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ENTIRE SOLUTIONS FOR A NONLINEAR DIFFERENTIAL EQUATION

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Abstract. In this article, we study the existence of solutions to the differential equation
$$f^n(z) + P(f) = P_1 e^{h_1} + P_2 e^{h_2},$$
where $n \geq 2$ is an positive integer, f is a transcendental entire function, $P(f)$ is a differential polynomial in f of degree less than or equal $n-1$, P_1, P_2 are small functions of e^z, h_1, h_2 are polynomials, and z is in the open complex plane $\mathbb{C}$. Our results extend those obtained by Li [6, 7, 8, 9].

1. Introduction and main results

Nevanlinna value distribution theory of meromorphic functions has been extensively applied to resolve growth (see[6]), value distribution [6], and solvability of meromorphic solutions of linear and nonlinear differential equations [4, 6, 10, 11]. Considering meromorphic functions f in the complex plane, we assume that the reader is familiar with the standard notations and results such as the proximity function $m(r, f)$, counting function $N(r, f)$, characteristic function $T(r, f)$, the first and second main theorems, lemma on the logarithmic derivatives etc. of Nevanlinna theory; see e.g. [3, 6]. Given a meromorphic function f, we shall call a meromorphic function $a(z)$ a small function of $f(z)$ if $T(r, a) = S(r, f)$, where $S(r, f)$ is used to denote any quantity that satisfies $S(r, f) = o(T(r, f))$ as $r \to \infty$, possibly outside a set of r of finite logarithmic measure. A differential polynomial $P(f)$ in f is a polynomial in f and its derivatives with small functions of f as the coefficients. The notation $\mathscr{F}$ is defined to the family of all meromorphic functions which satisfy $\overline{N}(r, \frac{1}{h}) + \overline{N}(r, h) = S(r, h)$. Note that all functions in family $\mathscr{F}$ are transcendental, and all functions of the form $be^{\lambda z}$ are functions in family $\mathscr{F}$, where λ is any nonzero constant and b is a rational function.

In 2006, Li and Yang [7, 11] obtain the following results.

Theorem 1.1. *Let $n \geq 4$ be an integer, and $P(f)$ denote an algebraic differential polynomial in f of degree $\leq n-3$. Let P_1, P_2 be two nonzero polynomials, α_1 and*

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α_2 be two nonzero constants with $\frac{\alpha_1}{\alpha_2} \neq$ rational. Then the differential equation

$$f^n(z) + P(f) = P_1 e^{\alpha_1 z} + P_2 e^{\alpha_2 z}$$

has no transcendental entire solutions.

Theorem 1.2. *Let $n \geq 3$ be an integer, and $P(f)$ be an algebraic differential polynomial in f of degree $\leq n-3$, $b(z)$ be a meromorphic function, and λ, c_1, c_2 and three nonzero constants, Then the differential equation*

$$f^n(z) + P(f) = b(z)(c_1 e^{\lambda z} + c_2 e^{-\lambda z})$$

has no transcendental entire solutions $f(z)$, satisfying $T(r, b) = S(r, f)$.

Recently, Considering the degree of the differential polynomial $P(f)$ of $n-2$ or $n-1$, P. Li [9] proved the following results which are improvements or complementarity of Theorems 1.1 and 1.2.

Theorem 1.3. *Let $n \geq 2$ be an integer. Let f be a transcendental entire function, $P(f)$ be a differential polynomial in f of degree $\leq n-1$. If*

$$f^n(z) + P(f) = P_1 e^{\alpha_1 z} + P_2 e^{\alpha_2 z}, \tag{1.1}$$

where $P_i (i = 1, 2)$ are nonvanshing small functions of e^z, $\alpha_i (i = 1, 2)$ are positive numbers satisfying $(n-1)\alpha_2 \geq n\alpha_1 > 0$, then there exists a small function γ of f such that

