A_0)U(-t)\Big(\prod_{k=2}^n F_k(A)-\prod_{k=2}^n F_k(A_0)\Big)V(t)\varphi,\\
J_7(s,t)=-(A_0-i)^{-2}\tilde{E}_n(A_0)F_1(A_0)U(-s)\Big(\prod_{k=2}^n F_k(A)-\prod_{k=2}^n F_k(A_0)\Big)V(s)\varphi,\\
J_8(s,t)=(A_0-i)^{-2}\tilde{E}_n(A_0)\tilde{F}_n(A_0)(W(t)-W(s))\varphi.
\end{gathered}
$$

Then (2.10), (2.11) and (2.12) imply

$$\|J_1(s,t)\|+\|J_4(s,t)\|+\|J_5(s,t)\|\le 6\varepsilon.$$

From (A2), we find that $F_1(A)\{(A-i)^{-1}\}^2-F_1(A_0)(A_0-i)^{-2}$ and $\prod_{k=2}^n F_k(A)-\prod_{k=2}^n F_k(A_0)$ are compact operators. Thus, using Lemma 2.5, we have

$$\lim_{s,t\to\infty}\|J_j(s,t)\|=0,$$

where $j=2,3,6,7$. Lemma 2.4 implies that

$$\lim_{s,t\to\infty}\|J_8(s,t)\|=0.$$

Therefore, we have

$$\limsup_{s,t\to\infty}\|(W(t)-W(s))f\|\le 6\varepsilon.$$

This indicates the existence of W.

Finally, we show that $W\not\equiv 0$ using the same argument as that used in [8], section 2. We assume that $W\equiv 0$; i.e.,

$$\lim_{t\to\infty}\|V(t)f\|=0$$

for any $f\in\mathcal{H}$. Then using the same argument as that used in [5], section 3, it follows that

$$\|G(A_0)f\|^2\le\|f\|\|(G(A)-G(A_0))f\|+\|f\|\Big(\frac{1}{2}\int_0^\infty\|\sqrt{B}U(t)G(A_0)f\|^2dt\Big)^{1/2}, \tag{2.13}$$

where $G(\lambda)=(\lambda-i)^{-1}$.

Let $n\in\mathbb{N}$. For any α,α',β, and β' satisfying $a_n<\alpha<\alpha'<\beta'<\beta<a_{n+1}$, we consider $\psi_{\alpha,\beta}(\lambda)\in C_0^\infty(\mathbb{R})$, $0\le\psi_{\alpha,\beta}\le 1$ such that $\psi_{\alpha,\beta}$ has support in $(-\beta,-\alpha)\cup(\alpha,\beta)$ and that $\psi_{\alpha,\beta}=1$ on $[-\beta',-\alpha']\cup[\alpha',\beta']$.

We put $f=U(s)\psi_{\alpha,\beta}(A_0)g$ for any $g\in\mathcal{H}$. Hence, it follows from (2.13) that

$$
\begin{aligned}
\|G(A_0)\psi_{\alpha,\beta}(A_0)g\|^2&\le\|f\|\|(G(A)-G(A_0))U(s)\psi_{\alpha,\beta}(A_0)g\|\\
&\quad+\|f\|\Big(\frac{1}{2}\int_s^\infty\|\sqrt{B}U(t)G(A_0)\psi_{\alpha,\beta}(A_0)g\|^2dt\Big)^{1/2}.
\end{aligned}
$$

Next, (A1) and (A2) imply

$$\lim_{s\to\infty} \|(G(A)-G(A_0))U(s)\psi_{\alpha,\beta}(A_0)g\| = 0.$$

Since $G(A_0)F_n(A_0)^{-1}\psi_{\alpha,\beta}(A_0)$ is bounded, $\sqrt{B}G(A_0)\psi_{\alpha,\beta}(A_0)$ is A_0-smooth by (A3). This implies

$$\lim_{s\to\infty}\int_s^\infty \|\sqrt{B}U(t)G(A_0)\psi_{\alpha,\beta}(A_0)g\|^2 dt = 0.$$

Therefore, we obtain $\psi_{\alpha,\beta}(A_0)g \equiv 0$. Using $E(\{\pm a_n\}) = E(\{\pm a_{n+1}\}) = 0$ and the decomposition of the identity on $(-a_{n+1}, -a_n) \cup (a_n, a_{n+1})$, we have $g \equiv 0$. This is a contradiction. □

3. Proof of Theorem 1.1

In this section, we will show that A_0 and B defined in section 1 satisfy (A1), (A2), and (A3) in section 2. (A2) is obtained from Rellich's theorem. (A1) is discussed as follows.

