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THE LEGENDRE EQUATION AND ITS SELF-ADJOINT OPERATORS

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Abstract. The Legendre equation has interior singularities at -1 and $+1$. The celebrated classical Legendre polynomials are the eigenfunctions of a particular self-adjoint operator in $L^2(-1,1)$. We characterize all self-adjoint Legendre operators in $L^2(-1,1)$ as well as those in $L^2(-\infty,-1)$ and in $L^2(1,\infty)$ and discuss their spectral properties. Then, using the 'three-interval theory', we find all self-adjoint Legendre operators in $L^2(-\infty,\infty)$. These include operators which are not direct sums of operators from the three separate intervals and thus are determined by interactions through the singularities at -1 and $+1$.

1. Introduction

The Legendre equation

$$-(py')' = \lambda y, \quad p(t) = 1 - t^2, \tag{1.1}$$

is one of the simplest singular Sturm-Liouville differential equations. Its potential function q is zero, its weight function w is the constant 1, and its leading coefficient p is a simple quadratic. It has regular singularities at the points ± 1 and at $\pm\infty$. The singularities at ± 1 are due to the fact that $1/p$ is not Lebesgue integrable in left and right neighborhoods of these points; the singularities at $-\infty$ and at $+\infty$ are due to the fact that the weight function $w(t) = 1$ is not integrable at these points.

The equation (1.1) and its associated self-adjoint operators exhibit a surprisingly wide variety of interesting phenomena. In this paper we survey these important points. Of course, one of the main reasons this equation is important in many areas of pure and applied mathematics stems from the fact that it has interesting solutions. Indeed, the Legendre polynomials $\{P_n\}_{n=0}^{\infty}$ form a complete orthogonal set of functions in $L^2(0,\infty)$ and, for $n \in \mathbb{N}_0$, $y = P_n(t)$ is a solution of (1.1) when $\lambda = \lambda_n = n(n+1)$. Properties of the Legendre polynomials can be found in several textbooks including the remarkable book of [18]. Most of our results can be inferred directly from known results scattered widely in the literature, others require some additional work. A few are new. It is remarkable that one can find some new results

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on this equation which has such a voluminous literature and a history of more than 200 years.

The equation (1.1) and its associated self-adjoint operators are studied on each of the three intervals

$$J_1 = (-\infty, -1), \quad J_2 = (-1, 1), \quad J_3 = (1, \infty), \tag{1.2}$$

and on the whole real line $J_4 = \mathbb{R} = (-\infty, \infty)$. The latter is based on some minor modifications of the 'two-interval' theory developed by Everitt and Zettl [8] in which the equation (1.1) is considered on the whole line $\mathbb{R}$ with singularities at the interior points -1 and $+1$. For each interval the corresponding operator setting is the Hilbert space $H_i = L^2(J_i)$, $i = 1, 2, 3, 4$ consisting of complex valued functions $f \in AC_{\rm loc}(J_i)$ such that

$$\int_{J_i} |f|^2 < \infty. \tag{1.3}$$

Since $p(t)$ is negative when $|t| > 1$ we let

$$r(t) = t^2 - 1. \tag{1.4}$$

Then (1.1) is equivalent to

$$-(ry')' = \xi y, \quad \xi = -\lambda. \tag{1.5}$$

Note that $r(t) > 0$ for $t \in J_1 \cup J_3$ so that (1.5) has the usual Sturm-Liouville form with positive leading coefficient r.

Before proceeding to the details of the study of the Legendre equation on each of the three intervals J_i, $i = 1, 2, 3$ and on the whole line $\mathbb{R}$ we make some general observations. (We omit the study of the two-interval Legendre problems on any two of the three intervals J_1, J_2, J_3 since this is similar to the three-interval case. The two-interval theory could also be applied to the two intervals $\mathbb{R}$ and J_i for any i.)

For $\lambda = \xi = 0$ two linearly independent solutions are given by

$$u(t) = 1, \quad v(t) = \frac{-1}{2} \ln\left(\left|\frac{1-t}{t+1}\right|\right) \tag{1.6}$$

Since these two functions u, v play an important role below we make some observations about them.

Observe that for all $t \in \mathbb{R}$, $t \neq \pm 1$, we have

$$(pv')(t) = +1. \tag{1.7}$$

Thus the quasi derivative (pv') can be continuously extended so that it is well defined and continuous on the whole real line $\mathbb{R}$ including the two singular points -1 and $+1$. It is interesting to observe that u, (pu') and (the extended) (pv') can be defined to be continuous on $\mathbb{R}$ and only v blows up at the singular points -1 and $+1$.

These simple observations about solutions of (1.1) when $\lambda = 0$ extend in a natural way to solutions for all $\lambda \in \mathbb{C}$ and are given in the next theorem whose proof may be of more interest than the theorem. It is based on a 'system regularization' of (1.1) using the above functions u, v.

The standard system formulation of (1.1) has the form

$$Y' = (P - \lambda W)Y \quad \text{on } (-1, 1), \tag{1.8}$$

where

$$Y = \begin{pmatrix} y \\ py' \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1/p \\ 0 & 0 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{1.9}$$

Let u and v be given by (1.6) and let

$$U = \begin{pmatrix} u & v \\ pu' & pv' \end{pmatrix} = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}. \tag{1.10}$$

Note that $\det U(t) = 1$, for $t \in J_2 = (-1, 1)$, and set

$$Z = U^{-1}Y. \tag{1.11}$$

Then

$$\begin{aligned} Z' &= (U^{-1})'Y + U^{-1}Y' = -U^{-1}U'U^{-1}Y + (U^{-1})(P - \lambda V)Y \\ &= -U^{-1}U'Z + (U^{-1})(P - \lambda W)UZ \\ &= -U^{-1}(PU)Z + U^{-1}(DU)Z - \lambda(U^{-1}WU)Z = -\lambda(U^{-1}WU)Z. \end{aligned}$$

Letting $G = (U^{-1}WU)$ we may conclude that

$$Z' = -\lambda GZ. \tag{1.12}$$

Observe that

$$G = U^{-1}WU = \begin{pmatrix} -v & -v^2 \\ 1 & v \end{pmatrix}. \tag{1.13}$$

Definition 1.1. We call (1.12) a ‘regularized’ Legendre system.

This definition is justified by the next theorem.

Theorem 1.2. *Let $\lambda \in \mathbb{C}$ and let G be given by* (1.13).

(1) *Every component of G is in $L^1(-1, 1)$ and therefore* (1.12) *is a* regular *system.*

(2) *For any $c_1, c_2 \in \mathbb{C}$ the initial value problem*

$$Z' = -\lambda GZ, \quad Z(-1) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \tag{1.14}$$

has a unique solution Z defined on the closed interval $[-1, 1]$.

(3) *If $Y = \begin{pmatrix} y(t,\lambda) \\ (py')(t,\lambda) \end{pmatrix}$ is a solution of* (1.8) *and $Z = U^{-1}Y = \begin{pmatrix} z_1(t,\lambda) \\ z_2(t,\lambda) \end{pmatrix}$, then Z is a solution of* (1.12) *and for all $t \in (-1, 1)$ we have*

$$y(t,\lambda) = uz_1(t,\lambda) + v(t)z_2(t,\lambda) = z_1(t,\lambda) + v(t)z_2(t,\lambda) \tag{1.15}$$

$$(py')(t,\lambda) = (pu')z_1(t,\lambda) + (pv')(t)z_2(t,\lambda) = -z_2(t,\lambda) \tag{1.16}$$

(4) *For every solution $y(t,\lambda)$ of the singular scalar Legendre equation* (1.1) *the quasi-derivative $(py')(t,\lambda)$ is continuous on the compact interval $[-1, 1]$. More specifically we have*

$$\lim_{t\to -1^+} (py')(t,\lambda) = -z_2(-1,\lambda), \quad \lim_{t\to 1^-} (py')(t,\lambda) = -z_2(1,\lambda). \tag{1.17}$$

Thus the quasi-derivative is a continuous function on the closed interval $[-1, 1]$ for every $\lambda \in \mathbb{C}$.

(5) *Let $y(t,\lambda)$ be given by* (1.15)*. If $z_2(1,\lambda) \neq 0$ then $y(t,\lambda)$ is unbounded at 1; if $z_2(-1,\lambda) \neq 0$ then $y(t,\lambda)$ is unbounded at -1.*

(6) *Fix* $t \in [-1,1]$, *let* $c_1, c_2 \in \mathbb{C}$. *If* $Z = \begin{pmatrix} z_1(t,\lambda) \\ z_2(t,\lambda) \end{pmatrix}$ *is the solution of* (1.12) *determined by the initial conditions* $z_1(-1,\lambda) = c_1,\ z_2(-1,\lambda) = c_2$, *then* $z_i(t,\lambda)$ *is an entire function of* λ, $i = 1,2$. *Similarly for the initial condition* $z_1(1,\lambda) = c_1,\ z_2(1,\lambda) = c_2$.

(7) *For each* $\lambda \in \mathbb{C}$ *there is a nontrivial solution which is bounded in a (two sided) neighborhood of* 1*; and there is a (generally different) nontrivial solution which is bounded in a (two sided) neighborhood of* -1.

(8) *A nontrivial solution* $y(t,\lambda)$ *of the singular scalar Legendre equation* (1.1) *is bounded at* 1 *if and only if* $z_2(1,\lambda) = 0$*; a nontrivial solution* $y(t,\lambda)$ *of the singular scalar Legendre equation* (1.1) *is bounded at* -1 *if and only if* $z_2(-1,\lambda) = 0$.

Proof. Part (1) follows from (1.13), (2) is a direct consequence of (1) and the theory of regular systems, $Y = UZ$ implies (3)$\Longrightarrow$(4) and (5); (6) follows from (2) and the basic theory of regular systems. For (7) determine solutions $y_1(t,\lambda)$, $y_{-1}(t,\lambda)$ by applying the Frobenius method to obtain power series solutions of (1.1) in the form: (see [2], page 5 with different notations)

$$y_1(t,\lambda) = 1 + \sum_{n=1}^{\infty} a_n(\lambda)(t-1)^n, \quad |t-1| < 2; \tag{1.18}$$

$$y_{-1}(t,\lambda) = 1 + \sum_{n=1}^{\infty} b_n(\lambda)(t+1)^n, \quad |t+1| < 2; \tag{1.19}$$

Item (8) follows from (1.15) that if $z_2(1,\lambda) \neq 0$, then $y(t,\lambda)$ is not bounded at 1. Suppose $z_2(1,\lambda) = 0$. If the corresponding $y(t,\lambda)$ is not bounded at 1 then there are two linearly unbounded solutions at 1 and hence all nontrivial solutions are unbounded at 1. This contradiction establishes (8) and completes the proof of the theorem. □

Remark 1.3. From Theorem (1.2) we see that, *for every* $\lambda \in \mathbb{C}$, the equation (1.1) has a solution y_1 which is bounded at 1 and has a solution y_{-1} which is bounded at -1.

It is well known that for $\lambda_n = n(n+1) : n \in \mathbb{N}_0 = \{0,1,2,\dots\}$ the Legendre polynomials P_n (see 1.6 below) are solutions on $(-1,1)$ and hence are bounded at -1 and at $+1$.

For later reference we introduce the primary fundamental matrix of the system (1.12).

Definition 1.4. Fix $\lambda \in \mathbb{C}$. Let $\Phi(\cdot,\cdot,\lambda)$ be the primary fundamental matrix of (1.12); i.e. for each $s \in [-1,1]$, $\Phi(\cdot,s,\lambda)$ is the unique matrix solution of the initial value problem:

$$\Phi(s,s,\lambda) = I \tag{1.20}$$

where I is the 2×2 identity matrix. Since (1.12) is regular, $\Phi(t,s,\lambda)$ is defined for all $t,s \in [-1,1]$ and, for each fixed t,s, $\Phi(t,s,\lambda)$ is an entire function of λ.

We now consider two point boundary conditions for (1.12); later we will relate these to singular boundary conditions for (1.1). Let $A, B \in M_2(\mathbb{C})$, the set of 2×2 complex matrices, and consider the boundary value problem

$$Z' = -\lambda G Z, \quad AZ(-1) + BZ(1) = 0. \tag{1.21}$$

Lemma 1.5. *A complex number $-\lambda$ is an eigenvalue of* (1.21) *if and only if*

$$\Delta(\lambda) = \det[A + B\Phi(1, -1, -\lambda)] = 0. \tag{1.22}$$

Furthermore, a complex number $-\lambda$ is an eigenvalue of geometric multiplicity two if and only if

$$A + B\Phi(1, -1, -\lambda) = 0. \tag{1.23}$$

Proof. Note that a solution for the initial condition $Z(-1) = C$ is given by

$$Z(t) = \Phi(t, -1, -\lambda)C, \quad t \in [-1, 1]. \tag{1.24}$$

The boundary value problem (1.21) has a nontrivial solution for Z if and only if the algebraic system

$$[A + B\Phi(1, -1, -\lambda)]Z(-1) = 0 \tag{1.25}$$

has a nontrivial solution for $Z(-1)$.