$$(f - \gamma)^n = P_2 e^{\alpha_2 z}. \tag{1.2}$$

Theorem 1.4. *Let $n \geq 2$ be an integer, α_1, α_2 be real numbers and $\alpha_1 < 0 < \alpha_2$. Let P_1, P_2 be small functions of e^z. If there exists a transcendental entire function f satisfying the differential equation (1.1), where $P(f)$ is a differential polynomial in f of degree not exceeding $n-2$, then $\alpha_1 + \alpha_2 = 0$, and there exist constants c_1, c_2 and small functions β_1, β_2 with respect to f such that*

$$f = c_1 \beta_1 e^{\alpha_1 z/n} + c_2 \beta_2 e^{\alpha_2 z/n}, \tag{1.3}$$

moreover, $\beta_i^n = P_i$, $i = 1, 2$.

Theorem 1.5. *Let $n \geq 2$ be an integer, α_1, α_2 be positive numbers satisfying $(n-1)\alpha_2 \geq n\alpha_1 > 0$. Let P_1, P_2 be small functions of e^z. If $\frac{\alpha_1}{\alpha_2}$ is irrational, then the differential equation (1.1) has no entire solutions, where $P(f)$ is a differential polynomial in f of degree $\leq n-1$.*

Remark 1.6. By an example, Li [9] pointed if the degree of $P(f)$ is $n-1$, then the solutions of (1.1) may not be the form in (1.3).

It is natural to ask whether $\alpha_1 z$ and $\alpha_2 z$ in (1.1) can be replaced by two polynomials. In this article, by the same method as in [9], we obtain the following results.

Theorem 1.7. *Let $n \geq 2$ be an integer. Let f be a transcendental entire function, $P(f)$ be a differential polynomial in f of degree $\leq n-1$. If*

$$f^n(z) + P(f) = P_1 e^{Q_1(z)} + P_2 e^{Q_2(z)}, \tag{1.4}$$

where $P_i (i = 1, 2)$ are nonvanshing small meromorphic functions of e^z, $Q_1(z) = \alpha_k z^k + \alpha_{k-1} z^{k-1} + \dots + \alpha_1 z + \alpha_0$, $Q_2(z) = \beta_k z^k + \beta_{k-1} z^{k-1} + \dots + \beta_1 z + \beta_0$ are two polynomials satisfying $(n-1)\beta_k \geq n\alpha_k > 0$ (where $\alpha_{k-1}, \dots \alpha_0, \beta_{k-1}, \dots \beta_0$

are finite constants and $k \geq 1$) is a positive integer, then there exists a small meromorphic function γ of f such that

$$(f-\gamma)^n = P_2 e^{Q_2}. \tag{1.5}$$

Theorem 1.8. *Let $n \geq 2$ be an integer and P_1, P_2 be small functions of e^z. If there exists a transcendental entire function f satisfying the differential equation (1.4), where $P(f)$ is a differential polynomial in f of degree not exceeding $n-2$ and $\alpha_k < 0 < \beta_k$, then $\alpha_k + \beta_k = 0$, and there exist constants c_1, c_2 and small functions β_1, β_2 with respect to f such that*

$$f = c_1\beta_1 e^{\frac{Q_1}{n}} + c_2\beta_2 e^{\frac{Q_2}{n}},$$

moreover, $\beta_i^n = P_i$, $i = 1, 2$.

Theorem 1.9. *Let $n \geq 2$ be an integer, P_1, P_2 be small functions of e^z. If $\frac{\alpha_k}{\beta_k}$ is irrational, then the differential equation (1.4) has no entire solutions, where $P(f)$ is a differential polynomial in f of degree $\leq n-1$ and $(n-1)\beta_k \geq n\alpha_k > 0$.*

Obviously, our results generalize the results in [6, 7, 8, 9].

2. Preliminary Lemmas

In order to prove our theorems, we need the following lemmas. First, we need the following well-known Clunie's lemma, which has been extensively applied in studying the value distribution of a differential polynomial $P(z, f)$, as well as the growth estimates of solutions and meromorphic solvability of differential equations in the complex plane.