Proposition 3.1. *For A_0 defined in section 1,* (A1) *is satisfied for $m = 1$:*

$$\sigma(A_0) = \sigma_{ac}(A_0) = (-\infty,-1] \cup [1,\infty).$$

Proof. We put

$$T_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{L_0} & i \\ \sqrt{L_0} & -i \end{pmatrix}.$$

Hence, T_0 is a unitary operator from $\mathcal{H} = \dot{H}_0(\Omega) \times L^2(\Omega)$ onto $L^2(\Omega) \times L^2(\Omega)$. Put

$$B_0 = T_0 A_0 T_0^{-1} = \begin{pmatrix} \sqrt{L_0} & 0 \\ 0 & -\sqrt{L_0} \end{pmatrix}.$$

A_0 and B_0 are unitary equivalent. As mentioned in section 1, since $\sigma(L_0) = \sigma_{ac}(L_0) = [1^2,\infty)$, we have

$$\sigma(B_0) = \sigma_{ac}(B_0) = (-\infty,-1] \cup [1,\infty).$$

Thus the proof is complete. □

To discuss (A3), we first give the following definitions. We define weighted L^2-spaces in the form

$$L^2_{Y_m}(\mathbb{R}^N) = \{f(x) : \int_{\mathbb{R}^N} |Y_m f(x)|^2 dx < \infty\},$$
$$L^2_{Y_m^{-1}}(\mathbb{R}^N) = \{f(x) : \int_{\mathbb{R}^N} |Y_m^{-1} f(x)|^2 dx < \infty\},$$

where

$$Y_m = \Big(\prod_{k=0}^m \log^{[k]}(e_m + |x|)\Big)^{-1/2} (\log^{[m]}(e_m + |x|))^{-\delta/2}$$

for the same m as in (1.3).

For the Hilbert space $\mathcal{E}$, we denote the inner product, the norm, and the operator norm from $\mathcal{E}$ to $\mathcal{E}$ by $\langle\cdot,\cdot\rangle_{\mathcal{E}}$, $\|\cdot\|_{\mathcal{E}}$, and $\|\cdot\|_{\mathcal{B}(\mathcal{E})}$, respectively. If $\mathcal{E} = \mathcal{H}$ defined in section 1, then we omit the suffix $\mathcal{H}$.

Lemma 3.2. *Let $n \in \mathbb{N}$. Then for every $\lambda \in (-\infty, -n) \cup (n, \infty)$, there exist limits $r_n(\lambda \pm i0) \in \mathcal{B}(L^2_{Y_m^{-1}}(\mathbb{R}^N), L^2_{Y_m}(\mathbb{R}^N))$ such that*

$$\langle r_n(\lambda \pm i0) Y_m u, Y_m v\rangle_{L^2(\mathbb{R}^N)} = \lim_{\kappa\downarrow 0} \langle r_n(z) Y_m u, Y_m v\rangle_{L^2(\mathbb{R}^N)}, \tag{3.1}$$

for any $u, v \in L^2(\mathbb{R}^N)$, where $z = \lambda \pm i\kappa$ with $\kappa > 0$. Moreover, there exists a positive constant C such that

$$\|Y_m r_n(\lambda \pm i0) Y_m\|_{\mathcal{B}(L^2(\mathbb{R}^N))} \le C|\lambda^2 - n^2|^{-1/2}, \tag{3.2}$$

where C is independent of λ.

The above lemma will be proved in section 4. Using this lemma, we demonstrate the following proposition.

Proposition 3.3. *Let $n \in \mathbb{N}$ and $\varepsilon \in (0, 1)$; further, let $f \in \mathcal{H}$. Then the following is observed:*

(1) *For any $\lambda \in (-n - \varepsilon, -n) \cup (n, n + \varepsilon)$, there exists a positive constant C_1, independent of λ, such that*
$$\frac{d}{d\lambda}\|E(\lambda)\sqrt{B} f\|^2 \le C_1(\lambda^2 - n^2)^{-1/2}\|f\|^2.$$
(2) *For any $\lambda \in (-n-1, -n-\varepsilon) \cup (n+\varepsilon, n+1)$, there exists a positive constant C_2, independent of λ, such that*
$$\frac{d}{d\lambda}\|E(\lambda)\sqrt{B} f\|^2 \le C_2\|f\|^2.$$

Proof. Put $\rho(z) = R(z) - R(\overline{z})$ with $z = \lambda + i\kappa$, $\kappa > 0$, and

$$(\rho(\lambda)\varphi)(x, y) = \frac{2}{\pi}\sum_{k=1}^{n} \sin ky \int_0^\pi \sin ky'((r_k(\lambda + i0) - r_k(\lambda - i0))\varphi)(x, y')dy',$$

for $\varphi \in C_0^\infty(\Omega)$. Therefore, by Lemma 3.2,

$$\lim_{\kappa\downarrow 0}\langle(\rho(z) - \rho(\lambda))Y_m u, Y_m v\rangle_{L^2(\mathbb{R}^N)} = 0$$

holds for any $u, v \in L^2(\mathbb{R}^N)$. Let $f \in \mathcal{H}$. Using

$$\frac{d}{d\lambda}\|E(\lambda)\sqrt{B} f\|^2 = \frac{1}{2\pi i}\langle\{(A_0 - (\lambda + i0))^{-1} - (A_0 - (\lambda - i0))^{-1}\}\sqrt{B} f, \sqrt{B} f\rangle$$

and

$$\langle\{(A_0 - (\lambda + i0))^{-1} - (A_0 - (\lambda - i0))^{-1}\}\sqrt{B} f, \sqrt{B} f\rangle = \lambda\langle\rho(\lambda)\sqrt{b} f_2, \sqrt{b} f_2\rangle_{L^2(\Omega)},$$

we have

$$\frac{d}{d\lambda}\|E(\lambda)\sqrt{B} f\|^2 = \frac{1}{2\pi i}\lambda\langle\sqrt{b}\rho(\lambda)\sqrt{b} f_2, f_2\rangle_{L^2(\Omega)}. \tag{3.3}$$

