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EXISTENCE AND ASYMPTOTIC BEHAVIOUR OF POSITIVE SOLUTIONS FOR SEMILINEAR ELLIPTIC SYSTEMS IN THE EUCLIDEAN PLANE

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Abstract. We study the semilinear elliptic system

$$\Delta u = \lambda p(x) f(v), \Delta v = \lambda q(x) g(u),$$

in an unbounded domain D in $\mathbb{R}^2$ with compact boundary subject to some Dirichlet conditions. We give existence results according to the monotonicity of the nonnegative continuous functions f and g. The potentials p and q are nonnegative and required to satisfy some hypotheses related on a Kato class.

1. Introduction

Semilinear elliptic systems of the form

$$\begin{aligned} \Delta u &= F(u, v), \\ \Delta v &= G(u, v), \end{aligned} \tag{1.1}$$

in $\mathbb{R}^n$ have been extensively treated recently. Lair and Wood [9] studied the semiliniar elliptic system

$$\begin{aligned} \Delta u &= p(|x|) v^\alpha, \\ \Delta v &= q(|x|) u^\beta, \end{aligned} \tag{1.2}$$

in $\mathbb{R}^n$ ($n \geq 3$). They showed the existence of entire positive radial solutions. More precisely, for the sublinear case where $\alpha, \beta \in (0, 1)$, they proved the existence of bounded solutions of (1.2) if p and q satisfy the decay conditions

$$\int_0^\infty tp(t)dt < \infty, \quad \int_0^\infty tq(t)dt < \infty, \tag{1.3}$$

and the existence of large solutions if

$$\int_0^\infty tp(t)dt = \infty, \quad \int_0^\infty tq(t)dt = \infty. \tag{1.4}$$

For the superlinear case, where $\alpha, \beta \in (1, +\infty)$. The authors proved the existence of an entire large positive solution of problem (1.2), provided that the functions p and q satisfy (1.3).

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Peng and Song [13] considered the semilinear elliptic system

$$\begin{aligned} \Delta u &= p(|x|)f(v), \\ \Delta v &= q(|x|)g(u), \end{aligned} \tag{1.5}$$

in $\mathbb{R}^n$ $(n \geq 3)$, under the assumptions:

(A1) The functions p and q satisfy condition (1.3).

(A2) The functions f and g are positive nondecreasing, satisfying the Keller-Osserman condition [8, 12]

$$\int_1^\infty \frac{1}{\sqrt{\int_0^s f(t)dt}}ds < \infty, \quad \int_1^\infty \frac{1}{\sqrt{\int_0^s g(t)dt}}ds < \infty. \tag{1.6}$$

(A3) The functions f and g are convex on $[0, +\infty)$.

The authors proved the existence of an entire large positive solution of problem (1.5). We remark that Peng and Song extended their results to the superlinear case in [9].

Cirstea and Radulescu [5] gave existence results for system (1.5). They adopted the assumptions (A1)-(A2) and the assumption

(A3') $f, g \in C^1[0, +\infty)$, $f(0) = g(0) = 0$, $\lim_{t\to+\infty} \inf \frac{f(t)}{g(t)} > 0$,

to prove the existence of entire large positive solutions.

Recently, Ghanmi et al [7] considered the semilinear elliptic system

$$\begin{aligned} \Delta u &= \lambda p(x)f(v), \\ \Delta v &= \mu q(x)g(u), \end{aligned}$$

in a domain D of $\mathbb{R}^n$ $(n \geq 3)$ with compact boundary subject to some Dirichlet conditions. They assumed that the functions f, g are nonnegative continuous monotone on $(0, \infty)$, the nonnegative potentials p and q are required to satisfy some hypotheses related to a Kato class [3, 10]. In particular, in the case where f and g are nondecreasing and for given positive constants λ_0, μ_0, they showed that for each $\lambda \in [0, \lambda_0)$ and $\mu \in [0, \mu_0)$, there exists a positive bounded solution (u, v) satisfying the boundary conditions

$$u\big|_{\partial_\infty D} = \varphi \mathbf{1}_{\partial D} + a\mathbf{1}_{\{\infty\}}, \quad v\big|_{\partial_\infty D} = \psi \mathbf{1}_{\partial D} + b\mathbf{1}_{\{\infty\}}$$

where φ and ψ are nontrivial nonnegative continuous functions on ∂D.