To prove the furthermore part, observe that two linearly independent solutions of the algebraic system (1.25) for $Z(-1)$ yield two linearly independent solutions $Z(t)$ of the differential system and conversely. □

Given any $\lambda \in \mathbb{R}$ and any solutions y, z of (1.1) the Lagrange form $[y, z](t)$ is defined by

$$[y, z](t) = y(t)(p\overline{z'})(t) - \overline{z}(t)(py')(t).$$

So, in particular, we have

$$[u, v](t) = +1, \quad [v, u](t) = -1, \quad [y, u](t) = -(py')(t), \quad t \in \mathbb{R},$$
$$[y, v](t) = y(t) - v(t)(py')(t), \quad t \in \mathbb{R},\ t \neq \pm 1.$$

We will see below that, although v blows up at ± 1, the form $[y, v](t)$ is well defined at -1 and $+1$ since the limits

$$\lim_{t \to -1} [y, v](t), \quad \lim_{t \to +1} [y, v](t)$$

exist and are finite from both sides. This for any solution y of (1.1) for any $\lambda \in \mathbb{R}$. Note that, since v blows up at 1, this means that y must blow up at 1 except, possibly when $(py')(1) = 0$. We will expand on this observation below in the section on 'Regular Legendre' equations.

Now we make the following additional observations: For definitions of the technical terms used here, see [21].

Proposition 1.6. *The following results are valid:*

1. *Both equations* (1.1) *and* (1.5) *are singular at $-\infty$, $+\infty$ and at -1, $+1$, from both sides.*
2. *In the L^2 theory the endpoints $-\infty$ and $+\infty$ are in the limit-point (LP) case, while -1^-, -1^+, 1^-, 1^+ are all in the limit-circle (LC) case. In particular both solutions u, v are in $L^2(-1, 1)$. Here we use the notation -1^- to indicate that the equation is studied on an interval which has -1 as its right endpoint. Similarly for -1^+, 1^-, 1^+.*
3. *For every $\lambda \in \mathbb{R}$ the equation* (1.1) *has a solution which is bounded at -1 and another solution which blows up logarithmically at -1. Similarly for $+1$.*

(4) *When $\lambda = 0$, the constant function u is a principal solution at each of the endpoints -1^-, -1^+, 1^-, 1^+ but u is a nonprincipal solution at both endpoints $-\infty$ and $+\infty$. On the other hand, v is a nonprincipal solution at -1^-, -1^+, 1^-, 1^+ but is the principal solution at $-\infty$ and $+\infty$. Recall that, at each endpoint, the principal solution is unique up to constant multiples but a nonprincipal solution is never unique since the sum of a principal and a nonprincipal solution is nonprincipal.*

(5) *On the interval $J_2 = (-1, 1)$ the equation (1.1) is nonoscillatory at -1^-, -1^+, 1^-, 1^+ for every real λ.*

(6) *On the interval $J_3 = (1, \infty)$ the equation (1.5) is oscillatory at ∞ for every $\lambda > -1/4$ and nonoscillatory at ∞ for every $\lambda < -1/4$.*

(7) *On the interval $J_3 = (1, \infty)$ the equation (1.1) is nonoscillatory at ∞ for every $\lambda < 1/4$ and oscillatory at ∞ for every $\lambda > 1/4$.*

(8) *On the interval $J_1 = (-\infty, -1)$ the equation (1.1) is nonoscillatory at $-\infty$ for every $\lambda < +1/4$ and oscillatory at $-\infty$ for every $\lambda > +1/4$.*

(9) *On the interval $J_1 = (-\infty, -1)$ the equation (1.5) is oscillatory at $-\infty$ for every $\lambda > -1/4$ and nonoscillatory at $-\infty$ for every $\lambda < -1/4$.*

(10) *The spectrum of the classical Sturm-Liouville problem (SLP) consisting of equation (1.1) on $(-1, 1)$ with the boundary condition*

$$(py')(-1) = 0 = (py')(+1)$$

is discrete and is given by

$$\sigma(S_F) = \{n(n+1) : n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}\}.$$

Here S_F denotes the classical Legendre operator; i.e., the self-adjoint operator in the Hilbert space $L^2(-1, 1)$ which represents the Sturm-Liouville problem (SLP) (1.1), (1.11). The notation S_F is used to indicate that this is the celebrated Friedrichs extension. It's orthonormal eigenfunctions are the Legendre polynomials $\{P_n : n \in \mathbb{N}_0\}$ given by:

$$P_n(t) = \sqrt{\frac{2n+1}{2}} \sum_{j=0}^{[n/2]} \frac{(-1)^j (2n-2j)!}{2^n j!(n-j)!(n-2j)!} t^{n-2j} \quad (n \in \mathbb{N}_0)$$

where $[n/2]$ denotes the greatest integer $\le n/2$.

The special (ausgezeichnete) operator S_F is one of an uncountable number of self-adjoint realizations of the equation (1.1) on $(-1, 1)$ in the Hilbert space $H = L^2(-1, 1)$. The singular boundary conditions determining the other self-adjoint realizations will be given explicitly below.

(11) *The essential spectrum of every self-adjoint realization of equation (1.1) in the Hilbert space $L^2(1, \infty)$ and of (1.1) in the Hilbert space $L^2(-\infty, -1)$ is given by*

$$\sigma_e = (-\infty, -1/4].$$

For each interval every self-adjoint realization is bounded above and has at most two eigenvalues. Each eigenvalue is $\ge -1/4$. The existence of $0, 1$ or 2 eigenvalues and their location depends on the boundary condition. There is no uniform bound for all self-adjoint realizations.

(12) *The essential spectrum of every self-adjoint realization of equation (1.5) in the Hilbert space $L^2(1, \infty)$ and of (1.5) in the Hilbert space $L^2(-\infty, -1)$ is*

given by

$$\sigma_e = [1/4, \infty).$$

For each interval every self-adjoint realization is bounded below and has at most two eigenvalues. There is no uniform bound for all self-adjoint realizations. Each eigenvalue is $\leq 1/4$. *The existence of* $0, 1$ *or* 2 *eigenvalues and their location depends on the boundary condition.*

Proof. Parts (1), (2), (4) are basic results in Sturm-Liouville theory [21]. The proof of (3) will be given below in the section on regular Legendre equations. For these and other basic facts mentioned below the reader is referred to the book "Sturm-Liouville Theory" [21]. Part (10) is the well known celebrated classical theory of the Legendre polynomials, see [15] for a characterization of the Friedrichs extension. In the other parts, the statements about oscillation, nonoscillation and the essential spectrum σ_e follow from the well known general fact that, when the leading coefficient is positive, the equation is oscillatory for all $\lambda > \inf \sigma_e$ and nonoscillatory for all $\lambda < \inf \sigma_e$. Thus $\inf \sigma_e$ is called the oscillation number of the equation. It is well known that the oscillation number of equation (1.5) on $(1, \infty)$ is $-1/4$. Since (1.5) is nonoscillatory at 1^+ for all $\lambda \in \mathbb{R}$ oscillation can occur only at ∞. The transformation $t \to -1$ shows that the same results hold for (1.5) on $(-\infty, -1)$. Since $\xi = -\lambda$ the above mentioned results hold for the standard Legendre equation (1.1) but with the sign reversed. To compute the essential spectrum on $(1, \infty)$ we first note that the endpoint 1 makes no contribution to the essential spectrum since it is limit-circle nonoscillatory. Note that $\int_2^\infty 1/\sqrt{r} = \infty$ and

$$\lim_{t\to\infty} \frac{1}{4}(r''(t) - \frac{1}{4}\frac{[r'(t)]^2}{r(t)}) = \lim_{t\to\infty} \frac{1}{4}(2 - \frac{1}{4}\frac{4t^2}{t^2-1}) = \frac{1}{4}.$$

From this and Theorem XIII.7.66 in Dunford and Schwartz [6], part (12) follows and part (11) follows from (12). Parts (6)-(10) follow from the fact that the starting point of the essential spectrum is the oscillation point of the equation; that is, the equation is oscillatory for all λ above the starting point and nonoscillatory for all λ below. (Note that there is a sign change correction needed in the statement of Theorem XIII.7.66 since $1 - t^2$ is negative when $t > 1$ and this theorem applies to a positive leading coefficient.) $\square$

Notation. $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex number fields respectively; $\mathbb{N}$ and $\mathbb{N}_0$ denote the positive and non-negative integers respectively; L denotes Lebesgue integration; $AC_{\text{loc}}(J)$ is the set of complex valued functions which are Lebesgue integrable on every compact subset of J; (a, b) and $[\alpha, \beta]$ represent open and compact intervals of $\mathbb{R}$, respectively; other notations are introduced in the sections below.

2. Regular Legendre Equations

In this section we construct *regular* Sturm-Liouville equations which are equivalent to the classical *singular equation* (1.1). This construction is based on a transformation used by Niessen and Zettl in [15]. We apply this construction to the Legendre problem on the interval $(-1, 1)$:

$$My = -(py') = \lambda y \quad \text{on } J_2 = (-1, 1), \quad p(t) = 1 - t^2, \quad -1 < t < 1. \tag{2.1}$$

This transformation depends on a modification of the function v given by (1.6). Note that v changes sign in $(-1,1)$ at 0 and we need a function which is positive on the entire interval $(-1,1)$ and is a nonprincipal solution at both endpoints.

This modification consists of using a multiple of v which is positive near each endpoint and changing the function v in the middle of J_2

$$v_m(t) = \begin{cases} \frac{-1}{2}\ln\left(\frac{1-t}{1+t}\right), & 1/2 \le t < 1 \\ m(t), & -1/2 \le t \le 1/2 \\ \frac{1}{2}\ln\left(\frac{1-t}{1+t}\right), & -1 \le t \le -1/2 \end{cases} \tag{2.2}$$

where the 'middle function' m is chosen so that the modified function v_m defined on $(-1,1)$ satisfies the following properties:

(1) $v_m(t) > 0$, $-1 < t < 1$.
(2) $v_m, (pv_m') \in AC_{\rm loc}(-1,1)$, $v_m, (pv_m') \in L^2(-1,1)$.
(3) v_m is a nonprincipal solution at both endpoints.

For later reference we note that

$$\begin{gathered} (pv_m')(t) = +1, \quad \frac{1}{2} \le t < 1, \\ (pv_m')(t) = -1, \quad -1 < t < \frac{-1}{2}, \\ [u, v_m](t) = u(t)(pv_m')(t) - v(t)(pu')(t) = (pv_m')(t) = 1, \quad \frac{1}{2} \le t < 1, \\ [u, v_m](t) = u(t)(pv_m')(t) - v(t)(pu')(t) = (pv_m')(t) = -1, \quad -1 < t < -\frac{1}{2}. \end{gathered}$$

Niessen and Zettl [15, Lemma 2.3 and Lemma 3.6], showed that such choices for m are possible in general. Although in the Legendre case studied here an explicit such m can be constructed we do not do so here since our focus is on boundary conditions at the endpoints which are independent of the choice of m.

Definition 2.1. Let M be given by (2.1). Define

$$P = v_m^2 p, \quad Q = v_m M v_m, \quad W = v_m^2, \quad \text{on } J_2 = (-1,1). \tag{2.3}$$

Consider the equation

$$Nz = -(Pz')' + Qz = \lambda W z, \quad \text{on } J_2 = (-1,1). \tag{2.4}$$

In (2.3), P denotes a scalar function; this notation should not be confused with P defined in (1.9) where P denotes a matrix.

Lemma 2.2. *Equation* (2.4) *is regular with $P > 0$ on J_2, $W > 0$ on J_2.*

Proof. The positivity of P and W are clear. To prove that (2.4) is regular on $(-1,1)$ we have to show that

$$\int_{-1}^{1} \frac{1}{P} < \infty, \quad \int_{-1}^{1} Q < \infty, \quad \int_{-1}^{1} W < \infty. \tag{2.5}$$

The third integral is finite since $v \in L^2(-1,1)$.