To show (1) and (2), we estimate $\|\sqrt{b}\rho(\lambda)\sqrt{b}\|^2_{\mathcal{B}(L^2(\Omega))}$. Applying Parseval's identity to the Fourier series, the Schwarz inequality, and (1.3), we have

$$\|\sqrt{b}\rho(\lambda)\sqrt{b}\|^2_{\mathcal{B}(L^2(\Omega))} \le C\sum_{k=1}^{n}\|Y_m(r_k(\lambda + i0) - r_k(\lambda - i0))Y_m\|^2_{\mathcal{B}(L^2(\mathbb{R}^N))}$$

for some positive constant C. Hence, we only have to estimate the right-hand side of the above inequality. By (3.2), there exists a positive constant $\tilde{C}_1$, independent of λ, such that

$$\begin{aligned}
&\sum_{k=1}^{n} \|Y_m(r_k(\lambda+i0)-r_k(\lambda-i0))Y_m\|^2_{\mathcal{B}(L^2(\mathbb{R}^N))}\\
&= \sum_{k=1}^{n-1} \|Y_m(r_k(\lambda+i0)-r_k(\lambda-i0))Y_m\|^2_{\mathcal{B}(L^2(\mathbb{R}^N))}\\
&\quad + \|Y_m(r_n(\lambda+i0)-r_n(\lambda-i0))Y_m\|^2_{\mathcal{B}(L^2(\mathbb{R}^N))}\\
&\le \tilde{C}_1(1+|\lambda^2-n^2|^{-1}).
\end{aligned}$$

for any $\lambda \in (-n-\varepsilon,-n)\cup(n,n+\varepsilon)$.

For any $\lambda \in (-n-1,-n-\varepsilon)\cup(n+\varepsilon,n+1)$, there exists a positive constant $\tilde{C}_2$, independent of λ, such that

$$\sum_{k=1}^{n} \|Y_m(r_k(\lambda+i0)-r_k(\lambda-i0))Y_m\|^2_{\mathcal{B}(L^2(\mathbb{R}^N))} \le \tilde{C}_2.$$

Thus,

$$\|\sqrt{b}\rho(\lambda)\sqrt{b}\|_{\mathcal{B}(L^2(\Omega))} \le \begin{cases} \tilde{C}_1|\lambda^2-n^2|^{-1/2} & \text{if } n<|\lambda|<n+\varepsilon,\\ \tilde{C}_2 & \text{if } n+\varepsilon<|\lambda|<n+1. \end{cases}$$

This together with (3.3) prove (1) and (2). $\square$

Proof of (A3). From Proposition 3.3, we may put $a_n = n$. We take $\varepsilon \in (0,1)$. Substitute $z=\mu+i\kappa$ for $\kappa>0$ and $K_\varepsilon(n)=\sqrt{B}F_n(A_0)E_{n,n+\varepsilon}(A_0)$. It follows from the well-known formula

$$\begin{aligned}
&\operatorname{Im}\langle (A_0-z)^{-1}K_\varepsilon(n)^*f, K_\varepsilon(n)^*f\rangle\\
&= \Big(\int_{-n-\varepsilon}^{-n} + \int_{n}^{n+\varepsilon}\Big)\frac{\kappa}{(\lambda-\mu)^2+\kappa^2}|(\lambda-i)^{-2}(\lambda^2-n^2)|^2\frac{d}{d\lambda}\|E(\lambda)\sqrt{B}f\|^2 d\lambda.
\end{aligned}$$

Thus, using Proposition 3.3 (1) and the well-known identity

$$\int_{-\infty}^{\infty}\frac{\kappa}{(\lambda-\mu)^2+\kappa^2}d\lambda=\pi,$$

we have

$$\sup_{\operatorname{Im} z\neq 0, f\in\mathcal{H}} |\operatorname{Im}\langle (A_0-z)^{-1}K_\varepsilon(n)^*f, K_\varepsilon(n)^*f\rangle| \le C\|f\|^2$$

for some $C>0$. This implies that $K_\varepsilon(n)=\sqrt{B}F_n(A_0)E_{n,n+\varepsilon}(A_0)$ is A_0-smooth (cf. Kato [6] or Reed-Simon [15]).