Lemma 2.1 ([1, 2])**.** *Let f be a transcendental meromorphic solution of*

$$f^n A(z, f) = B(z, f),$$

where $A(z, f)$, $B(z, f)$ are differential polynomials in f and its derivatives with small meromorphic coefficients a_λ, in the sense of $T(r, a_\lambda) = S(r, f)$ for all $\lambda \in I$, where I is an index set. If the total degree of $B(z, f)$ as a polynomial in f and its derivatives is less than or equal n, then $m(r, A(z, f)) = S(r, f)$.

Lemma 2.2 ([3])**.** *Suppose that f is a nonconstant meromorphic function and $F = f^n + Q(f)$, where $Q(f)$ is a differential polynomial in f with degree $\leq n-1$. If $N(r, f) + N(r, \frac{1}{F}) = S(r, f)$, then*

$$F = (f+\gamma)^n,$$

whereby γ is meromorphic and $T(r, \gamma) = S(r, f)$

Lemma 2.3 ([8])**.** *Suppose that h is a function in family $\mathscr{F}$. Let $f = a_0h^p + a_1h^{p-1} + \cdots + a_p$, and $g = b_0h^q + b_1h^{q-1} + \cdots + b_q$ be polynomials in h with all coefficients being small functions of h and $a_0b_0a_p \neq 0$ If $q \leq p$, then $m(r, \frac{g}{f}) = S(r, h)$.*

3. Proofs of main theorems

Proof of Theorem 1.7. First of all, we write $P(f)$ as

$$P(f)=\sum_{j=0}^{n-1} b_j M_j(f), \tag{3.1}$$

where b_j are small functions of f, $M_0(f)=1$, $M_j(f)(j=1,2,\dots,n-1)$ are homogeneous differential monomials in f of degree j. Without loss of generality, we assume that $b_0\not\equiv 0$, otherwise, we do the transformation $f=f_1+c$ for a suitable constant c. From (1.4), we have

$$\frac{1}{P_1e^{Q_1}+P_2e^{Q_2}-b_0}+\sum_{j=1}^{n-1}\frac{b_j}{P_1e^{Q_1}+P_2e^{Q_2}-b_0}\frac{M_j(f)}{f^j}(\frac{1}{f})^{n-j}=(\frac{1}{f})^n. \tag{3.2}$$

Note that $m(r,\frac{M_j(f)}{f^j})=S(r,f)$,

$$\begin{aligned} &m(r,\frac{1}{P_1e^{Q_1(z)}+P_2e^{Q_2(z)}-b_0})\\ &=m(r,\frac{1}{P_1e^{\alpha_{k-1}z^{k-1}+\dots+\alpha_0}e^{\alpha_k z^k}+P_2e^{\beta_{k-1}z^{k-1}+\dots+\beta_0}e^{\beta_k z^k}-b_0}),\end{aligned}$$

where P_1, P_2, $e^{\alpha_{k-1}z^{k-1}+\dots+\alpha_0}$, $e^{\beta_{k-1}z^{k-1}+\dots+\beta_0}$ are small functions of e^{z^k}.

We take $h=e^{z^k}$, $q=0,p=\beta_k$, by Lemma 2.3, we obtain

$$\begin{aligned} &m(r,\frac{1}{P_1e^{Q_1(z)}+P_2e^{Q_2(z)}-b_0})\\ &=S(r,e^{z^k})=S(r,P_1e^{Q_1(z)}+P_2e^{Q_2(z)}-b_0)=S(r,f(z)).\end{aligned}$$