Since v_m is a nonprincipal solution at both endpoints, it follows from SL theory [21] that

$$\int_{-1}^{c} \frac{1}{pv_m^2} < \infty, \quad \int_{d}^{1} \frac{1}{pv_m^2} < \infty,$$

for some c, d, $-1 < c < d < 1$. By (2), $1/v_m^2$ is bounded on $[c, d]$ and therefore

$$\int_c^d \frac{1}{p} < \infty$$

so we can conclude that the first integral (2.5) is finite. The middle integral is finite since Mv_m is identically zero near each endpoint and $v_m, pv_m' \in AC_{\rm loc}(-1, 1)$. $\square$

Corollary 2.3. *Let $\lambda \in \mathbb{C}$. For every solution z of* (2.4)*, the limits*

$$\begin{gathered} z(-1) = \lim_{t \to -1^+} z(t), \quad z(1) = \lim_{t \to 1^-} z(t), \\ (Pz')(-1) = \lim_{t \to -1^+} (Pz')(t), \quad (Pz')(1) = \lim_{t \to 1^-} (Pz')(t) \end{gathered} \tag{2.6}$$

exist and are finite.

Proof. This follows directly from SL theory [21]; every solution and its quasi-derivative have finite limits at each regular endpoint. $\square$

We call equation (2.4) a 'regularized Legendre equation'. It depends on the function v which depends on m. The key property of v is that it is a positive nonprincipal solution at each endpoint. Note that v_m in (2.2) is 'patched together' from two different nonprincipal solutions, one from each endpoint, the 'patching' function m plays no significant role in this paper.

Note that (2.4) is also defined on $(-1, 1)$ but can be considered on the compact interval $[-1, 1]$ in contrast to the singular Legendre equation (1.1). A significant consequence of this, as shown by (2.6), is that, for each $\lambda \in \mathbb{C}$, every solution z of (2.4) and its quasi-derivative (Pz') can be continuously extended to the endpoints ± 1. We use the notation (Pz') to remind the reader that the product (Pz') has to be considered as one function when evaluated at ± 1 since P is not defined at -1 and at 1.

Remark 2.4. Note that we are using the theory of *quasi-differential* equations. The conditions (2.5) show that the equation (2.4) is a regular quasi-differential equation. We take full advantage of this fact in this paper.

Let $S_{\min}(N)$ and $S_{\max}(N)$ denote the minimal and maximal operators associated with (2.4), and denote their domains by $D_{\min}(N)$, $D_{\max}(N)$, respectively. Note that these are operators in the weighted Hilbert space with weight function v_m^2 which we denote by $L^2(v_m) = L^2(J_2, v_m^2)$. A self-adjoint realization $S(N)$ of (2.4) is an operator in $L^2(v_m)$ which satisfies

$$S_{\min}(N) \subset S(N) = S^*(N) \subset S_{\max}(N). \tag{2.7}$$

Applying the theory of self-adjoint regular Sturm-Liouville problems to the regularized Legendre equation (2.4) we obtain the following result.

Theorem 2.5. *If A, B are 2×2 complex matrices satisfying the following two conditions:*

$$\operatorname{rank}(A : B) = 2, \tag{2.8}$$

$$AEA^* = BEB^*, \tag{2.9}$$

then the set of all $z \in D_{\max}(N)$ satisfying

$$A \begin{pmatrix} z(-1) \\ (Pz')(-1) \end{pmatrix} + B \begin{pmatrix} z(1) \\ (Pz')(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.10}$$

is a self-adjoint domain. Conversely, given any self-adjoint realization of (2.4) *in the space* $L^2(v)$*; i.e., any operator* $S(N)$ *satisfying* (2.7)*, there exist* 2×2 *complex matrices* A, B *satisfying* (2.8) *and* (2.9) *such that the domain of* $S(N)$ *is the set of all* $z \in D_{\max}(N)$ *satisfying* (2.10)*. Here* (A, B) *is the* 2×4 *matrix whose first two columns are the columns of* A *and whose last two columns are those of* B*.*

For a proof of the above theorem, see [21]. It is convenient to divide the self-adjoint boundary conditions (2.10) into two disjoint mutually exclusive classes: the separated conditions and the coupled ones. The former have the well known canonical representation

$$\begin{gathered}\cos(\alpha)z(-1) + \sin(\alpha)(Pz')(-1) = 0, \quad 0 \le \alpha < \pi, \\ \cos(\beta)z(1) + \sin(\beta)(Pz')(1) = 0, \quad 0 < \beta \le \pi.\end{gathered} \tag{2.11}$$

The latter have the not so well known canonical representation

$$\begin{pmatrix} z(1) \\ (Pz')(1) \end{pmatrix} = e^{i\gamma} K \begin{pmatrix} z(-1) \\ (Pz')(-1) \end{pmatrix}, \quad -\pi < \gamma \le \pi. \tag{2.12}$$

Examples of separated conditions are the well known Dirichlet condition

$$z(-1) = 0 = z(1) \tag{2.13}$$

and the Neumann condition

$$(Pz')(-1) = 0 = (Pz')(1). \tag{2.14}$$

Examples of coupled conditions are the periodic conditions

$$z(-1) = z(1) \tag{2.15}$$

$$(Pz')(-1) = (Pz')(1) \tag{2.16}$$

and the semi-periodic (also called anti-periodic) conditions

$$\begin{gathered} z(-1) = -z(1) \\ (Pz')(-1) = -(Pz')(1) \end{gathered} \tag{2.17}$$

Note, however, that when $\gamma \neq 0$ we have complex matrices A, B defining regular self-adjoint operators.

Next we explore the relationship between solutions y of the singular equation (2.1) and solutions z of the regularized Legendre equation (1.1).

Lemma 2.6. *For any* $\lambda \in \mathbb{C}$*, the solutions* $y(\cdot, \lambda)$ *of the singular equation* (1.1) *and the solutions* $z(\cdot, \lambda)$ *of the regular equation* (2.4) *are related by*

$$\frac{y(t, \lambda)}{v_m(t)} = z(t, \lambda), \quad -1 < t < 1,\ \lambda \in \mathbb{C} \tag{2.18}$$

and the correspondence $y(\cdot, \lambda) \to z(\cdot, \lambda)$ *is* $1 - 1$ *onto. Note that there is the same* λ *on both sides.*

Proof. Fix $\lambda \in \mathbb{C}$ and simplify the notation for this proof so that $v = v_m$ and let

$$z = \frac{y}{v} \quad \text{on } (-1, 1).$$

Then $z' = \frac{vy' - yv'}{v^2}$ and $((pv^2)z')' = (v(py') - y(pv'))' = v(py')' + v'py' - y'pv' - y(pv')' = v(-\lambda y) + y(Mv) = -\lambda v^2 \frac{y}{v} + \frac{y}{v} vMv = -\lambda v^2 z + Qz$ and (2.4) follows. Reversing the steps shows that the correspondence is $1 - 1$. $\square$

Remark 2.7. We comment on the relationship between the classical singular Legendre equation (1.1) and its regularizations (2.4); this remark will be amplified below after we have discussed the self-adjoint operators generated by the singular Legendre equation (1.1). In particular, we will see below that the operator $S(N)$ determined by the Dirichlet condition (2.13), which we denote by $S_F(N)$, is a regular representation of the celebrated classical singular Friedrichs operator, denoted by S_F below, whose eigenvalues are $\{n(n+1) : n \in \mathbb{N}_0\}$ and whose eigenfunctions are the classical Legendre polynomials P_n given above. Note that the solutions $y(t,\lambda)$ and $z(t,\lambda)$ have exactly the same zeros in the open interval $(-1,1)$ but not in the closed interval $[-1,1]$ since z may be zero at the endpoints and y may not be defined there.

Remark 2.8. Each solution z and its quasi-derivative (Pz') is continuous on the compact interval $[-1,1]$. Note that $v(t)$ does not depend on λ. Therefore the singularity of every solution $y(t,\lambda)$ for all $\lambda \in \mathbb{C}$ is contained in v, in other words, the nature of the singularities of the solutions $y(t,\lambda)$ are invariant with respect to λ. Although $v(t)$ does not exist for $t=-1$ and $t=1$ and $y(t)$ also may not exist for $t=-1$ and $t=1$ the limits

$$\lim_{t\to -1^+} \frac{y(t,\lambda)}{v(t)} = z(-1,\lambda), \quad \lim_{t\to 1^-} \frac{y(t,\lambda)}{v(t)} = z(1,\lambda) \tag{2.19}$$

exist for all solutions $y(t,\lambda)$ of the Legendre equation (1.1). If $z(1,\lambda) \neq 0$, then $y(t,\lambda)$ blows up logarithmically as $t \to 1$; similarly at -1.

Remark 2.9. Applying the correspondence (2.18) to the Legendre polynomials we obtain a factorization of these polynomials:

$$P_n(t) = v(t) z_n(t), \quad -1 < t < 1,\ n \in \mathbb{N}_0. \tag{2.20}$$

Since P_n is continuous at -1 and at 1 and v blows up at these points it follows that $z_n(-1) = 0 = z_n(1)$, $n \in \mathbb{N}_0$. Note that z_n has exactly the same zeros as P_n in the open interval $(-1,1)$. However, also note that this is not the case for the closed interval $[-1,1]$ since $z_n(-1) = 0 = z_n(1)$ but $P_n(1) \neq 0 \neq P_n(-1)$ for each $n \in \mathbb{N}_0$.

Remark 2.10. Below, following the characterization of the self-adjoint Legendre realizations S of the singular Legendre equation (1.1) using singular SL theory, we will specify a $1-1$ correspondence between the self-adjoint realizations $S(N)$ of the regularized Legendre equation (2.4) and the self-adjoint operators of the singular classical Legendre equation (1.1). In particular, we will see that the operator $S_D(N)$ determined by the regular Dirichlet boundary condition

$$z(-1) = 0 = z(1) \tag{2.21}$$

corresponds to the celebrated classical Friedrichs Legendre operator S_F determined by the singular boundary condition

$$(py')(-1) = 0 = (py')(1)$$

whose eigenvalues are $\{n(n+1),\ n \in \mathbb{N}_0\}$ and whose eigenfunctions are the classical Legendre polynomials P_n given by part (10) of Proposition (1.6). The Dirichlet operator $S_D(N)$ has the same eigenvalues as S_F but its eigenfunctions are given by

$$z_n = \frac{P_n}{v^2}, \quad n \in \mathbb{N}_0.$$

Note that each z_n has exactly the same zeros in the open interval $(-1,1)$ but not in the closed interval $[-1,1]$ since $z_n(-1) = 0 = z_n(1)$. Also note that S_F is a self-adjoint operator in the space $L^2(-1,1)$ and $S_D(N)$ is a self-adjoint operator in the weighted Hilbert space $L^2((-1,1),v^2)$.

3. Self-Adjoint Operators in $L^2(-1,1)$

By a self-adjoint operator associated with equation (1.1) in $H_2 = L^2(-1,1)$ or a self-adjoint realization of equation (1.1) in H_2 we mean a self-adjoint restriction of the maximal operator $S_{\max}$ associated with (1.1). This is defined as follows:

$$D_{\max} = \{f : (-1,1) \to \mathbb{C} \mid f,\, pf' \in AC_{\mathrm{loc}}(-1,1); f,\, pf' \in H_2\} \tag{3.1}$$

$$S_{\max} f = -(pf')', \quad f \in D_{\max} \tag{3.2}$$

We refer the reader to the classic texts of Akhiezer and Glazman [1], Dunford and Schwartz [6], Naimark [14], and Titchmarsh [19] for general, and specific, information on the theory of self-adjoint extensions of symmetric differential operators. We also refer to the excellent account of [7] on the right-definite self-adjoint theory of the Legendre expression (1.1).

Note that all bounded continuous functions on $(-1,1)$ are in $D_{\max}$; in particular all polynomials are in $D_{\max}$. (More precisely the restriction of every polynomial to $(-1,1)$ is in $D_{\max}$). However $D_{\max}$ also contains functions which are not bounded on $(-1,1)$, e.g. $f(t) = \ln(1-t)$.

Lemma 3.1. *The operator $S_{\max}$ is densely defined in H_2 and therefore has a unique adjoint in H_2 denoted by $S_{\min}$:*

$$S_{\max}^* = S_{\min}. \tag{3.3}$$

Furthermore, the minimal operator $S_{\min}$ in H_2 is symmetric, closed, densely defined and

$$S_{\min}^* = S_{\max} \tag{3.4}$$

Moreover, if S is a self-adjoint extension of $S_{\min}$ then S is also a self-adjoint restriction of $S_{\max}$ and conversely. Thus we have

$$S_{\min} \subset S = S^* \subset S_{\max}. \tag{3.5}$$

The above lemma is part of basic Sturm-Liouville theory; see for example [21].

It is clear from (3.5) that each self-adjoint operator S is determined by its domain $D(S)$. The operators S satisfying (3.5) are called self-adjoint realizations of (1.1) in H_2 or on $(-1,1)$. We will also refer to these as Legendre operators in H_2 or on $(-1,1)$.