By Proposition 3.3 (2), we can show that $\sqrt{B}E_{n+\varepsilon,n+1}(A_0)$ is A_0-smooth in the same manner as that mentioned above. Since $F_n(A_0)$ is bounded, the operator $\sqrt{B}F_n(A_0)E_{n+\varepsilon,n+1}(A_0)$ is also A_0-smooth. Thus $\sqrt{B}F_n(A_0)E_{n,n+1}(A_0)$ is A_0-smooth because of $E(\{\pm(n+\varepsilon)\})=0$ ($\sigma_p(A_0)=\emptyset$). $\square$

Remark 3.4. From [5, Proposition 3.5] and [17, Lemma 5], the following statements are obtained.

(1) Let $N \geq 3$ and $s \geq 1$. Then for any $\lambda \in (-n-1, -n) \cup (n, n+1)$, there exists a positive constant C, independent of λ, such that

$$\|Z_s r_n(\lambda \pm i0) Z_s\|_{\mathcal{B}(L^2(\mathbb{R}^N))} \leq C,$$

where $Z_s = (1+|x|^2)^{-s/2}$.

(2) Let $s > 1$. Then for any $\lambda \in (-n-1, -n) \cup (n, n+1)$, there exists a positive constant C, independent of λ, such that

$$\|Z_s\{r_n(\lambda + i0) - r_n(\lambda - i0)\} Z_s\|_{\mathcal{B}(L^2(\mathbb{R}^2))} \leq C.$$

Thus, assuming (1.4) with $\delta \geq 1$ (if $N \geq 3$) and $\delta > 1$ (if $N = 2$), we need not insert $F_n(A_0)$ in assumption (A3).

Proof. We omit the proof of (1), refer to [5], and provide a brief sketch of the proof of (2). Put

$$E_0(\sqrt{\lambda^2 - n^2})(x', x) = \frac{i}{4} H_0^+(\sqrt{\lambda^2 - n^2}|x - x'|) - \frac{i}{4} H_0^-(\sqrt{\lambda^2 - n^2}|x - x'|) - \frac{i}{2},$$

where $H_0^\pm$ are the Hankel functions of order zero with $H_0^- = \overline{H_0^+}$. Let $g \in C_0^\infty(\mathbb{R}^2)$. By [17, Lemma 5], for any $\varepsilon > 0$, there exists a $C > 0$ such that

$$|E_0(\sqrt{\lambda^2 - k^2})(x', x)| \leq C(\lambda^2 - k^2)^{\varepsilon/2} |x' - x|^\varepsilon.$$

In addition, using

$$((r_k(\lambda + i0) - r_k(\lambda - i0))g)(x) = \int_{\mathbb{R}^2} \Big(\frac{i}{2} + E_0(\sqrt{\lambda^2 - k^2})(x', x)\Big) g(x') dx',$$

we obtain the desired estimate. $\square$

Remark 3.5. In the case of $N = 1$, for $n \in \mathbb{N}$, we provide a concrete example of $BE((n, n+1))$, which is not A_0-smooth. We put $b(x, y) = \chi_{(0,\pi)}(x)$ and $f = {}^t(0, \chi_{(0,\pi)}(x) \sin ny)$. Then we have

$$\sup_{\kappa \neq 0} |\operatorname{Im}\langle (A_0 - (n - i\kappa))^{-1} E((n, n+1))\sqrt{B} f, E((n, n+1))\sqrt{B} f\rangle| = \infty$$

Indeed, since Green's function of $r_n(z)$ is

$$\frac{i}{2\sqrt{z^2 - n^2}} e^{i\sqrt{z^2 - n^2}|x|},$$

where $\operatorname{Im}\sqrt{z^2 - n^2} > 0$. By (3.3), we have

$$\frac{d}{d\lambda}\|E(\lambda)\sqrt{B} f\|^2 = \frac{\pi^2}{2} \frac{\sin^2 \frac{\sqrt{\lambda^2 - n^2}}{2}\pi}{\big(\frac{\sqrt{\lambda^2 - n^2}}{2}\pi\big)^2} \frac{\lambda}{\sqrt{\lambda^2 - n^2}}.$$

Therefore, we can obtain the estimate

$$\begin{aligned}
&|2\operatorname{Im}\langle (A_0-(n-i\kappa))^{-1}E((n,n+1))\sqrt{B}f, E((n,n+1))\sqrt{B}f\rangle| \\
&= \big|\big\langle E((n,n+1))\sqrt{B}f, \\
&\quad \{(A_0-(n+i\kappa))^{-1}-(A_0-(n-i\kappa))^{-1}\}E((n,n+1))\sqrt{B}f\big\rangle\big| \\
&= \int_n^{n+1} \frac{\pi^2\kappa}{(\lambda-n)^2+\kappa^2} \frac{\sin^2\frac{\sqrt{\lambda^2-n^2}}{2}\pi}{\big(\frac{\sqrt{\lambda^2-n^2}}{2}\pi\big)^2} \frac{\lambda}{\sqrt{\lambda^2-n^2}} d\lambda \\
&\ge Cn\int_n^{n+1} \frac{\kappa}{(\lambda-n)^2+\kappa^2}\frac{1}{\sqrt{\lambda^2-n^2}}d\lambda
\end{aligned}$$

for some $C>0$. Integrating by parts and using Fatou's lemma, we have

$$\liminf_{\kappa\to 0}\int_n^{n+1}\frac{\kappa}{(\lambda-n)^2+\kappa^2}\frac{1}{\sqrt{\lambda^2-n^2}}d\lambda \ge \frac{\pi}{2\sqrt{2n+1}}+\int_n^{n+1}\frac{\pi\lambda}{2(\lambda^2-n^2)^{\frac{3}{2}}}d\lambda=\infty$$