Therefore, the left-hand side of (3.2) is a polynomial in $1/f$ of degree at most $n-1$ with coefficients being small proximate functions of $1/f$. Hence

$$m(r,\frac{1}{f})=S(r,f). \tag{3.3}$$

Taking the derivatives in both sides of (1.4) gives

$$nf^{n-1}f'+(P(f))'=(P_1'+Q_1'P_1)e^{Q_1}+(P_2'+Q_2'P_2)e^{Q_2}. \tag{3.4}$$

By eliminating e^{Q_1} and e^{Q_2}, respectively from (1.4) and the above equation, we obtain

$$(P_2'+Q_2'P_2)f^n-P_2nf^{n-1}f'+(P_2'+Q_2'P_2)P(f)-P_2(P(f))'=\beta e^{Q_1} \tag{3.5}$$

$$(P_1'+Q_1'P_1)f^n-P_1nf^{n-1}f'+(P_1'+Q_1'P_1)P(f)-P_1(P(f))'=-\beta e^{Q_2}, \tag{3.6}$$

where $\beta=P_1P_2'-P_2P_1'+(Q_2'-Q_1')P_1P_2$ which is a small function of f. We note that β cannot vanish identically, otherwise, by integration we obtain $e^{Q_2-Q_1}=C\frac{P_1}{P_2}$ for a constant, which is impossible. From (3.5) and (3.6), we obtain

$$m(r,e^{Q_j})\le nT(r,f)+S(r,f),\quad j=1,2. \tag{3.7}$$

On the other hand, from (1.4), we have

$$nT(r,f)=m(r,f^n)=m(r,f^n+P(f))\le T(r,P_1e^{Q_1}+P_2e^{Q_2})+S(r,f). \tag{3.8}$$

Therefore, $S(r,e^{Q_1}) = S(r,e^{Q_2}) = S(r,f) := S(r)$. From (3.2), we have

$$\frac{e^{Q_i}}{p_1e^{Q_1}+p_2e^{Q_2}-b_0} + \sum_{j=1}^{n-1}\frac{b_je^{Q_i}}{p_1e^{Q_1}+p_2e^{Q_2}-b_0}\frac{M_j(f)}{f^j}\frac{1}{f^{n-j}} = \frac{e^{Q_i}}{f^n}, \quad i=1,2.$$

It follows that

$$m(r,\frac{e^{Q_i}}{f^n}) = S(r), \quad i=1,2. \tag{3.9}$$

Next, we prove

$$m(r,\frac{e^{Q_1}}{f^{n-1}}) = S(r). \tag{3.10}$$

For a fixed $r>0$, let $z=re^{i\theta}$. The interval $[0,2\pi)$ can be expressed as the union of the following three disjoint sets:

$$\begin{gathered}
E_1 = \{\theta\in[0,2\pi) | \frac{|f(z)|}{|e^{Q_2(z)-Q_1(z)}|} \le 1\},\\
E_2 = \{\theta\in[0,2\pi) | \frac{|f(z)|}{|e^{Q_2(z)-Q_1(z)}|} > 1, |e^{z^k}| \le 1\},\\
E_3 = \{\theta\in[0,2\pi) | \frac{|f(z)|}{|e^{Q_2(z)-Q_1(z)}|} > 1, |e^{z^k}| > 1\}.
\end{gathered}$$

By the definition of the proximate function, we have

$$m(r,\frac{e^{Q_1(z)}}{f^{n-1}(z)}) = \frac{1}{2\pi}\int_0^{2\pi}\log^+|\frac{e^{Q_1(z)}}{f^{n-1}(z)}|d\theta = I_1+I_2+I_3,$$

where

$$I_j = \frac{1}{2\pi}\int_{E_j}\log^+|\frac{e^{Q_1(z)}}{f^{n-1}(z)}|d\theta, \quad (j=1,2,3).$$