Next we describe these self-adjoint domains. It is remarkable that all self-adjoint Legendre operators can be described explicitly in terms of two-point singular boundary conditions. For this the functions u, v given by (1.6) play an important role, in a sense they form a basis for all self-adjoint boundary conditions [21]. Let

$$My = -(py')'. \tag{3.6}$$

Of critical importance in the characterization of all self-adjoint boundary conditions is the Lagrange sesquilinear form $[\cdot,\cdot]$, now defined for all maximal domain functions,

$$[f,g] = fp(\overline{g}') - gp(\overline{f}') \quad (f,g \in D_{\max}), \tag{3.7}$$

and the associated Green's formula

$$\int_a^b \{\overline{g}Mf - \overline{f}Mg\} = [f,g](b) - [f,g](a), \quad f,g \in D_{\max},\ -1 < a < b < 1 \tag{3.8}$$

From this inequality it follows that the limits

$$\lim_{a\to -1^+}[f,g](t), \quad \lim_{b\to +1^-}[f,g](t) \tag{3.9}$$

exist and are finite.

We can now give a characterization of all self-adjoint Legendre operators in $L^2(-1,1)$.

Theorem 3.2. *Let u,v be given by* (1.6). *Let A,B be 2×2 complex matrices satisfying the following two conditions:*

$$\operatorname{rank}(A:B) = 2, \tag{3.10}$$

$$AEA^* = BEB^*, \quad E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.11}$$

Define $D(S) = \{y \in D_{\max}\}$ such that

$$A\begin{pmatrix} (-py')(-1) \\ (ypv' - v(py'))(-1) \end{pmatrix} + B\begin{pmatrix} (-py')(1) \\ (ypv' - v(py'))(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.12}$$

Then $D(S)$ is a self-adjoint domain. Furthermore all self-adjoint domains are generated this way. Here $(A:B)$ denotes the 2×4 matrix whose first two columns are those of A and whose last two columns are the columns of B.

The proof of the above theorem is given in [21, pages 183-185].

Remark 3.3. We comment on some aspects of this remarkable characterization of all self-adjoint Legendre operators in $L^2(-1,1)$.

(1) Just as in the regular case, the singular self-adjoint boundary conditions (3.12) are explicit since v is explicitly given near the endpoints by (1.6).

(2) Note that $[y,u] = -py'$ and $[y,v] = y(pv') - v(py')$. Hence $-py'$ and $(ypv' - v(py'))$ exist as finite limits at -1 and at 1 for all maximal domain functions y. In particular, these limits exist and are finite for all solutions y of equation (1.1) for any λ. Thus a number λ is an eigenvalue of the singular boundary value problem (1.1), (3.12) if and only if the equation (1.1) has a nontrivial solution y satisfying (3.12). Note that the separate terms $y(pv')$ and $v(py')$ may not exist at -1 or at $+1$, they may blow up or oscillate wildly at these points but the combination $[y,v]$ has a finite limit at -1 and at $+1$ for any maximal domain functions y,v.

(3) Choose $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then (3.11) holds and (3.12) reduces to

$$(py')(-1) = 0 = (py')(1) \tag{3.13}$$

This is the boundary condition which determines, among the uncountable number of self-adjoint conditions, the special ('ausgezeichnete') Friedrichs extension S_F. It is interesting to observe that, even though (3.13) has the appearance of a regular Neumann condition, in fact it is the singular analogue of the regular Dirichlet condition. It is well known [21] that. in general, the Dirichlet boundary condition determines the Friedrichs extension S_F of regular SLP and that for singular non-oscillatory limit-circle problems, in general, the Friedrichs extension S_F is determined by the conditions

$$[y, u_a](a) = 0 = [y, u_b](b)$$

where u_a is the principal solution at the left endpoint a and u_b is the principal solution at the right endpoint b. Since the constant function $u = 1$ is the principal solution *at both endpoints* -1 and 1 in the Legendre case we have $[y, u] = -(py')$ and (3.13) follows.

(4) The condition (3.12) includes separated and coupled conditions. Below we will give a canonical form for these two classes of conditions which is analogous to the regular case. We will also see below that (3.12) includes *complex* boundary conditions. These are coupled; it is known that all separated self-adjoint conditions can be taken as real; i.e., each complex separated condition (3.12) is equivalent to a real such condition.

(5) Since each endpoint is LCNO (limit-circle nonoscillatory) it is well known that the spectrum σ of every self-adjoint extension S, $\sigma(S)$ is discrete, bounded below and unbounded above with no finite cluster point. For S_F we have the celebrated result that

$$\sigma(S_F) = n(n+1), \ (n \in \mathbb{N}_0)$$

and the corresponding orthonormal eigenfunctions are polynomials given by (1.12). For other self-adjoint Legendre operators S the eigenvalues and eigenfunctions are not known in closed form. However they can be computed numerically with the FORTRAN code SLEIGN2, developed by Bailey, Everitt and Zettl [3]; this code, and a number of files related to it, can be downloaded from www.math.niu.edu/SL2. It comes with a user friendly interface.

(6) It is known from general Sturm-Liouville theory that the eigenfunctions of every self-adjoint Legendre realization S are dense in $L^2(-1,1)$. In particular the Legendre polynomials (1.12) are dense in $L^2(-1,1)$.

(7) If S is generated by a separated boundary condition (3.12), then the $n-th$ eigenfunction of S has exactly n zeros in the open interval $(-1,1)$ for each $n \in \mathbb{N}_0$. In particular, this is true for the Legendre polynomials (1.12).

(8) The self-adjoint boundary conditions (3.12) depend on the function v given by (1.6). But note that only the values of v near the endpoints play a role in (3.12) and therefore v can be replaced by any function which is asymptotically equivalent to it, in particular v can be replaced by any function which has the same values as v in a neighborhood of -1 and of 1.

Now that we have determined all the self-adjoint singular Legendre operators with Theorem 3.2, we compare these with the self-adjoint operators determined by the regularized Legendre equation which are given by Theorem 2.5. In making this comparison it is important to keep in mind that these operators act in different Hilbert spaces: $L^2(-1,1)$ for the singular classical case and $L^2(v^2) = L^2((-1,1), v^2)$ for the regularized case.

But first we show that the correspondence

$$\frac{y}{v} = z, \quad y = vz \tag{3.14}$$

extends from solutions to functions in the domains of the operator realizations of the classical Legendre equation and its regularization. Since we now compare operator realizations of the singular equation (1.1) and its regularization (2.4) with each other we use the notation $S(M)$ for operators associated with the former and $S(N)$ for those of the latter. It is important to remember that the operators $S(M)$ are operators in the Hilbert space $L^2(-1,1)$ and the operators $S(N)$ are operators in the Hilbert space $L(v^2) = L^2((-1,1), v^2)$.

We denote the Lagrange forms associated with equations (1.1) and (2.4) by

$$[y,f]_M = yp\overline{f'} - \overline{f}py', \quad y,f \in D_{\max}(M), \tag{3.15}$$

and by

$$[z,g]_N = zP\overline{g}' - \overline{g}Pz', \quad z,g \in D_{\max}(N),\ P = v^2p, \tag{3.16}$$

respectively.

Notation. We say that $D(N)$ is a self-adjoint domain for (2.4) if the operator with this domain is a self-adjoint realization of (2.4) in the Hilbert space $L(v^2)$. Similarly, $D(M)$ is a self-adjoint domain for (1.1) if the operator with this domain is a self-adjoint realization of (2.1) in the Hilbert space $L^2(-1,1)$.

Theorem 3.4. *Let* (1.1) *and* (2.4) *hold; let v be given by* (1.6).

1. *A function $z \in D_{\max}(N)$ if and only if $vz \in D_{\max}(M)$.*
2. *$D(N)$ is a self-adjoint domain for* (2.4) *if and only if $D(M) = \{y = vz : z \in D(M)\}$.*
3. *In particular we have a new characterization of the Friedrichs domain for* (1.1)
$$D(S_F(M)) = \{vz : z \in D_{\max}(N) :\ z(-1) = 0 = z(1)\}.$$

Proof. Assume $y,\ f \in D_{\max}(M)$. Let $z = \frac{y}{v},\ g = \frac{f}{v}$. Then we have

$$\begin{aligned}
{}_N = [\frac{y}{v}, \frac{f}{v}]_N &= \frac{y}{v}P(\frac{\overline{f}}{v})' - \frac{\overline{f}}{v}P(\frac{y}{v})' \\
&= \frac{y}{v}pv^2\frac{v\overline{f'} - \overline{f}v'}{v_2} - \frac{\overline{f}}{v}pv^2\frac{vy' - yv'}{v_2} \\
&= yp\overline{f'} - \frac{y}{v}p\overline{f}v' - \overline{f}py' + \frac{y}{v}p\overline{f}v' \\
&= yp\overline{f'} - \overline{f}py' = [y,f]_M
\end{aligned} \tag{3.17}$$

Part (2) follows from (1) and (3.17). To prove (1). Assume

$$y \in D_{\max}(M) = \{y \in L^2(J_2) : py' \in AC_{\mathrm{loc}}(J_2),\ My = (py')' \in L^2(J_2)\}.$$

We must show that

$$z \in D_{\max}(N) = \{z \in L^2(v^2),\ Pz' \in AC_{\mathrm{loc}}(J_2),\ Nz = (Pz')' \in L^2(v^2)\}.$$

Note that

$$\int_{-1}^{1} |z^2|v^2 = \int_{-1}^{1} |y^2| < \infty,$$
$$Pz' = v(py') - y(pv') = v(py') - y \in AC_{\mathrm{loc}}(J_2),$$
$$(Pz')' = v'py' + v(py')' - y'(pv') - y(pv)' = vMy \in L^2(v^2).$$

The converse follows similarly by reversing the steps in this argument. $\square$

3.1. **Eigenvalue Properties.** In this subsection we study the variation of the eigenvalues as functions of the boundary conditions for the Legendre problem consisting of the equation

$$My = -(py')' = \lambda y \quad \text{on } J_2 = (-1,1),\ p(t) = 1 - t^2,\ -1 < t < 1, \tag{3.18}$$

together with the boundary conditions

$$A \begin{pmatrix} (-py')(-1) \\ (ypv' - v(py'))(-1) \end{pmatrix} + B \begin{pmatrix} (-py')(1) \\ (ypv' - v(py'))(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.19}$$

Here v is given by (1.6) near the endpoints and the matrices A, B satisfy (3.10), (3.11).

Since the homogeneous boundary conditions (3.19) are invariant under multiplication by a nonsingular matrix, to study the dependence of the eigenvalues on the boundary conditions it is very useful to have a canonical represention of them. For such a representation it is convenient to classify the boundary conditions into two mutually exclusive classes: separated and coupled. The separated conditions have the form [21]

$$\begin{gathered} \cos(\alpha)[y,u](-1) + \sin(\alpha)[y,v](-1) = 0, \quad 0 \le \alpha < \pi, \\ \cos(\beta)[y,u](1) + \sin(\beta)[y,v]\,(1) = 0, \quad 0 < \beta \le \pi. \end{gathered} \tag{3.20}$$

The coupled conditions have the canonical representation [21]

$$Y(1) = e^{i\gamma} K\, Y(-1), \tag{3.21}$$

where

$$Y = \begin{pmatrix} [y,u] \\ [y,v] \end{pmatrix}, \quad -\pi < \gamma \le \pi,\ K \in SL_2(\mathbb{R}); \tag{3.22}$$

i.e., $K = (k_{ij})$, $k_{ij} \in \mathbb{R}$, and $\det(K) = 1$.

Definition 3.5. The boundary conditions (3.20) are called *separated* and (3.21) are *coupled*; if $\gamma = 0$ we say they are *real coupled* and with $\gamma \neq 0$ they are *complex coupled*.

Theorem 3.6. *Let S be a self-adjoint Legendre operator in $L^2(-1,1)$ according to Theorem* (3.2) *and denote its spectrum by $\sigma(S)$.*

(1) *Then the boundary conditions determining S are either given by* (3.20) *or by* (3.21) *and each such boundary condition determines a self-adjoint Legendre operator in $L^2(-1,1)$.*
(2) *The spectrum $\sigma(S) = \{\lambda_n : n \in N_0 = \{0,1,2,\cdots\}\}$ is real, discrete, and can be ordered to satisfy*

$$-\infty < \lambda_0 \le \lambda_1 \le \lambda_2 \le \dots \tag{3.23}$$

Here equality cannot hold for two consecutive terms.
(3) *If the boundary conditions are separated, then strict inequality holds everywhere in* (3.23) *and if u_n is an eigenfunction of λ_n then u_n is unique up to constant multiples and has exactly n zeros in the open interval $(-1,1)$ for each $n = 0,1,2,3,\dots$*
(4) *If the boundary conditions are coupled and real $(\gamma = 0)$ and u_n is a real eigenfunction of λ_n, then the number of zeros of u_n in the open interval $(-1,1)$ is 0 or 1 if $n = 0$ and $n-1$ or n or $n+1$, if $n \ge 1$. (Note that,*

although there may be eigenvalues of multiplicity 2 *and therefore some ambiguity in the indexing of the eigenvalues, the eigenfunctions* u_n *are uniquely defined, up to constant multiples.)*

(5) *If the boundary conditions are coupled and complex* ($\gamma \neq 0$) *then all eigenvalues are simple and strict inequality holds in* (3.23). *If* u_n *is an eigenfunction of* λ_n, *then* u_n *has no zero in the closed interval* $[-1,1]$. *The number of zeros of both the real part* $\mathrm{Re}(u_n)$ *and of the imaginary part* $\mathrm{Im}(u_n)$ *in the half-open interval* $[-1,1)$ *is* 0 *or* 1 *if* $n=0$ *and is* $n-1$ *or* n *or* $n+1$ *if* $n \geq 1$.