Therefore, $\sqrt{B}E((n,n+1))$ is not A_0-smooth.

4. Proof of Lemma 3.2

Without loss of generality, we may assume $n=0$. Hence, we only have to prove the following lemma.

Lemma 4.1. *For every $\lambda\in\mathbb{R}\backslash\{0\}$, there exist limits*

$$r_0(\lambda\pm i0)\in\mathcal{B}(L^2_{Y_m^{-1}}(\mathbb{R}^N), L^2_{Y_m}(\mathbb{R}^N))$$

such that

$$\langle r_0(\lambda\pm i0)Y_m u, Y_m v\rangle_{L^2(\mathbb{R}^N)}=\lim_{\kappa\downarrow 0}\langle r_0(z)Y_m u, Y_m v\rangle_{L^2(\mathbb{R}^N)},$$

for any $u,v\in L^2(\mathbb{R}^N)$, where $z=\lambda\pm i\kappa$ with $\kappa>0$. Moreover, there exists a positive constant C such that

$$\|Y_m r_0(\lambda\pm i0)Y_m\|_{\mathcal{B}(L^2(\mathbb{R}^N))}\le C|\lambda|^{-1},$$

where C is independent of λ.

To prove Lemma 4.1, we define a Besov space (introduced by Agmon-Hörmander [2])

$$B_{1/2}(\mathbb{R}^N)=\big\{f(x):\|f\|_{B_{1/2}}=\sum_{j\ge 1}R_j^{1/2}\{\int_{D_j}|f(x)|^2dx\}^{1/2}<\infty\big\},$$

where $R_{-1}=0$, $R_j=2^{j-1}$ $(j=1,2,3\dots)$ and $D_j=\{x\in\mathbb{R}^N : R_{j-2}<|x|<R_{j-1}\}$. The dual space of $B_{1/2}(\mathbb{R}^N)$ with respect to $L^2(\mathbb{R}^N)$ is denoted by $B^*_{1/2}(\mathbb{R}^N)$.

The following result is well known (cf. Agmon [1, Theorems 3.1 and 3.2]).

Lemma 4.2. *For every $\lambda\in\mathbb{R}\backslash\{0\}$, there exist limits*

$$r_0(\lambda\pm i0)\in\mathcal{B}(B_{1/2}(\mathbb{R}^N), B^*_{1/2}(\mathbb{R}^N))$$

such that

$$\langle r_0(\lambda\pm i0)u, v\rangle_{L^2(\mathbb{R}^N)}=\lim_{\kappa\downarrow 0}\langle r_0(z)u, v\rangle_{L^2(\mathbb{R}^N)}$$

for any $u, v \in B^{1/2}(\mathbb{R}^N)$, where $z = \lambda \pm i\kappa$ with $\kappa > 0$. Moreover, there exists a positive constant C such that

$$\|r_0(\lambda \pm i0)\|_{\mathcal{B}(B_{1/2}(\mathbb{R}^N), B^*_{1/2}(\mathbb{R}^N))} \le C|\lambda|^{-1},$$

where C is independent of λ.

Remark 4.3. The proof of (2) for $N = 1$ was not mentioned in [1]. However, it is not difficult. The proof of (2), according to Isozaki [4, section 5.3], we can done as follows.

$$\begin{aligned} |\langle (\frac{d^2}{dx^2} - z^2)^{-1} u, v\rangle_{L^2(\mathbb{R})}| &= \Big| \iint_{\mathbb{R}^2} \frac{i}{2z} e^{iz|x-y|} u(y) dy \overline{v(x)} dx \Big| \\ &\le \frac{1}{2|z|} \int_{\mathbb{R}} |u(y)| dy \int_{\mathbb{R}} |v(x)| dx \\ &= \frac{1}{2|z|} \Big(\sum_{j=1}^{\infty} \int_{D_j} |u(y)| dy \Big) \Big(\sum_{j=1}^{\infty} \int_{D_j} |v(x)| dx \Big) \\ &\le \frac{1}{|z|} \|u\|_{B_{1/2}} \|v\|_{B_{1/2}}. \end{aligned}$$