For $\theta\in E_1$, we have $|f(z)|\le|e^{Q_2(z)-Q_1(z)}|$. Since $\frac{e^{Q_1(z)}}{f^{n-1}(z)} = \frac{e^{Q_2(z)}}{f^n(z)}\frac{f(z)}{e^{Q_2(z)-Q_1(z)}}$, we obtain

$$I_1 \le m(r,\frac{e^{Q_2}}{f^n}) = S(r).$$

For $\theta\in E_2$, we have $|e^{Q_1(z)}| = |e^{\alpha_k z^k(1+o(1))}| \le 1$, and thus $|\frac{e^{Q_1(z)}}{f^{n-1}(z)}| \le \frac{1}{|f^{n-1}(z)|}$. It follows from (3.3) that

$$I_2 \le m(r,\frac{1}{f^{n-1}}) = S(r).$$

For $\theta\in E_3$, we have $|f(z)| > |e^{Q_2(z)-Q_1(z)}|$. Therefore,

$$\begin{aligned}
|\frac{e^{Q_1(z)}}{f^{n-1}(z)}| &\le \frac{|e^{Q_1(z)}|}{|e^{(n-1)(Q_2(z)-Q_1(z))}|}\\
&= \frac{1}{|e^{(n-1)Q_2(z)-nQ_1(z)}|} = \frac{1}{|e^{((n-1)\beta_k-n\alpha_k)z^k(1+o(1))}|}.
\end{aligned}$$

By the assumption $(n-1)\beta_k \ge n\alpha_k > 0$, we obtain $|\frac{e^{Q_1(z)}}{f^{n-1}(z)}| \le 1$. Therefore, we have $I_3 = 0$. Hence (3.10) holds.

It follows from (3.5) that

$$f^{n-1}\varphi = \beta\frac{e^{Q_1}}{f^{n-1}}f^{n-1} - R(f),$$

where $\varphi = (P_2' + P_2Q_2')f - nP_2f'$, and

$$R(f) = (P_2' + P_2Q_2')P(f) - P_2P'(f)$$

which is a differential polynomial in f of degree at most $n-1$. By Lemma 2.1, we obtain $m(r, \varphi) = S(r, f)$. Note that since φ is entire, we have $N(r, \varphi) = S(r, \varphi) = S(r, f)$. Hence $T(r, \varphi) = S(r, f)$, i.e., φ is a small function of f, By the definition of φ, we obtain

$$f' = \frac{P_2' + Q_2'P_2}{nP_2}f - \frac{\varphi}{nP_2}.$$

Substituting the above equation into (3.6) gives

$$f^n - \frac{nP_1\varphi}{\beta}f^{n-1} - \frac{P_2(P_1' + Q_1'P_1)}{\beta}P(f) + \frac{P_1P_2}{\beta}(P(f))' = P_2e^{Q_2}.$$

By Lemma 2.2, we see that there exists a small function γ of f such that $(f-\gamma)^n = P_2e^{Q_2}$. This also completes the proof of Theorem 1.7. $\square$

Proof of Theorem 1.8. We discuss only the case $\alpha_k + \beta_k \geq 0$. The case $\alpha_k + \beta_k \leq 0$ can be discussed similarly. Suppose that f is a transcendental entire solution of (1.4). Similar to the proof of Theorem 1.7, we can still get (3.3)-(3.9). For a fixed $r > 0$, let $z = re^{i\theta}$. We can express the interval $[0, 2\pi)$ as the union of the following three disjoint sets:

$$\begin{gathered}
E_1 = \{\theta \in [0, 2\pi) | \frac{|f^2(z)|}{|e^{Q_2(z)-Q_1(z)}|} \leq 1|\},\\
E_2 = \{\theta \in [0, 2\pi) | \frac{|f^2(z)|}{|e^{Q_2(z)-Q_1(z)}|} > 1, |e^{z^k}| \leq 1\},\\
E_3 = \{\theta \in [0, 2\pi) | \frac{|f^2(z)|}{|e^{Q_2(z)-Q_1(z)}|} > 1, |e^{z^k}| > 1\}.
\end{gathered}$$

By the definition of the proximate function, we have

$$m(r, \frac{e^{Q_1(z)+Q_2(z)}}{f^{2n-2}(z)}) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |\frac{e^{Q_1(z)+Q_2(z)}}{f^{2n-2}(z)}|d\theta = I_1 + I_2 + I_3,$$

where

$$I_j = \frac{1}{2\pi}\int_{E_j} \log^+ |\frac{e^{Q_1(z)+Q_2(z)}}{f^{2n-2}(z)}|d\theta, \quad j = 1, 2, 3.$$