In this article, we consider an unbounded domain D in $\mathbb{R}^2$ with compact nonempty boundary ∂D consisting of finitely many Jordan curves. We are concerned with the semilinear elliptic system

$$\begin{gathered} \Delta u = \lambda p(x)f(v), \quad \text{in } D \\ \Delta v = \mu q(x)g(u), \quad \text{in } D \\ u\big|_{\partial D} = a\varphi, \quad v\big|_{\partial D} = b\psi, \\ \lim_{|x|\to+\infty} \frac{u(x)}{\ln|x|} = \alpha, \quad \lim_{|x|\to+\infty} \frac{v(x)}{\ln|x|} = \beta, \end{gathered} \tag{1.7}$$

where a, b, α and β are nonnegative constants such that $a + \alpha > 0$, $b + \beta > 0$. The functions φ and ψ are nontrivial nonnegative and continuous on ∂D. We will give two existence results according to the monotoniciy of the functions f and g.

Throughout this paper, we denote by $H_D\varphi$ the bounded continuous solution of the Dirichlet problem

$$\begin{gathered}\Delta w = 0 \quad \text{in } D,\\ w\big|_{\partial D} = \varphi, \quad \lim_{|x|\to+\infty} \frac{w(x)}{\ln|x|} = 0,\end{gathered} \tag{1.8}$$

where φ is a nonnegative continuous function on ∂D.

We remark that the solution $H_D\varphi$ of (1.8) belongs to $\mathcal{C}(\overline{D}\cup\{\infty\})$ and satisfies $\lim_{|x|\to+\infty} H_D\varphi(x) = C > 0$ (See [6, p. 427]).

For the sake of simplicity we denote

$$\widetilde{\varphi} := aH_D\varphi + \alpha h, \quad \widetilde{\psi} := bH_D\psi + \beta h, \tag{1.9}$$

where h is the harmonic function defined by (2.2), below.

The outline of this paper is as follows. In section 2, we will give some notions related to the Green function G_D of the domain D associated to the Laplace operator Δ and properties of the functions belonging to a some Kato class $K(D)$ (See [10, 14]). In section 3, we will first give an example and then we give the proof of the existence result for the problem (1.7). More precisely, we adopt in section 3 the following hypotheses

- (H1) The functions $f,\ g : [0,\infty) \to [0,\infty)$ are nondecreasing and continuous.
- (H2) The functions $\widetilde{p} := pf(\widetilde{\psi})$ and $\widetilde{q} := qg(\widetilde{\varphi})$ belong to the Kato class $K(D)$.
- (H3) $\lambda_0 := \inf_{x\in D} \frac{\widetilde{\varphi}(x)}{V(\widetilde{p})(x)} > 0$ and $\mu_0 := \inf_{x\in D} \frac{\widetilde{\psi}(x)}{V(\widetilde{q})(x)} > 0$, where V is the Green kernel defined by (2.1) below.

We prove the following result.

Theorem 1.1. *Assume* (H1)–(H3), *then for each* $\lambda \in [0,\lambda_0)$ *and* $\mu \in [0,\mu_0)$, *problem* (1.7) *has a positive continuous solution* (u,v) *satisfying, on* D,

$$\begin{gathered}(1-\frac{\lambda}{\lambda_0})[aH_D\varphi + \alpha h] \le u \le aH_D\varphi + \alpha h,\\ (1-\frac{\mu}{\mu_0})[bH_D\psi + \beta h] \le v \le bH_D\psi + \beta h.\end{gathered}$$

In the last section, we fix $\lambda = \mu = 1$ and a nontrivial nonnegative continuous function Φ on ∂D and we note $h_0 = H_D\Phi$. Then we give an existence result for problem (1.7) with $a = 1$ and $b = 1$, under the following hypotheses:

- (H4) The functions $f, g : [0,\infty) \to [0,\infty)$ are nonincreasing and continuous.
- (H5) The functions $p_0 := p\frac{f(h_0)}{h_0}$ and $q_0 := q\frac{g(h_0)}{h_0}$ belong to the Kato class $K(D)$.

More precisely, we obtain the following result.

Theorem 1.2. *Assume* (H4)–(H5), *then there exists a constant* $c > 1$ *such that if* $\varphi \ge c\Phi$ *and* $\psi \ge c\Phi$ *on* ∂D, *then problem* (1.7) *with* $a = 1$ *and* $b = 1$ *has a positive continuous solution* (u,v) *satisfying, on* D,

$$\begin{gathered}h_0 + \alpha h \le u \le H_D\varphi + \alpha h,\\ h_0 + \beta h \le v \le H_D\psi + \beta h.\end{gathered}$$

Note that this result generalizes those by Athreya [2] and Toumi and Zeddini [14], stated for semilinear elliptic equations.

2. Preliminaries

In the reminder of this paper, we will adopt the following notation.

$\mathcal{C}(\overline{D}\cup\{\infty\}) = \{f \in \mathcal{C}(\overline{D}) : \lim_{|x|\to+\infty} f(x) \text{ exists}\}$. We note that $\mathcal{C}(\overline{D}\cup\{\infty\})$ is a Banach space endowed with the uniform norm $\|f\|_\infty = \sup_{x\in D}|f(x)|$.