Now Lemma 4.1 follows from Lemma 4.2 and the relation between $L^2_{Y_m} = L^2_{Y_m}(\mathbb{R}^N)$ and $B_{1/2} = B_{1/2}(\mathbb{R}^N)$ (cf. Roach-Zhang [16] ($m = 1$) and Nakazawa [11] ($m \ge 1$)):

$$L^2_{Y_0^{-1}} \subset L^2_{Y_1^{-1}} \subset \dots \subset L^2_{Y_m^{-1}} \subset B_{1/2} \subset L^2 \subset B^*_{1/2} \subset L^2_{Y_m} \subset \dots \subset L^2_{Y_1} \subset L^2_{Y_0}. \tag{4.1}$$

Proof. Since (4.1) follows from the duality and

$$L^2_{Y_m^{-1}} \subset L^2_{Y_{m+1}^{-1}} \subset B_{1/2}, \tag{4.2}$$

we only have to prove only (4.2). First, for any $k \in \mathbb{N} \cup \{0\}$ satisfying $k \le m - 1$ and an arbitrarily fixed positive number δ, we have:

$$\log^{[k]}(e_m + r) \le \frac{e_{m-k}}{e_{m-k-1}} \log^{[k]}(e_{m-1} + r), \tag{4.3}$$

$$[\log^{[m]}(e_m + r)]^{1+\delta} \le \frac{1+\delta}{\delta} [\log^{[m-1]}(e_{m-1} + r)]^{\delta}. \tag{4.4}$$

Indeed, substituting

$$f(r) = \frac{e_{m-k}}{e_{m-k-1}} \log^{[k]}(e_{m-1} + r) - \log^{[k]}(e_m + r)$$

and

$$g(r) = \frac{1+\delta}{\delta} [\log^{[m-1]}(e_{m-1} + r)]^{\delta} - [\log^{[m]}(e_m + r)]^{1+\delta},$$

we can easily verify that $f'(r), g'(r) \ge 0$ and $f(0) = 0, g(0) = 1/\delta > 0$. Therefore, (4.3) and (4.4) hold. Hence, there exists a positive number $M_m \ge \{e_m(1 + \delta)\}/\delta$ such that

$$\begin{aligned} &\Big[\prod_{k=0}^{m-1} \log^{[k]}(e_m + r) \Big] \Big[\log^{[m]}(e_m + r) \Big]^{1+\delta} \\ &\le M_m \begin{cases} (1 + r)^{1+\delta} & \text{if } m = 1, \\ [\prod_{k=0}^{m-2} \log^{[k]}(e_{m-1} + r)][\log^{[m-1]}(e_{m-1} + r)]^{1+\delta} & \text{if } m \ge 2. \end{cases} \end{aligned}$$

Thus, we obtain

$$L^2_{Y_m^{-1}} \subset L^2_{Y_{m+1}^{-1}}. \tag{4.5}$$

Put $\varphi(r) = \{\prod_{k=0}^m \log^{[k]}(e_m + r)\}^{1/2}\{\log^{[m]}(e_m + r)\}^{\delta/2}$. From the Schwarz inequality, for any $f \in C_0^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned}
\|f\|_{B_{1/2}} &= \sum_{j\geq 1} R_j^{1/2}\Big(\int_{D_j} \varphi(r)^{-2}\cdot|\varphi(r)f(x)|^2 dx\Big)^{1/2} \\
&\leq \big\{\sum_{j\geq 1} R_j\big(\max_{D_j}\varphi(r)^{-2}\big)\big\}^{1/2}\Big(\sum_{j\geq 1}\int_{D_j}|\varphi(r)f(x)|^2 dx\Big)^{1/2} \\
&= M(\varphi)\|f\|_\varphi,
\end{aligned}$$

where

$$M(\varphi) \equiv \big\{\sum_{j\geq 1} R_j\big(\max_{D_j}\varphi(r)^{-2}\big)\big\}^{1/2}.$$

Hence, we have

$$L^2_{Y_m^{-1}} \subset B_{1/2} \tag{4.6}$$

because

$$\begin{aligned}
M(\varphi)^2 &= \sum_{j\geq 1} R_j\varphi(R_{j-1})^{-2} \\
&= \varphi(0)^{-2} + 2^2\sum_{j\geq 2} 2^{j-3}\varphi(R_{j-1})^{-2} \\
&\leq \varphi(0)^{-2} + 2^2\int_0^\infty \varphi(r)^{-2} dr < \infty.
\end{aligned}$$

Thus, from (4.5) and (4.6) we obtain (4.2). $\square$

5. Total energy decay

In this section, we assume that the function $b(x,y)$ satisfies

$$b_0\Big(\prod_{k=0}^m \log^{[k]}(e_m + r)\Big)^{-1} \leq b(x,y) \leq b_1 \tag{5.1}$$

for some $b_0, b_1 > 0$ and $m \in \mathbb{N}\cup\{0\}$.

Under assumption (5.1), the operator $-iA$ defined in section 1 generates a contraction semi-group $V(t)(t \geq 0)$. Hence, we obtain the following theorem.

Theorem 5.1. *For any $f \in \mathcal{H}$, $\lim_{t\to\infty}\|V(t)f\| = 0$.*

The above theorem is an immediate consequence of the usual density argument and the following proposition.