(6) *If the boundary condition is the classical condition*
$$(py')(-1) = 0 = (py')(1), \tag{3.24}$$
then the eigenvalues are given by
$$\lambda_n = n(n+1), \quad n \in \mathbb{N}_0 = \{0,1,2,3,\dots\}$$
and the normalized eigenfunctions are the classical Legendre polynomials.

(7) *For any boundary conditions, separated, real coupled or complex coupled, we have*
$$\lambda_n \leq n(n+1), \quad n \in \mathbb{N}_0 = \{0,1,2,3,\dots\}. \tag{3.25}$$
In other words, the eigenvalues of the self-adjoint Legendre operator determined by the classical boundary conditions (3.24) *maximize the eigenvalues of all other self-adjoint Legendre operators.*

(8) *For any self-adjoint boundary conditions, separated, real coupled or complex coupled, we have*
$$n(n+1) \leq \lambda_{n+2},\ n \in \mathbb{N}_0 = \{0,1,2,3,\dots\}. \tag{3.26}$$
In other words, the n-th eigenvalue of the self-adjoint Legendre operator determined by the classical boundary conditions (3.24) *is a lower bound of* λ_{n+2} *for all other self-adjoint Legendre operators. These bounds are precise:*

(9) *The range of* $\lambda_0(S) = (-\infty, 0]$ *as* S *varies over all self-adjoint Legendre operators in* $L^2(-1,1)$.

(10) *The range of* $\lambda_1(S) = (-\infty, 0]$ *as* S *varies over all self-adjoint Legendre operators in* $L^2(-1,1)$.

(11) *The range of* $\lambda_n(S) = ((n-2)(n-1), n\,(n+1)]$ *as* S *varies over all self-adjoint Legendre operators in* $L^2(-1,1)$.

(12) *The last three statements about the range of the eigenvalues are still valid if the operators* S *are restricted to those determined by real boundary condition only.*

(13) *Assume* S *is any self-adjoint Legendre operator in* $L^2(-1,1)$, *determined by separated, real coupled or complex coupled, boundary conditions and let* $\sigma(S) = \{\lambda_n : n \in N_0 = \{0,1,2,\dots\}\}$ *denote its spectrum. Then*
$$\frac{\lambda_n}{n^2} \to 1, \quad \text{as } n \to \infty.$$

Proof. Part (7); i.e., (3.25), is the well known classical result about the Legendre equation and its polynomial solutions. All the other parts follow from applying the known corresponding results for regular problems, see [21, Chapter 4], to the above regularization of the singular Legendre equation. □

3.2. **Legendre Green's Function.** In this subsection we construct the Legendre Green's function. This seems to be new even though, as mentioned in the Introduction, the Legendre equation

$$-(py')' = \lambda y,\ p(t) = 1 - t^2, \quad \text{on } J = (-1, 1) \tag{3.27}$$

is one of the simplest singular differential equations. Its potential function q is zero, its weight function w is the constant 1, and its leading coefficient p is a simple quadratic. It is singular at both endpoints -1 and $+1$. These singularities are due to the fact that $1/p$ is not Lebesgue integrable in left and right neighborhoods of these points.

Our construction of the Legendre Green's functions is a five step procedure:

(1) Formulate the singular second order scalar equation (3.27) as a first order singular system.
(2) 'Regularize' this singular system by constructing regular systems which are equivalent to it.
(3) Construct the Green's matrix for boundary value problems of the regular system.
(4) Construct the singular Green's matrix for the equivalent singular system from the regular one.
(5) Extract the upper right corner element from the singular Green's matrix. This is the Green's function for singular scalar boundary value problems for equation (3.27).

For the convenience of the reader we present these five steps here even though some of them were given above.

For $\lambda = 0$ recall the two linearly independent solutions u, v of (3.27) given by

$$u(t) = 1, \quad v(t) = \frac{-1}{2} \ln(|\frac{1-t}{t+1}|) \tag{3.28}$$

The standard system formulation of (3.27) has the form

$$Y' = (P - \lambda W)Y, \quad \text{on } (-1, 1) \tag{3.29}$$

where

$$Y = \begin{pmatrix} y \\ py' \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1/p \\ 0 & 0 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{3.30}$$

Let

$$U = \begin{pmatrix} u & v \\ pu' & pv' \end{pmatrix} = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}. \tag{3.31}$$

Note that $\det U(t) = 1$, for $t \in J = (-1, 1)$, and set

$$Z = U^{-1}Y. \tag{3.32}$$

Then

$$\begin{aligned} Z' &= (U^{-1})'Y + U^{-1}Y' = -U^{-1}U'U^{-1}Y + (U^{-1})(P - \lambda W)Y \\ &= -U^{-1}U'Z + (U^{-1})(P - \lambda W)UZ \\ &= -U^{-1}(PU)Z + U^{-1}(PU)Z - \lambda(U^{-1}WU)Z = -\lambda(U^{-1}WU)Z. \end{aligned}$$

Letting $G = (U^{-1}WU)$ we may conclude that

$$Z' = -\lambda GZ, \tag{3.33}$$

where

$$G = U^{-1}WU = \begin{pmatrix} -v & -v^2 \\ 1 & v \end{pmatrix}, \tag{3.34}$$

Note that (3.34) is the regularized Legendre system of Section 1.

The next theorem summarizes the properties of (3.34) and their relationship to (3.27).

Theorem 3.7. *Let $\lambda \in \mathbb{C}$ and let G be given by* (3.34)*.*

(1) *Every component of G is in $L^1(-1,1)$ and therefore* (3.33) *is a* regular *system.*
(2) *For any $c_1, c_2 \in \mathbb{C}$ the initial value problem*
$$Z' = -\lambda GZ, \; Z(-1) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \tag{3.35}$$
has a unique solution Z defined on the closed interval $[-1,1]$.
(3) *If $Y = \begin{pmatrix} y(t,\lambda) \\ (py')(t,\lambda) \end{pmatrix}$ is a solution of* (3.29) *and $Z = U^{-1}Y = \begin{pmatrix} z_1(t,\lambda) \\ z_2(t,\lambda) \end{pmatrix}$, then Z is a solution of* (3.33) *and for all $t \in (-1,1)$ we have*
$$y(t,\lambda) = uz_1(t,\lambda) + v(t)z_2(t,\lambda) = z_1(t,\lambda) + v(t)z_2(t,\lambda) \tag{3.36}$$
$$(py')(t,\lambda) = (pu')z_1(t,\lambda) + (pv')(t)z_2(t,\lambda) = z_2(t,\lambda) \tag{3.37}$$
(4) *For every solution $y(t,\lambda)$ of the singular scalar Legendre equation* (3.27) *the quasi-derivative $(py')(t,\lambda)$ is continuous on the compact interval $[-1,1]$. More specifically we have*
$$\lim_{t\to -1^+} (py')(t,\lambda) = z_2(-1,\lambda), \quad \lim_{t\to 1^-} (py')(t,\lambda) = z_2(1,\lambda). \tag{3.38}$$
Thus the quasi-derivative is a continuous function on the closed interval $[-1,1]$ for every $\lambda \in \mathbb{C}$.
(5) *Let $y(t,\lambda)$ be given by* (3.36)*. If $z_2(1,\lambda) \neq 0$ then $y(t,\lambda)$ is unbounded at 1; If $z_2(-1,\lambda) \neq 0$ then $y(t,\lambda)$ is unbounded at -1.*
(6) *Fix $t \in [-1,1]$, let $c_1, c_2 \in \mathbb{C}$. If $Z = \begin{pmatrix} z_1(t,\lambda) \\ z_2(t,\lambda) \end{pmatrix}$ is the solution of* (3.33) *determined by the initial conditions $z_1(-1,\lambda) = c_1$, $z_2(-1,\lambda) = c_2$, then $z_i(t,\lambda)$ is an entire function of λ, $i = 1,2$. Similarly for the initial condition $z_1(1,\lambda) = c_1$, $z_2(1,\lambda) = c_2$.*
(7) *For each $\lambda \in \mathbb{C}$ there is a nontrivial solution which is bounded in a (two sided) neighborhood of 1; and there is a (generally different) nontrivial solution which is bounded in a (two sided) neighborhood of -1.*
(8) *A nontrivial solution $y(t,\lambda)$ of the singular scalar Legendre equation* (3.27) *is bounded at 1 if and only if $z_2(1,\lambda) = 0$; A nontrivial solution $y(t,\lambda)$ of the singular scalar Legendre equation* (3.27) *is bounded at -1 if and only if $z_2(-1,\lambda) = 0$.*

Proof. Part (1) follows from (3.34), (2) is a direct consequence of (1) and the theory of regular systems, $Y = UZ$ implies (3)$\Longrightarrow$(4) and (5); (6) follows from (2) and the basic theory of regular systems. For (7) determine solutions $y_1(t,\lambda)$, $y_{-1}(t,\lambda)$ by applying the Frobenius method to obtain power series solutions of (1.1) in the form: (see [10], page 5 with different notations)

$$y_1(t,\lambda) = 1 + \sum_{n=1}^{\infty} a_n(\lambda)(t-1)^n, \quad |t-1| < 2; \tag{3.39}$$

$$y_{-1}(t,\lambda) = 1 + \sum_{n=1}^{\infty} b_n(\lambda)(t+1)^n, \quad |t+1| < 2; \tag{3.40}$$

To prove (8) it follows from (3.36) that if $z_2(1,\lambda) \neq 0$, then $y(t,\lambda)$ is not bounded at 1. Suppose $z_2(1,\lambda) = 0$. If the corresponding $y(t,\lambda)$ is not bounded at 1 then there are two linearly unbounded solutions at 1 and hence all nontrivial solutions are unbounded at 1. This contradiction establishes (8) and completes the proof of the theorem. □

Remark 3.8. From Theorem 3.7 we see that, for every $\lambda \in \mathbb{C}$, the equation (3.27) has a solution y_1 which is bounded at 1 and has a solution y_{-1} which is bounded at -1. It is well known that for $\lambda_n = n(n+1) : n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ the Legendre polynimials P_n are solutions on $(-1, 1)$ and hence are bounded at -1 and at $+1$.

We now consider two point boundary conditions for (3.33); later we will relate these to singular boundary conditions for (3.27).

Let $A, B \in M_2(\mathbb{C})$, the set of 2×2 complex matrices, and consider the boundary value problem

$$Z' = -\lambda G Z, \quad AZ(-1) + BZ(1) = 0. \tag{3.41}$$

Recall that $\Phi(t, s, -\lambda)$ is the primary fundamental matrix of the system $Z' = -\lambda GZ$ constructed in Section 1.

Lemma 3.9. *A complex number $-\lambda$ is an eigenvalue of* (3.41) *if and only if*

$$\Delta(\lambda) = \det(A + B\Phi(1, -1, -\lambda) = 0. \tag{3.42}$$

Furthermore, a complex number $-\lambda$ is an eigenvalue of geometric multiplicity two if and only if

$$A + B\Phi(1, -1, -\lambda) = 0. \tag{3.43}$$

Proof. Note that a solution for the initial condition $Z(-1) = C$ is given by

$$Z(t) = \Phi(t, -1, -\lambda)C, \ t \in [-1, 1]. \tag{3.44}$$

The boundary value problem (3.41) has a nontrivial solution for Z if and only if the algebraic system

$$[A + B\Phi(1, -1, -\lambda)]Z(-1) = 0 \tag{3.45}$$

has a nontrivial solution for $Z(-1)$.