For $\theta \in E_1$, we have

$$|\frac{e^{Q_1(z)+Q_2(z)}}{f^{2n-2}(z)}| = |\frac{e^{2Q_2(z)}}{f^{2n}(z)}\frac{f^2(z)}{e^{Q_2(z)-Q_1(z)}}| \leq |\frac{e^{Q_2(z)}}{f^n(z)}|^2.$$

Thus by (3.9), we obtain $I_1 \leq S(r)$. For $\theta \in E_2$, it follows from $|e^{z^k}| \leq 1$ and $\alpha_k + \beta_k \geq 0$ that $|e^{(\alpha_k+\beta_k)z^k(1+o(1))}| \leq 1$. Therefore,

$$|\frac{e^{Q_1(z)+Q_2(z)}}{f^{2n-2}(z)}| \leq \frac{1}{|f^{2n-2}(z)|}.$$

Then by (3.3), we obtain $I_2 \le S(r)$. For $\theta \in E_3$, we have $|f^2(z)| > |e^{Q_2(z)-Q_1(z)}|$. Thus

$$\Big|\frac{e^{Q_1(z)+Q_2(z)}}{f^{2n-2}(z)}\Big| < \frac{|e^{Q_1(z)+Q_2(z)}|}{|e^{(n-1)(Q_2(z)-Q_1(z))}|} = \frac{1}{|e^{(n-2)Q_2(z)-nQ_1(z)}|}$$
$$= \frac{1}{|e^{[(n-2)\beta_k - n\alpha_k]z^k(1+o(1))}|} \le 1.$$

It follows that $I_3 \le S(r)$. Hence we have

$$m(r, \frac{e^{Q_1+Q_2}}{f^{2n-2}}) = S(r,f). \tag{3.11}$$

Multiplying (3.5) by (3.6) gives

$$f^{2n-2}\varphi + Q(f) = -\beta^2 e^{Q_1+Q_2}, \tag{3.12}$$

where $Q(f)$ is a differential polynomial in f of degree at most $2n-2$, and

$$\varphi = ((P_1' + Q_1'P_1)f - nP_1f')((P_2' + Q_2'P_2)f - nP_2f')). \tag{3.13}$$

From (3.12) and by Lemma 2.1, we obtain $m(r,\varphi) = S(r,f)$. Therefore, $T(r,\varphi) = S(r,f)$.

If $(P_1' + Q_1'P_1)f - nP_1f' \equiv 0$, then by integration we obtain $f^n = cP_1e^{Q_1}$, for a nonzero constant c. Therefore, $f = ae^{\frac{Q_1}{n}}$ for a small function a of f. Thus we see that the left-hand side of (1.4) is a polynomial in $e^{\frac{Q_1}{n}}$ of degree n. However, the right-hand side of (1.4) cannot be a polynomial in $e^{\frac{Q_1}{n}}$. Hence $(P_1' + Q_1'P_1)f - nP_1f' \not\equiv 0$. Similarly, we have $(P_2' + Q_2'P_2)f - nP_2f' \not\equiv 0$. Therefore, $\varphi \not\equiv 0$. Let

$$(P_2' + Q_2'P_2)f - nP_2f' = h. \tag{3.14}$$

Then we have

$$(P_1' + Q_1'P_1)f - nP_1f' = \frac{\varphi}{h}. \tag{3.15}$$

By eliminating f' and f, respectively from (3.14) and (3.15), we obtain

$$f = \frac{P_1}{\beta}h - \frac{\varphi P_2}{\beta}\frac{1}{h}, \tag{3.16}$$

$$f' = \frac{P_1' + Q_1'P_1}{n\beta}h - \frac{P_2' + Q_2'P_2}{n\beta}\frac{\varphi}{h}, \tag{3.17}$$