Proposition 5.2. *Let ε satisfy $0 < \varepsilon \leq \min\{1, b_0/2\}$. Assume the initial data $f = {}^t(f_1, f_2) \in C_0^\infty(\Omega)\times C_0^\infty(\Omega)$. Then*

$$\|V(t)f\| \leq C_2\{\log^{[m]}(e_m + t)\}^{-\varepsilon/2}$$

for a positive constant $C_2 = C_2(f_1, f_2, b_0, b_1, \varepsilon) > 0$.

This result is proved using the same arguments as those used in [10, section 2]. Here, we provide a brief summary of the proof.

Let u be the unique solution of (1.1) with initial data $f = {}^t(f_1, f_2) \in C_0^\infty(\Omega) \times C_0^\infty(\Omega) \subset D(A)$. Let φ be a function defined by $\varphi(s) = \{\log^{[m]}(e_m + s)\}^\varepsilon$ $(0 < \varepsilon \leq 1)$. Multiplying both sides of (1.1) with $\partial_t\{\varphi(r+t)\overline{u}\}$, we obtain

$$\partial_t X(x,y,t) + \nabla \cdot Y(x,y,t) + Z(x,y,t) = 0, \tag{5.2}$$

where

$$\begin{gathered} X(x,y,t) = \frac{\varphi}{2}(|\partial_t u|^2 + |\nabla u|^2) + \frac{\varphi' b - \varphi''}{2}|u|^2 + \varphi' \operatorname{Re}(\partial_t u \overline{u}), \\ Y(x,y,t) = -\{\varphi \operatorname{Re}(\nabla u \overline{\partial_t u}) + \varphi' \operatorname{Re}(\nabla u \overline{u})\}, \\ \begin{aligned} Z(x,y,t) &= \big(\varphi b - \frac{3\varphi'}{2}\big)|\partial_t u|^2 + \frac{\varphi'}{2}|\nabla u|^2 + \varphi' \operatorname{Re}(\partial_r u \overline{\partial_t u}) + \varphi'' \operatorname{Re}(\partial_r u \overline{u}) \\ &\quad + \frac{\varphi''' - \varphi'' b}{2}|u|^2. \end{aligned} \end{gathered}$$

To prove Proposition 5.2, we state the following four lemmas.

Lemma 5.3. *Let ε satisfy $0 < \varepsilon \leq \min\{1, b_0/2\}$. Then*

$$Z(x,y,t) \geq -\partial_t\big(\frac{\varphi''|u|^2}{2}\big).$$

Proof. It holds that

$$Z(x,y,t) \geq (b\varphi - 2\varphi')|\partial_t u|^2 + \frac{1}{2}\big(2\varphi''' - b\varphi'' - \frac{\varphi''^2}{\varphi'}\big)|u|^2 - \partial_t\big(\frac{\varphi''|u|^2}{2}\big).$$

From the assumption of $b(x,y)$ and the definition of φ, we can easily verify that $b\varphi - 2\varphi'$ and $2\varphi''' - \frac{\varphi''^2}{\varphi'}$ are non-negative if ε is chosen as $\frac{b_0}{2} \geq \varepsilon$. This provides the conclusion. $\square$

Lemma 5.4. *Let ε satisfy $0 < \varepsilon \leq \min\{1, b_0/2\}$. Then*

$$\int_\Omega \Big(X - \frac{\varphi''|u|^2}{2}\big|_{t=\tau}\Big)\,dx\,dy \leq \int_\Omega \Big(X - \frac{\varphi''|u|^2}{2}\big|_{t=0}\Big)\,dx\,dy.$$

Proof. From (5.2) and Lemma 5.3, we obtain

$$\partial_t\Big(X(x,y,t) - \frac{\varphi''|u|^2}{2}\Big) + \nabla \cdot Y(x,y,t) \leq 0. \tag{5.3}$$

Since $V(t)f = {}^t(u(t), \partial_t u(t)) \in D(A)$, we have $u(x,0,t) = \partial_t u(x,0,t) = u(x,\pi,t) = \partial_t u(x,\pi,t) = 0$ in the trace sense. Thus, integration of (5.3) by parts over $\Omega \times [0,\tau]$ provides the conclusion. $\square$

Lemma 5.5. *Let ε satisfy $0 < \varepsilon \leq \min\{1, b_0/2\}$ and μ satisfy $1/2 \leq \mu < 1$. Then*

$$\int_\Omega \Big(X - \frac{\varphi''|u|^2}{2}\big|_{t=\tau}\Big)\,dx\,dy \geq \frac{(1-\mu)}{2}\{\log^{[m]}(e_m + \tau)\}^\varepsilon \|V(\tau)f\|^2.$$

Proof. Using (5.1) and the definition of $\varphi(r+t)$, we find

$$X - \frac{\varphi''|u|^2}{2}\big|_{t=\tau} \geq \frac{(1-\mu)\varphi}{2}(|\partial_t u|^2 + |\nabla u|^2).$$

For more details, refer the reader to [10, Lemmas 2.1 and 2.2]. This provides the conclusion. $\square$

The proof of the following lemma is obvious and is omitted.