For $x \in D$, we denote by $\delta_D(x)$ the distance from x to ∂D, by $\rho_D(x) := min(1, \delta_D(x))$ and by $\lambda_D(x) := \delta_D(x)(1+\delta_D(x))$.

Let f and g be two positive functions on a set S. We denote $f \sim g$ if there exists a constant $c > 0$ such that

$$\frac{1}{c}g(x) \le f(x) \le cg(x) \quad \text{for all } x \in S.$$

For a Borel measurable and nonnegative function f on D, we denote by Vf the Green kernel of f defined on D by

$$Vf(x) = \int_D G_D(x,y)f(y)dy. \tag{2.1}$$

We recall that if $f \in L^1_{\rm loc}(D)$ and $Vf \in L^1_{\rm loc}(D)$, then we have $\Delta(Vf) = -f$ in D, in the distributional sense (See [4, p 52]).

We note that the Green function satisfies

$$G_D(x,y) \sim \ln(1 + \frac{\lambda_D(x)\lambda_D(y)}{|x-y|^2})$$

on D^2 (See [11]).

Definition 2.1. A Borel measurable function q in D belongs to the Kato class $K(D)$ if q satisfies

$$\lim_{\alpha\to 0}\Big(\sup_{x\in D}\int_{D\cap B(x,\alpha)} \frac{\rho_D(y)}{\rho_D(x)} G_D(x,y)|q(y)|dy\Big) = 0,$$

and

$$\lim_{M\to+\infty}\Big(\sup_{x\in D}\int_{D\cap(|y|\ge M)} \frac{\rho_D(y)}{\rho_D(x)} G_D(x,y)|q(y)|dy\Big) = 0.$$

Example 2.2. Let $p > 1$ and $\gamma, \theta \in \mathbb{R}$ such that $\gamma < 2 - \frac{2}{p} < \theta$. Then using the Hölder inequality and the same arguments as in [14, Proposition 3.4] it follows that for each $f \in L^p(D)$, the function defined in D by $\frac{f(x)}{(1+|x|)^{\theta-\gamma}(\delta_D(x))^\gamma}$ belongs to $K(D)$.

Throughout this article, h will be the function defined on D by

$$h(x) = 2\pi \lim_{|y|\to+\infty} G_D(x,y) \tag{2.2}$$

Proposition 2.3 ([15]). *The function h defined by* (2.2) *is harmonic positive in D and satisfies*

$$\lim_{x\to z\in\partial D} h(x) = 0, \qquad \lim_{|x|\to+\infty} \frac{h(x)}{\ln|x|} = 1.$$

In the sequel, we use the notation

$$\|q\|_D = \sup_{x\in D}\int_D \frac{\rho_D(y)}{\rho_D(x)} G_D(x,y)|q(y)|dy, \tag{2.3}$$

$$\alpha_q = \sup_{x,y\in D}\int_D \frac{G_D(x,z)G_D(z,y)}{G_D(x,y)}|q(z)|dz. \tag{2.4}$$

It is shown in [14], that if $q \in K(D)$, then $\|q\|_D < \infty$, and $\alpha_q \sim \|q\|_D$. For stating our results we need the following result.

Proposition 2.4 ([14]). *Let q in $K(D)$, then the following assertions hold*

(i) *For any nonnegative superharmonic function w in D, we have*
$$V(wq)(x) = \int_D G_D(x,y)w(y)|q(y)|dy \le \alpha_q w(x), \forall x \in D. \tag{2.5}$$
(ii) *The potential $Vq \in \mathcal{C}(\overline{D} \cup \infty)$ and $\lim_{x \to z \in \partial D} Vq(x) = 0$.*

(iii) *Let $\Lambda_q = \{p \in K(D) : |p| \le q\}$. Then the family of functions*
$$\mathfrak{F}_q = \{\int_D G_D(.,y)h_0(y)p(y)dy : p \in \Lambda_q\}$$
is uniformly bounded and equicontinuous in $\overline{D} \cup \{\infty\}$. Consequently, it is relatively compact in $\mathcal{C}(\overline{D} \cup \{\infty\})$.

3. Proof of Theorem 1.1

Before stating the proof, we give an example where (H2) and (H3) are satisfied.

Example 3.1. Let $D = \overline{B(0,1)}^c$ be the exterior of the unit closed disk. Let $\alpha = b = 1$ and $\beta = a = 0$. Assume that $\psi \ge c_1 > 0$ on ∂D. Let $p_1, \widetilde{q}$ be nonnegative functions in $K(D)$ such that the function $p := p_1 h$ is in $K(D)$. Then using the fact that the function f is continuous and $H_D\psi$ is bounded on D we obtain that $\widetilde{p} := pf(H_D\psi) \in K(D)$ and so the hypothesis (H2) is satisfied. Now, since $\widetilde{p_1} := p_1 f(H_D\psi) \in K(D)$ then by Proposition 2.4 (i) we obtain
$$V(\widetilde{p}) \le \alpha_{\widetilde{p_1}} h.$$
Therefore, for each $x \in D$
$$\frac{h(x)}{V(\widetilde{p})(x)} \ge \frac{1}{\alpha_{p_1}} > 0;$$
that is, $\lambda_0 > 0$. On the other hand we have
$$\frac{H_D\psi(x)}{V(\widetilde{q})(x)} \ge \frac{c_1}{\alpha_{\widetilde{q}}} > 0,$$
which yields $\mu_0 > 0$. Thus the hypothesis (H3) is satisfied.