To prove the furthermore part, observe that two linearly independent solutions of the algebraic system (1.25) for $Z(-1)$ yield two linearly independent solutions $Z(t)$ of the differential system and conversely. □

Given any $\lambda \in \mathbb{R}$ and any solutions y, z of (3.27) the Lagrange form $[y, z](t)$ is defined by

$$[y, z](t) = y(t)(p\overline{z'})(t) - \overline{z}(t)(py')(t).$$

So, in particular, we have

$$[u, v](t) = +1, [v, u](t) = -1, [y, u](t) = -(py')(t), \ t \in \mathbb{R},$$
$$[y, v](t) = y(t) - v(t)(py')(t), \quad t \in \mathbb{R}, \ t \neq \pm 1.$$

We will see below that, although v blows up at ± 1, the form $[y,v](t)$ is well defined at -1 and $+1$ since the limits

$$\lim_{t\to -1}[y,v](t), \quad \lim_{t\to +1}[y,v](t)$$

exist and are finite from both sides. This for any solution y of (3.27) for any $\lambda \in \mathbb{R}$. Note that, since v blows up at 1, this means that y must blow up at 1 except, possibly when $(py')(1) = 0$.

We are now ready to construct the Green's function of the singular scalar Legendre problem consisting of the equation

$$My = -(py')' = \lambda y + h \quad \text{on } J = (-1,1), \quad p(t) = 1 - t^2,\ -1 < t < 1, \tag{3.46}$$

together with two point boundary conditions

$$A\begin{pmatrix} (-py')(-1) \\ (ypv' - v(py'))(-1) \end{pmatrix} + B\begin{pmatrix} (-py')(1) \\ (ypv' - v(py'))(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3.47}$$

where u, v are given by (3.28) and A, B are 2×2 complex matrices. This construction is based on the system regularization discussed above and we will use the notation from above. Consider the regular nonhomogeneous system

$$Z' = -\lambda GZ + F, \quad AZ(-1) + BZ(1) = 0. \tag{3.48}$$

where

$$F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad f_j \in L^1(J,\mathbb{C}),\ j = 1,2. \tag{3.49}$$

Theorem 3.10. *Let $-\lambda \in \mathbb{C}$ and let $\Delta(-\lambda) = [A + B\Phi(1,-1,-\lambda)]$. Then the following statements are equivalent:*

(1) *For $F = 0$ on $J = (-1,1)$, the homogeneous problem* (3.48) *has only the trivial solution.*
(2) *$\Delta(-\lambda)$ is nonsingular.*
(3) *For every $F \in L^1(-1,1)$ the nonhomogeneous problem* (3.48) *has a unique solution Z and this solution is given by*

$$Z(t,-\lambda) = \int_{-1}^{1} K(t,s,-\lambda)\,F(s)ds, \quad -1 \le t \le 1, \tag{3.50}$$

where

$$K(t,s,-\lambda) = \begin{cases} \Phi(t,-1,-\lambda)\Delta^{-1}(-\lambda)(-B)\Phi(1,s,-\lambda), \\ \quad \text{if } -1 \le t < s \le 1, \\ \Phi(t,-1,-\lambda)\Delta^{-1}(-\lambda)(-B)\Phi(1,s,-\lambda) + \phi(t,s-\lambda), \\ \quad \text{if } -1 \le s < t \le 1, \\ \Phi(t,-1,-\lambda)\Delta^{-1}(-\lambda)(-B)\Phi(1,s,-\lambda) + \frac{1}{2}\phi(t,s-\lambda), \\ \quad \text{if } -1 \le s = t \le 1. \end{cases}$$

The proof is a minor modification of the Neuberger construction given in [16]; see also [21].

From the regular Green's matrix we now construct the singular Green's matrix and from it the singular scalar Legendre Green's function.

Definition 3.11. Let

$$L(t,s,\lambda) = U(t)K(t,s,-\lambda)U^{-1}(s), \quad -1 \le t,s \le 1. \tag{3.51}$$

The next theorem shows that L_{12}, the upper right component of L, is the Green's function of the singular scalar Legendre problem (3.46), (3.47).

Theorem 3.12. *Assume that* $[A + B\Phi(1, -1, -\lambda)]$ *is nonsingular. Then for every function* h *satisfying*

$$h,\, vh \in L^1(J, \mathbb{C}), \tag{3.52}$$

the singular scalar Legendre problem (3.46)*,* (3.47) *has a unique solution* $y(\cdot, \lambda)$ *given by*

$$y(t, \lambda) = \int_{-1}^{1} L_{12}(t, s)\, h(s) ds,\ -1 < t < 1. \tag{3.53}$$

Proof. Let

$$F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = U^{-1} H, \quad H = \begin{pmatrix} 0 \\ -h \end{pmatrix}. \tag{3.54}$$

Then $f_j \in L^1(J_2, \mathbb{C})$, $j = 1, 2$. Since $Y(t, \lambda) = U(t)Z(t, -\lambda)$, from (3.50) we obtain

$$\begin{aligned} Y(t, \lambda) = U(t)Z(t, -\lambda) &= U(t) \int_{-1}^{1} K(t, s, -\lambda)\, F(s) ds \\ &= \int_{-1}^{1} U(t)K(t, s, -\lambda)U^{-1}(s)\, H(s) ds \\ &= \int_{-1}^{1} L(t, s, \lambda) H(s) ds, \quad -1 < t < 1. \end{aligned} \tag{3.55}$$

Therefore,

$$y(t, \lambda) = -\int_{-1}^{1} L_{12}(t, s, \lambda) h(s) ds,\ -1 < t < 1. \tag{3.56}$$

□

An important property of the Friedrichs extension S_F is the well known fact that it has the same lower bound as the minimal operator $S_{\min}$. But this fact does not characterize the Friedrichs extension of $S_{\min}$. Haertzen, Kong, Wu and Zettl [11] characterized all self-adjoint regular Sturm-Liouville operators which preserve the lower bound of the minimal operator, see also [21, Proposition 4.8.1]. The next theorem, characterizes the Legendre Friedrichs extension S_F uniquely.

Theorem 3.13 (Everitt, Littlejohn and Marić)**.** *Suppose that* $S \neq S_F$ *is a self-adjoint Legendre operator in* $L^2(-1, 1)$*. Then there exists* $f \in D(S)$ *such that*

$$pf'' \notin L^2(-1, 1) \quad \text{and} \quad f' \notin L^2(-1, 1).$$

A proof can be found in [10].

4. Maximal and Friedrichs Domains

In this section we develop properties of the maximal and Friedrichs domains including various characterizations of them. Recall that the maximal domain $D_{\max}$ is defined as follows. Let $H = L^2(-1, 1)$ and

$$D_{\max} = \{y \in H : (py') \in AC_{\text{loc}}(-1, 1),\ (py')' \in H.$$

The next lemma describes maximal domain functions and their quasi-derivatives.

Lemma 4.1. *Let v be given by* (1.6)*. For every $y \in D_{\max}$ there exist two constants $c, d \in \mathbb{C}$ and a function $g \in H$ such that*

$$y(t) = c + d\, v(t) + \int_{-1}^{t} [v(t) - v(s)]\, g(s) ds, \quad -1 < t < 1. \tag{4.1}$$

$$(py')(t) = d + \int_{-1}^{t} g(s)\, ds, \quad -1 < t < 1. \tag{4.2}$$

Conversely, for every $c, d \in \mathbb{C}$ and $g \in H$, the function y defined by (4.1) *is in $D_{\max}$.*

Proof. Suppose $y \in D_{\max}$. Then $(py')' \in H$. Let $(py')' = g$. Since u, v are linearly independent solutions of $(py')' = 0$, (4.1) follows directly from the variation of parameters formula. (The integrals exist since $v \in H$ and $v \in L^1(-1,1)$.) Differentiating (4.1) yields, for almost all $t \in (-1,1)$,

$$y'(t) = d\, v'(t) + v'(t) \int_{-1}^{t} g(s) ds.$$

Multiplying by $p(t)$, noting that $(pv')(t) = 1$, yields (4.2) . To prove the converse statement note that y is in H since each term of (4.1) is in $L^2(-1,1)$. Clearly, $(py') \in AC_{\rm loc}(-1,1)$, and $(py')' = g \in H$. $\square$

Corollary 4.2. *The quasi-derivative (py') of every maximal domain function y can be continuously extended to the compact interval $[-1,1]$ and is therefore continuous and bounded on $[-1,1]$.*

The proof of the above corollary follows directly from (4.2).

Lemma 4.3. *Let v be given by* (1.6)*. For every $y \in D_{\max}$ we have:*

(1) *Both limits*

$$\lim_{t \to -1^+} \frac{y(t)}{v(t)}, \quad \text{and} \quad \lim_{t \to 1^-} \frac{y(t)}{v(t)} \tag{4.3}$$

exist and are finite.

(2) *For any c, d, $-1 < c < 0 < d < 1$,*

$$(\sqrt{p}) v\Big(\frac{y}{v'}\Big) \in L^2(-1,c), \quad (\sqrt{p}) v\Big(\frac{y}{v'}\Big) \in L^2(d,1). \tag{4.4}$$

Proof. In Section 2 we showed that $z = y/v_m$ is a solution of the regular Legendre equation (2.4). Therefore z can be continuously extended to both endpoints. Since v_m agrees with v near both endpoints (4.3) follows. Part (2) follows from [15, Theorem 4.2, Page 558]. $\square$

Recall the definition of the Friedrichs domain D_F:

$$D_F = \{y \in D_{\max} : (py')(-1) = 0 = (py')(1)\}. \tag{4.5}$$

The next theorem gives a number of equivalent characterizations of the Friedrichs domain; see also [10] and [12].

Theorem 4.4. *Let v be given by* (1.6)*. For any $y \in D_{\max}$, the following statements are equivalent:*

(i) *In* (4.1) *of Lemma* (4.1) *the constant $d = 0$.*

(ii) *y is bounded on $(-1,1)$.*

(iii) *The limits*
$$\frac{y(t)}{v(t)} \to 0 \quad \text{as } t \to -1^+, \text{ and as } t \to +1$$
exist and are finite.

(iv)
$$\lim_{t\to -1^+}(py')(t) = 0 = \lim_{t\to 1^-}(py')(t).$$

(v) *The limits*
$$\lim_{t\to -1^+} y(t), \quad \lim_{t\to 1^-} y(t)$$
exist and are finite.

(vi) $y \in AC[-1,1]$;

(vii) $y' \in L^2(-1,1)$. *Furthermore, this result is best possible in that there exists* $g \in D(S_F)$ *such that* $g' \notin L^q(-1,1)$ *for any* $q > 2$ *and where* g *is independent of* q.

(viii) $p^{1/2}y' \in L^2(-1,1)$;

(ix) *For any* $-1 < c < 0 < d < 1$ *we have*
$$\frac{y}{(\sqrt{p})\, v} \in L^2(-1,c), \quad \frac{y}{(\sqrt{p})\, v} \in L^2(d,1), \quad -1 < c < 0 < d < 1.$$

(x) $y, y' \in AC_{\rm loc}(-1,1)$ *and* $py'' \in L^2(-1,1)$. *Furthermore, this result is best possible in the sense that there exists* $g \in D(S_F)$ *such that* $pg'' \notin L^q(-1,1)$ *for any* $q > 2$, *and where* g *is independent of* q.

Proof. The equivalence of (i), (ii), (iii), (v) and (vi) is clear from (4.1) of Lemma (4.1) and the definition of $v(t)$ in (1.6). We now prove the equivalence of (ii) and (iv) by using the method used to construct regular Legendre equations above. In particular we use the 'regularizing' function v_m and other notation from Section 2. Recall that v_m agrees with v near both endpoints and is positive on $(-1,1)$. As in Section 2, $[\cdot,\cdot]_M$ and $[\cdot,\cdot]_N$ denote the Lagrange brackets of M and N, respectively. Let $z = y/v$ and $x = u/v$. Then
$$\begin{aligned}
-(py')(1) &= [\frac{y}{v_m}, \frac{u}{v_m}]_M(1) = [z,x]_N(1)\\
&= \lim_{t\to 1} z(t) \lim_{t\to 1}(Px')(1) - \lim_{t\to 1} x(t) \lim_{t\to 1}(Pz')(1)\\
&= \lim_{t\to 1} z(t) \lim_{t\to 1}(Px')(1) = 0.
\end{aligned}$$
All these limits exist and are finite since N is a regular problem. Since u is a principal solution and v is a nonprincipal solution it follows that $\lim_{t\to 1} x(t) = 0$. The proof for the endpoint -1 is entirely similar. Thus we have shown that (ii) implies (iv). The converse is obtained by reversing the steps. Thus we conclude that (i) through (vi) are equivalent. Proofs of (vii), $(viii)$, (ix) and (x) can be found in [2]. □

5. Results on the Intervals $(-\infty,-1)$ and $(1,+\infty)$

Here we expand on the observations of Proposition 1.6 regarding the interval $(1,\infty)$. Similar remarks apply to $(-\infty,-1)$ as can be seen from the change of variable $t \to -t$. Consider
$$My = -(py')' = \lambda y \quad \text{on } J_3 = (1,\infty),\ p(t) = 1 - t^2. \tag{5.1}$$

Note that $p(t) < 0$ for $t > 1$; so to conform to the standard notation for Sturm-Liouville problems we study the equivalent equation

$$Ny = -(ry')' = \xi y \quad \text{on } J_3 = (1, \infty), \quad r(t) = t^2 - 1 > 0, \quad \xi = -\lambda. \tag{5.2}$$

Recall from (1.6) that for $\lambda = \xi = 0$ two linearly independent solutions are given by

$$u(t) = 1, \quad v(t) = \frac{1}{2}\ln(|\frac{t-1}{t+1}|) \tag{5.3}$$

Although we focus on the interval $(1, \infty)$ in this section we make the following general observations: For all $t \in \mathbb{R}$, $t \neq \pm 1$, we have

$$(pv')(t) = -1, \tag{5.4}$$

so for any $\lambda \in \mathbb{R}$ and any solution y of (1.1), we have the following Lagrange forms:

$$[y, u] = -py', \quad [y, v] = -y - v(py'), \quad [u, v] = -1, \quad [v, u] = 1. \tag{5.5}$$

These play an important role in the theory of self-adjoint Legendre operators and problems. Observe that, although v blows up at -1 and at $+1$ from both sides it turns out that these forms are defined and finite at all points of $\mathbb{R}$ including -1 and $+1$ provided we define the appropriate one sided limits:

$$[y, u](1^+) = \lim_{t \to 1^+} [y, u](t), \quad [y, u](1^+) = \lim_{t \to -1^-} [y, u](t) \tag{5.6}$$

for all $y \in D_{\max}(J_3)$. Since $u \in L^2(1, 2)$ and $v \in L^2(1, 2)$ it follows from general Sturm-Liouville theory that 1, the left endpoint of J_3 is limit-circle non-oscillatory (LCNO). In particular, all solutions of equations (5.1), (5.2) are in $L^2(1, 2)$ for each $\lambda \in \mathbb{C}$.