where $\beta = P_1P_2' - P_2P_1' + (Q_2' - Q_1')P_1P_2$ which is a small function of f, and cannot vanish identically. From (3.16), we see that

$$2T(r,h) = T(r,f) + S(r,f).$$

Therefore, any small function of f is also a small function of h. And from the definition of φ we see that h is a function in family $\mathscr{F}$. Thus $\frac{h'}{h}$ is a small function of f. By taking derivative in both sides of (3.16), we obtain

$$f' = ((\frac{P_1}{\beta})' + \frac{P_1}{\beta}\frac{h'}{h})h - ((\frac{\varphi P_2}{\beta})' - \frac{\varphi P_2}{\beta}\frac{h'}{h})\frac{1}{h}. \tag{3.18}$$

Comparing the coefficients of the right-hand side of (3.17) and (3.18), we deduce that

$$\frac{P_1'+Q_1'P_1}{n\beta}=(\frac{P_1}{\beta})'+\frac{P_1}{\beta}\frac{h'}{h}, \tag{3.19}$$

$$\frac{(P_2'+Q_2'P_2)\varphi}{n\beta}=(\frac{\varphi P_2}{\beta})'-\frac{\varphi P_2}{\beta}\frac{h'}{h}. \tag{3.20}$$

By integrating (3.19) and (3.20), respectively, we obtain

$$P_1e^{Q_1}=d_1(\frac{P_1}{\beta}h)^n,\quad P_2e^{Q_2}=d_2(\frac{\varphi P_2}{\beta}\frac{1}{h})^n, \tag{3.21}$$

where d_1 and d_2 are two nonzero constants. From the above two equations, we see that there exist two small functions β_1 and β_2 of e^z such that $P_i=\beta_i^n$, $i=1,2$, and

$$P_1P_2e^{Q_1+Q_2}=d_1d_2(\frac{P_1P_2\varphi}{\beta^2})^n. \tag{3.22}$$

The right-hand side of the above equation is a small function of f, and thus a small function of e^{z^k}. Therefore, the above equation holds only when $\alpha_k+\beta_k\equiv 0$. Furthermore, from (3.21), we see that there exist two nonzero constants c_1 and c_2 such that

$$\frac{P_1}{\beta}h=c_1\beta_1e^{\frac{Q_1}{n}},\quad \frac{P_2\varphi}{\beta}\frac{1}{h}=-c_2\beta_2e^{\frac{Q_2}{n}}. \tag{3.23}$$

Finally, from (3.16), we obtain (1.8). □

Proof of Theorem 1.9. If f is a transcendental entire solution of (1.4), then by Theorem 1.7, there exists a small function γ of f such that (1.5) holds. And thus $N(r,\frac{1}{f-\gamma})=S(r,f)$, i.e., γ is an exceptional small function of f. Equation (1.5) also shows that there exist two small functions ω_1 and ω_2 of f such that $f'=\omega_1f+\omega_2$. By substituting this equation into (1.4), we see that $P_1e^{Q_1}$ is a polynomial in f of degree $t<n$. By Lemma 2.2, there exist two small functions a and γ_1 of f such that

$$a(f-\gamma_1)^t=P_1e^{Q_1}.$$

Therefore, γ_1 is also an exceptional small function of f. Since any transcendental entire function cannot have two exceptional small functions, we deduce that $\gamma_1=\gamma$. From (1.5) and above equation, we obtain

$$e^{nQ_1-tQ_2}=\frac{P_2^ta^n}{P_1^n}. \tag{3.24}$$

The right-hand side of the above equation is small function of f, and thus a small function of e^z. Hence we obtain $nQ_1-tQ_2\equiv 0$. Therefore, $\lim_{z\to\infty}\frac{Q_1}{Q_2}=\frac{\alpha_k}{\beta_k}=\frac{t}{n}$ must be a rational number, which contradicts the assumption. This also completes the proof of Theorem 1.9. □

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