Lemma 5.6. *There exists a positive constant $C_1 = C_1(b_1, \varepsilon)$ such that*

$$\begin{aligned}
&\int_\Omega \Big(X - \frac{\varphi''|u|^2}{2}\big|_{t=0} \Big)\, dx\, dy \\
&\le C_1 \Big(\int_\Omega \{\log^{[m]}(e_m + r)\}^\varepsilon \{|\nabla f_1(x,y)|^2 + |f_2(x,y)|^2\}\, dx\, dy + \|f_1\|^2_{L^2(\Omega)} \Big).
\end{aligned}$$

Then Proposition 5.2 follows from Lemmas 5.4, 5.5, and 5.6.

References

[1] S. Agmon; *A representation theorem for solutions of the Helmholtz equation and resolvent estimates for the Laplacian*, In P. H. Rabinowitz and E. Zehnder eds., Analysis, et cetera, Academic Press, Boston, 1990, pp. 39–76.

[2] S. Agmon and L. Hörmander; *Asymptotic properties of solutions of differential equations with simple characteristics*, J. Analyse. Math.,**30**, 1976, pp. 1–38.

[3] M. Ben-Artzi; *On spectral properties of acoustic propagator in a layered band*, J. Differential Equations, **136**, 1997, pp. 115–135.

[4] H. Isozaki; *Many Body Shrödinger Equations*, (in Japanese), Springer, Tokyo, 2004.

[5] M. Kadowaki; *Resolvent estimates and scattering states for dissipative systems*, Publ. RIMS Kyoto Univ., **38**, 2002, pp. 191–209.

[6] T. Kato; *Wave operators and similarity for some non–selfadjoint operators*, Math. Ann. **162**, 1966, pp. 258–279.

[7] S. T. Kuroda; *An Introduction to Scattering Theory*, Lecture Note Series N^o 51, Matematisk Institut, Aarhus University, 1980.

[8] K. Mochizuki; *Scattering theory for wave equations with dissipative terms*, Publ. RIMS Kyoto Univ., **12**, 1976, pp. 383–390.

[9] K. Mochizuki; *Scattering Theory for Wave Equations*, (in Japanese), Kinokuniya, Tokyo, 1984.

[10] K. Mochizuki and H. Nakazawa; *Energy decay and asymptotic behavior of solutions to the wave equations with linear dissipation*, Publ. RIMS Kyoto Univ. **32**, 1996, pp. 401–414.

[11] H. Nakazawa; *On wave equations with dissipations*, Proceedings of the 4th International Conference AMADE 2006, 13-19th of September 2006, Minsk, Belarus, Volume 3: Differential Equations, 2006, pp. 102–110

[12] V. Petkov; *Scattering Theory for Hyperbolic Operators*, North Holland, Amsterdam, New York, Oxford, Tokyo, 1989.

[13] A. G. Ramm and P. Werner; *On the limit amplitude principle for a layer*, J. für die reine und angewandte Mathematik, **360**, 1986, pp. 19–46.

[14] M. Reed and B. Simon; *Methods of Modern Mathematical Physics, II. Fourier Analysis, Self-Adjointness*, Academic Press, New York, San Francisco, London, 1975

[15] M. Reed and B. Simon; *Methods of Modern Mathematical Physics, IV*, Analysis of Operators, Academic Press, New York, San Francisco, London, 1978.

[16] G. F. Roach and B. Zhang; *On Sommerfeld radiation conditions for the diffraction problem with two unbounded media*, Proc. Roy. Soc. Edinburgh, **121A**, 1992, pp. 149–161.

[17] W. Schlag; *Dispersive estimates for Schrödinger operators in dimension two*, Comm. Math. Phys. **257**, 2005, pp. 87–117.

[18] B. Simon; *Phase space analysis of simple scattering systems: extensions of some work of Enss*, Duke. Math. J. **46**, 1979, pp. 119–168.

[19] A. G. Sveshnikov; *The principle of radiation*, Dokl. Akad. Nauk SSSR, **73**, 1950, pp. 917–920.

[20] A. G. Sveshnikov; *The limiting absorption principle for a wave guide*, Dokl. Akad. Nauk SSSR, **80**, 1951, pp. 345–347.

[21] P. Werner; *Ein Resonanzphänomen in der Theorie akustischer und elektromagnetischer Wellen*, Math. Meth. Appl. Sci., **6**, 1984, pp. 104–128.

[22] R. Weder; *The limiting absorption principle at thresholds*, Journal de Math. Pures et Appl., **67**, 1988, pp. 313–338.

[23] R. Weder; *Spectral and Scattering Theory for Wave Propagation in Perturbed Stratified Media*, Applied Mathematical Sciences **87**, Springer-Verlag, New York, Berlin, Heidelberg, 1991.

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