In the mathematics and physics literature, when a singular Sturm-Liouville problem is studied on a half line (a, ∞), it is generally assumed that the endpoint a regular. Here the left endpoint $a = 1$ is singular. Therefore regular conditions such as $y(a) = 0$ or, more generally,

$$A_1 y(a) + A_2(py')(a) = 0, \quad A_1, A_2 \in \mathbb{R}, \ (A_1, A_2) \neq (0, 0)$$

do not make sense. Interestingly, as pointed out above, in the Legendre case studied here, while the Dirichlet condition

$$y(1) = 0$$

does not make sense, the Neumann condition

$$(py')(1) = 0, \tag{5.7}$$

does in fact determine a self-adjoint Legendre operator in $L^2(1, \infty)$ - the Friedrichs extension! So while (5.7) has the appearance of a regular Neumann condition it is in fact, in the Legendre case, the analogue of the Dirichlet condition!

By a self-adjoint operator associated with equation (5.2) in $H_3 = L^2(1, \infty)$; i.e., a self-adjoint realization of equation (5.2) in H_3 we mean a self-adjoint restriction of the maximal operator $S_{\max}$ associated with (5.2). This is defined as follows:

$$D_{\max} = \{f : (-1, 1) \to \mathbb{C} \mid f, \ pf' \in AC_{\mathrm{loc}}(-1, 1); f, \ pf' \in H_3\} \tag{5.8}$$

$$S_{\max} f = -(rf')', \quad f \in D_{\max} \tag{5.9}$$

Note that, in contrast to the $(-1, 1)$ case, the Legendre polynomials are not in $D_{\max}$; nor are solutions of (5.2) in general. As in the case for $(-1, 1)$ the following basic lemma holds:

Lemma 5.1. *The operator $S_{\max}$ is densely defined in H_3 and therefore has a unique adjoint in H_3 denoted by $S_{\min}$:*

$$S_{\max}^* = S_{\min}.$$

The minimal operator $S_{\min}$ in H_3 is symmetric, closed, densely defined, and satisfies

$$S_{\min}^* = S_{\max}.$$

Its deficiency index $d = d(S_{\min}) = 1$. If S is a self-adjoint extension of $S_{\min}$, then S is also a self-adjoint restriction of $S_{\max}$ and conversely. Thus we have:

$$S_{\min} \subset S = S^* \subset S_{\max}.$$

The statements in the above lemma are well known facts from Sturm-Liouville theory; for details, see [21].

It is clear from Lemma (5.1) that each self-adjoint operator S is determined by its domain. Next we describe these self-adjoint domains. For this the functions u, v given by (5.3) play an important role, in a sense they form a basis for all self-adjoint boundary conditions [21].

The Legendre operator theory for the interval $(1, \infty)$ is similar to the theory on $(-1, 1)$ except for the fact that the endpoint ∞ is in the limit-point case and therefore there are no boundary conditions required or allowed at ∞.

Thus all self-adjoint Legendre operators in $H_3 = L^2(1, \infty)$ are generated by separated singular self-adjoint boundary conditions at 1. These have the form

$$A_1[y, u](1) + A_2[y, v](1) = 0, \quad A_1, A_2 \in \mathbb{R}, \quad (A_1, A_2) \neq (0, 0). \tag{5.10}$$

Theorem 5.2. *Let $A_1, A_2 \in \mathbb{R}$, $(A_1, A_2) \neq (0, 0)$ and define a linear manifold $D(S)$ to consist of all $y \in D_{\max}$ satisfying* (5.10)*. Then the operator S with domain $D(S)$ is self-adjoint in $L^2(1, \infty)$. Moreover, given any operator S satisfying $S_{\min} \subset S = S^* \subset S_{\max}$ there exist $A_1, A_2 \in \mathbb{R}$, $(A_1, A_2) \neq (0, 0)$ such that $D(S)$, the domain of S, is given by* (5.10)*.*

The proof of the above theorem is based on the next three lemmas.

Lemma 5.3. *Suppose $S_{\min} \subset S = S^* \subset S_{\max}$. Then there exists a function $g \in D(S) \subset D_{\max}$ satisfying*

(1) *g is not in $D_{\min}$ and*
(2) *$[g, g](1) = 0$ such that $D(S)$ consists of all $y \in D_{\max}$ satisfying*
(3)
$$[y, g](1) = 0. \tag{5.11}$$

Conversely, given $g \in D_{\max}$ which satisfies conditions (1) and (2), the set $D(S) \subset D_{\max}$ consisting of all y satisfying (3) is a self-adjoint extension of $S_{\min}$.

The proof of the above lemma follows from the GKN theory (see [1] and [14]) applied to (5.2). The next lemma plays an important role and is called the 'Bracket Decomposition Lemma' in [21].

Lemma 5.4 (Bracket Decomposition Lemma)**.** *For any $y, z \in D_{\max}$ we have*

$$[y, z](1) = [y, v](1)[\overline{z}, u](1) - [y, u](1)[\overline{z}, v](1). \tag{5.12}$$

For a proof of the above lemma, see [21, Pages 175-176].

Lemma 5.5 ([21]). *For any $\alpha, \beta \in \mathbb{C}$ there exists a function $g \in D_{\max}(J_3)$ such that*
$$[g,u](1^+) = \alpha, [g,v](1^+) = \beta. \tag{5.13}$$

Armed with these lemmas we can now proceed to the proof.

Proof of Theorem 5.2. Let $A_1, A_2 \in \mathbb{R}$, $(A_1, A_2) \neq (0,0)$. By Lemma (5.5) there exists a $g \in D_{\max}(J_3)$ such that
$$[g,u](1^+) = A_2, [g,v](1^+) = -A_1. \tag{5.14}$$
From (5.12) we get that for any $y \in D_{\max}$ we have
$$[y,g](1) = [y,v](1)[g,u](1) - [y,u](1)[g,v](1) = A_1[y,u](1) + A_2[y,v](1). \tag{5.15}$$
Now consider the boundary condition
$$A_1[y,u](1) + A_2[y,v](1) = 0. \tag{5.16}$$
If (5.16) holds for all $y \in D_{\max}$, then it follows from Lemma 10.4.1, p175 of [21] that $g \in D_{\min}$. But this implies, also by Lemma 10.4.1, that $(A_1, A_2) \neq (0,0)$ which is a contradiction. From (5.14) it follows that
$$\begin{aligned}[g,g](1) &= [g,v](1)[g,u](1) - [g,u](1)[g,v](1)\\ &= A_1[g,u](1) + A_2[g,v](1) = A_1A_2 - A_2A_1 = 0.\end{aligned}$$
Therefore g satisfies conditions (1) and (2) of Lemma 5.3 and consequently
$$[y,g](1) = A_1[y,u](1) + A_2[y,v](1) = 0 \tag{5.17}$$
is a self-adjoint boundary condition. To prove the converse, reverse the steps in this argument. □

It is clear from Theorem (5.2) that there are an uncountable number of self-adjoint Legendre operators in $L^2(1,\infty)$. It is also clear that the Legendre polynomials P_n are not eigenfunctions of any such operator since they are not in the maximal domain and therefore not in the domain of any self-adjoint restriction S of $D_{\max}$.

Next we study the spectrum of the self-adjoint Legendre operators in $H_3 = L^2(1,\infty)$.

Theorem 5.6. *Let $S_{\min} \subset S = S^* \subset S_{\max}$ where $S_{\min}$ and $S_{\max}$ are the minimal and maximal operators in $L^2(1,\infty)$ associated with* (1.1)*. Then*
- *S has no discrete spectrum.*
- *The essential spectrum $\sigma_e(S)$ is given by $\sigma_e(S) = (-\infty, -\frac{1}{4}]$.*

The proof of the above lemma is given in Proposition 1.6. The next theorem gives the version of Theorem (5.6) for the Legendre equation in the more commonly used form (1.5).

Theorem 5.7. *Let $S_{\min} \subset S = S^* \subset S_{\max}$ where $S_{\min}$ and $S_{\max}$ are the minimal and maximal operators in $L^2(1,\infty)$ associated with the equation* (1.5)*. Then*
- *S has no discrete spectrum.*
- *The essential spectrum $\sigma_e(S)$ is given by $\sigma_e(S) = [\frac{1}{4}, \infty)$.*

The above theorem is obtained from the preceding theorem simply by changing the sign.

6. Legendre operators on the whole line

In this section we study the Legendre equation (1.1) on the whole real line $\mathbb{R}$ and note that, in addition to its singular points at $-\infty$ and $+\infty$, it also has singularities at the interior points -1 and $+1$;we refer to the paper of Zettl [22] for further details in this setting. Since we are studying the equation on both sides of these interior singularities there are in effect interior singularities at -1^-, -1^+ and at $+1^-$, $+1^+$. Our approach is based on the direct sum method developed by Everitt and Zettl [8] for one interior singular point. The modifications needed to apply this approach to two interior singularities, as we do here, is straightforward. This method yields, in a certain natural sense, all self-adjoint Legendre operators in the Hilbert space $L^2(\mathbb{R})$ which we identify with the direct sum

$$L^2(\mathbb{R}) = L^2(-\infty,-1) \dotplus L^2(-1,1) \dotplus L^2(1,\infty).$$

One method for getting such operators is to simply take the direct sum of three operators, one from each of the three separate spaces. However it is interesting to note that not all self-adjoint operators in $L^2(\mathbb{R})$ are generated by such direct sums. This is what makes the three-interval theory interesting: there are many other self-adjoint operators. These are generated by interactions *through* the interior singularities.

As above, let

$$J_1 = (-\infty,-1), \quad J_2 = (-1,1), \quad J_3 = (1,\infty), \quad J_4 = \mathbb{R} = (-\infty,\infty).$$

Let $S_{\min}(J_i)$, $S_{\max}(J_i)$ denote the minimal and maximal operators in $L^2(J_i)$, $i = 1,2,3$ and denote their domains by $D_{\min}(J_i)$, $D_{\max}(J_i)$, respectively.

Definition 6.1. The minimal and maximal Legendre operators $S_{\min}$ and $S_{\max}$ in $L^2(\mathbb{R})$ and their domains $D_{\min}$, $D_{\max}$ are defined as follows:

$$\begin{gathered}
D_{\min} = D_{\min}(J_1) \dotplus D_{\min}(J_2) \dotplus D_{\min}(J_3) \\
D_{\max} = D_{\max}(J_1) \dotplus D_{\max}(J_2) \dotplus D_{\max}(J_3) \\
S_{\min} = S_{\min}(J_1) \dotplus S_{\min}(J_2) \dotplus S_{\min}(J_3) \\
S_{\min} = S_{\max}(J_1) \dotplus S_{\max}(J_2) \dotplus S_{\max}(J_3).
\end{gathered}$$

Lemma 6.2. *The minimal operator $S_{\min}$ is a closed, densely defined, symmetric operator in $L^2(\mathbb{R})$ satisfying*

$$S_{\min}^* = S_{\max}, \; S_{\max}^* = S.$$

Its deficiency index, $d = d(S_{\min}) = 4$. Each self-adjoint extension S of $S_{\min}$ is a restriction of $S_{\max}$; i.e., we have

$$S_{\min} \subset S = S^* \subset S_{\max}.$$

Proof. The adjoint properties follow from the corresponding properties of the component operators and it follows that

$$\operatorname{def}(S_{\min}) = \operatorname{def}(S_{\min}(J_1)) + \operatorname{def}(S_{\min}(J_2)) + \operatorname{def}(S_{\min}(J_3)) = 1+2+1 = 4,$$

since $-\infty$ and $+\infty$ are LP and -1^-, -1^+, $+1^-$, $+1^+$ are all LC. For more details, see [8]. $\square$

Remark 6.3. Although the minimal and maximal operators $S_{\min}$, $S_{\max}$ are the direct sums of the corresponding operators on each of the three intervals we will see below that there are many self-adjoint extensions S of $S_{\min}$ other than those which are simply direct sums of operators from the three intervals.

For $y, z \in D_{\max}$, $y = (y_1, y_2, y_3)$, $z = (z_1, z_2, z_3)$ we define the "three interval" or "whole line" Lagrange sesquilinear from $[\cdot,\cdot]$ as follows:

$$\begin{aligned} &= [y_1, z_1]_1(-1^-) - [y_1, z_1]_1(-\infty) + [y_2, z_2]_2(+1^-) - [y_2, z_2]_2(-1^+) \\ &+ [y_3, z_3]_3(+\infty) - [y_3, z_3]_3(+1^+) \\ &= [y_1, z_1]_1(-1^-) + [y_2, z_2]_2(+1^-) - [y_2, z_2]_2(-1^+) - [y_3, z_3]_3(+1^+). \end{aligned} \tag{6.1}$$

Here $[y_i, z_i]_i$ denotes the Lagrange form on the interval J_i, $i = 1, 2, 3$. In the last step we noted that the Lagrange forms evaluated at $-\infty$ and at $+\infty$ are zero because these are LP endpoints. The fact that each of these one sided limits exists and is finite follows from the one interval theory.

As noted above in (1.6) for $\lambda = 0$ the Legendre equation

$$My = -(py')' = \lambda y \tag{6.2}$$

has two linearly independent solutions

$$u(t) = 1, \quad v(t) = -\frac{1}{2}\ln\left(\left|\frac{t-1}{t+1}\right|\right).$$

Observe that u is defined on all of $\mathbb{R}$ but v blows up logarithmically at the two interior singular points from both sides. Observe that

$$[u, v](t) = u(t)(pv')(t) - v(t)(pu')(t) = 1, \quad -\infty < t < \infty \tag{6.3}$$

where we have taken appropriate one sided limits at ± 1 and for all $y \in D$ we have

$$[y, u] = -py', \quad [y, v] = y - v(py') \tag{6.4}$$

and again by taking appropriate one sided limits, if necessary, $[y, u](t)$ is defined (finitely) for all $t \in \mathbb{R}$. Similarly the vector

$$Y = \begin{pmatrix} [y, u] \\ [y, v] \end{pmatrix} = \begin{pmatrix} -py' \\ y - v(py') \end{pmatrix} \tag{6.5}$$

is well defined. In particular,

$$Y(-1^-), \quad Y(-1^+), \quad Y(1^-), \quad Y(1^+) \tag{6.6}$$

are all well defined and finite. Note also that $Y(-\infty)$ and $Y(-\infty)$ are well defined and

$$Y(-\infty) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = Y(\infty). \tag{6.7}$$

Remark 6.4. For any $y \in D_{\max}$ the one sided limits of py' and of $y - v(py')$ exist and are finite at -1 and at 1. Hence if py' has a nonzero finite limit then y must blow up logarithmically.

Now we can state the theorem giving the characterization of all self-adjoint extensions S of the minimal operator $S_{\min}$; recall that these are all operators S satisfying $S_{\min} \subset S = S^* \subset S_{\max}$.

Theorem 6.5. *Suppose $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$, $D = (d_{ij})$, are 4×2 complex matrices satisfying the following two conditions:*
$$\operatorname{rank}(A, B, C, D) = 4, \tag{6.8}$$
$$AEA^* - BEB^* + CEC^* - DED^* = 0, \quad E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{6.9}$$
Componentwise, conditions (6.9) *are written for $j, k = 1, 2, 3, 4$, as*
$$(a_{j1}\overline{a}_{k2} - a_{j2}\overline{a}_{k1}) - (b_{j1}\overline{b}_{k2} - b_{j2}\overline{b}_{k1}) + (c_{j1}\overline{c}_{k2} - c_{j2}\overline{c}_{k1}) - (d_{j1}\overline{d}_{k2} - d_{j2}\overline{d}_{k1}) = 0.$$
Define D to be the set of all $y \in D_{\max}$ satisfying
$$AY(-1^-) + BY(-1^+) + CY(1^-) + DY(1^+) = 0, \tag{6.10}$$
where
$$Y = \begin{pmatrix} -py' \\ y - v(py') \end{pmatrix}.$$
Then D is the domain of a self-adjoint extension S of the three interval minimal operator $S_{\min}$.

Conversely, given any self-adjoint operator S satisfying $S_{\min} \subset S = S^* \subset S_{\max}$ with domain $D = D(S)$, there exist 2×4 complex matrices $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$, $D = (d_{ij})$, satisfying conditions (6.8) *and* (6.9)*, such that $D(S)$ is given by* (6.10)*.*

The proof of the above theorem is based on three lemmas which we establish next. Also see Example [21, 13.3.4, pp. 273-275].

Remark 6.6. The boundary conditions are given by (6.10); (6.8) determines the *number* of independent conditions and (6.9) specifies the conditions on the boundary conditions needed for self-adjointness.

Using the three interval Lagrange form the next lemma gives an extension of the GKN characterization for the whole line Legendre problem.

Lemma 6.7. *Suppose $S_{\min} \subset S = S^* \subset S_{\max}$. Then there exist $v_1, v_2, v_3, v_4 \in D(S) \subset D_{\max}$ satisfying conditions*

1. *v_1, v_2, v_3, v_4 are linear independent modulo $D_{\min}$; i.e., no nontrivial linear combination is in $D_{\min}$;*
2. *$[v_i, v_j] = 0$, $i, j = 1, 2, 3, 4$, such that $D(S)$ consists of all $y \in D_{\max}$ satisfying*
3. *$[y, v_j] = 0$, $j = 1, 2, 3, 4$.*

Conversely, given $v_1, v_2, v_3, v_4 \in D_{\max}$ which satisfy conditions (1) and (2) the set $D(S) \subset D_{\max}$ consisting of all y satisfying (3) is a self-adjoint extension of $S_{\min}$.

The above lemma follows from [8, Theorem 3.3] extended to three intervals and applied to the Legendre equation. The next lemma is called the 'Bracket Decomposition' Lemma in [21]. It applies to each of the intervals J_i, $i = 1, 2, 3$ but for simplicity of notation we omit the subscripts.

Lemma 6.8 (Bracket Decomposition Lemma)**.** *Let $J_i = (a, b)$, let $y, z, u, v \in D_{\max} = D_{\max}(J_i)$, $J_i = (a, b)$ and assume that $[v, u](c) = 1$ for some c, $a \le c \le b$, then*
$$[y, z](c) = [y, v](c)[\overline{z}, \overline{u}](c) - [y, \overline{u}](c)[\overline{z}, v](c). \tag{6.11}$$

For a proof of the above lemma, see [21, Pages 175-176]. The next lemma extends [21, Proposition 10.4.2, Pages 185-186] from the one interval case to the three intervals J_i, $i = 1, 2, 3$.

For this lemma we extend the definitions of the functions u, v given by (1.6) but we will continue to use the same notation.

$$u(t) = \begin{cases} 1 & -1 < t < 1,\ -2 < t < -1,\ 1 < t < 2, \\ 0 & |t| > 3, \end{cases} \tag{6.12}$$

$$v(t) = \begin{cases} -\frac{1}{2}\ln(\frac{t-1}{t+1}) & -1 < t < 1,\ -2 < t < -1,\ 1 < t < 2, \\ 0 & |t| > 3, \end{cases} \tag{6.13}$$

and define both functions on the intervals $[-3, -2]$, $[2, 3]$ so that they are continuously differentiable on these intervals.

Lemma 6.9. *Let $\alpha, \beta, \gamma, \delta \in \mathbb{C}$.*

- *There exists a $g \in D_{\max}(J_2)$ which is not in $D_{\min}(J_2)$ such that*
$$[g, u](-1^+) = \alpha, [g, v](-1^+) = \beta, [g, u](1^+) = \gamma, [g, v](1^+) = \delta. \tag{6.14}$$
- *There exists a $g \in D_{\max}(J_1)$ which is not in $D_{\min}(J_1)$ such that*
$$[g, u](-1^-) = \alpha, [g, v](-1^-) = \beta. \tag{6.15}$$
- *There exists a $g \in D_{\max}(J_3)$ which is not in $D_{\min}(J_3)$ such that*
$$[g, u](1^+) = \gamma, [g, v](1^+) = \delta. \tag{6.16}$$

Proof of Theorem 6.5. The method is the same as the method used in the proof of Theorem (5.2) but the computations are longer; it consists in showing that each part of Theorem (6.5) is equivalent to the corresponding part of Lemma (6.7). For more details, see [21]. □

Example 6.10. A Self-Adjoint Legendre Operator on the whole real line. The boundary condition

$$(py')(-1^-) = (py')(-1^+) = (py')(1^-) = (py')(1^+) = 0 \tag{6.17}$$

satisfies the conditions of Theorem (6.5) and therefore determines a self-adjoint operator S_L in $L^2(\mathbb{R})$. Let S_1 in $L^2(-\infty, -1)$ be determined by $(py')(-1^-) = 0$, $S_2 = S_F$ in $(-1, 1)$ by $(py')(-1^+) = (py')(1^-) = 0$, and S_3 by $(py')(1^+) = 0$. then each S_i is self-adjoint and the direct sum:

$$S = S_1 \dotplus S_2 \dotplus S_3 \tag{6.18}$$

is a self-adjoint operator in $L^2(-\infty, \infty)$. It is well known that the essential spectrum of a direct sum of operators is the union of the essential spectra of these operators. From this, Proposition (1.6), and the fact that the spectrum of S_2 is discrete we have

$$\sigma_e(S) = (-\infty, -1/4].$$

Note that the Legendre polynomials satisfy all four conditions of (6.17). Therefore the triples

$$P_L = (0, P_n, 0)\ (n \in \mathbb{N}_0), \tag{6.19}$$

are eigenfunctions of S_L with eigenvalues

$$\lambda_n = n(n+1)\ (n \in \mathbb{N}_0). \tag{6.20}$$

Thus we may conclude that

$$(-\infty, -1/4] \cup \{\lambda_n = n(n+1),\ n \in \mathbb{N}_0\} \subset \sigma(S). \tag{6.21}$$

We conjecture that

$$(-\infty, -1/4] \cup \{\lambda_n = n(n+1) :\ n \in \mathbb{N}_0\} = \sigma(S). \tag{6.22}$$

Example 6.11. By using equation (1.1) on the interval $(-1, 1)$ and equation (1.5) on the intervals $(-\infty, -1)$ and $(1, \infty)$, in other words by using $p(t) = 1 - t^2$ for $-1 < t < 1$ and $p(t) = t^2 - 1$ for $-\infty < t < -1$ and for $1 < t < \infty$ and applying the three-interval theory as in Example (1) we obtain an operator whose essential spectrum is $[1/4, \infty)$ and whose discrete spectrum contains the classical Legendre eigenvalues:

$$\{\lambda_n = n(n+1) :\ n \in \mathbb{N}_0\}.$$

Note that $\lambda_0 = 0$ is below the essential spectrum and all other eigenvalues λ_n for $n > 0$ are embedded in the essential spectrum. Each triple

$$(0, P_n, 0) \text{ when } n \in \mathbb{N}_0$$

is an eigenfunction with eigenvalue λ_n for $n \in \mathbb{N}_0$.

Conclusion. In this paper we have studied spectral theory in Hilbert spaces of square-integrable functions associated with the Legendre expression (1.1), this is known as the right-definite theory. There is also a left-definite theory, stemming from the work of Pleijel [17], see also [9], [20], [21] and the references in these papers. This takes place in the setting of Hilbert-Sobolev spaces. There is a third approach, developed by Littlejohn and Wellman [13], and used in [9] for (1.1) - also called 'left-definite' by these authors - which takes place in the setting of an infinite number of Hilbert-Sobolev spaces. We plan to write a sequel to this paper discussing these other two approaches.

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