Proof of Theorem 1.1. Let $\lambda \in [0, \lambda_0)$ and $\mu \in [0, \mu_0)$. We intend to prove that the problem (1.7) has a positive continuous solution. To this aim we define the sequences $(u_k)_{k\in\mathbb{N}}$ and $(v_k)_{k\in\mathbb{N}}$ as follows:
$$\begin{gathered} v_0 = \widetilde{\psi}, \\ u_k = \widetilde{\varphi} - \lambda V(pf(v_k)), \\ v_{k+1} = \widetilde{\psi} - \mu V(qg(u_k)), \end{gathered}$$
where $\widetilde{\varphi}$ and $\widetilde{\psi}$ are defined by (1.9). We shall prove by induction that for each $k \in \mathbb{N}$,
$$\begin{gathered} 0 < \big(1 - \frac{\lambda}{\lambda_0}\big)\widetilde{\varphi} \le u_k \le u_{k+1} \le \widetilde{\varphi}, \\ 0 < \big(1 - \frac{\mu}{\mu_0}\big)\widetilde{\psi} \le v_{k+1} \le v_k \le \widetilde{\psi}. \end{gathered}$$

First, using hypothesis (H3) we obtain, on D,

$$\lambda_0 V(pf(\widetilde{\psi})) \le \widetilde{\varphi}.$$

Then by the monotonicity of f, it follows that

$$\widetilde{\varphi} \ge u_0 = \widetilde{\varphi} - \lambda V(pf(\widetilde{\psi})) \ge \big(1 - \frac{\lambda}{\lambda_0}\big)\widetilde{\varphi} > 0.$$

So

$$v_1 - v_0 = -\mu V(qg(u_0)) \le 0$$

and consequently

$$u_1 - u_0 = \lambda V(p[f(v_0) - f(v_1)]) \ge 0.$$

Moreover, the hypothesis (H3) yields

$$\mu_0 V(qg(\widetilde{\varphi})) \le \widetilde{\psi}.$$

Then using the fact that the function g is nondecreasing we have

$$v_1 \ge \widetilde{\psi} - \mu V(qg(\widetilde{\varphi})) \ge \big(1 - \frac{\mu}{\mu_0}\big)\widetilde{\psi} > 0.$$

In addition, we have $u_1 \le \widetilde{\varphi}$, then it follows that

$$u_0 \le u_1 \le \widetilde{\varphi} \quad \text{and} \quad v_1 \le v_0 \le \widetilde{\psi}.$$

Suppose that

$$u_k \le u_{k+1} \le \widetilde{\varphi} \quad \text{and} \quad (1 - \frac{\mu}{\mu_0})\widetilde{\psi} \le v_{k+1} \le v_k.$$

Therefore,

$$\begin{gathered} v_{k+2} - v_{k+1} = \mu V(q[g(u_k) - g(u_{k+1})]) \le 0, \\ u_{k+2} - u_{k+1} = \lambda V(p[f(v_{k+1}) - f(v_{k+2})]) \ge 0. \end{gathered}$$

Furthermore, since $u_{k+1} \le \widetilde{\varphi}$ the monotonicity of the function g yields

$$v_{k+2} \ge \widetilde{\psi} - \lambda V(qg(\widetilde{\varphi})) \ge (1 - \frac{\mu}{\mu_0})\widetilde{\psi} > 0.$$

Thus, we obtain

$$u_{k+1} \le u_{k+2} \le \widetilde{\varphi} \quad \text{and} \quad \big(1 - \frac{\mu}{\mu_0}\big)\widetilde{\psi} \le v_{k+2} \le v_{k+1}.$$

Hence, the sequences (u_k) and (v_k) converge respectively to two functions u and v satisfying

$$0 < \big(1 - \frac{\lambda}{\lambda_0}\big)\widetilde{\varphi} \le u \le \widetilde{\varphi}, \quad 0 < \big(1 - \frac{\mu}{\mu_0}\big)\widetilde{\psi} \le v \le \widetilde{\psi}.$$

Furthermore, for each $k \in \mathbb{N}$, we have

$$f(v_k) \le f(\widetilde{\psi}), \quad g(u_k) \le g(\widetilde{\varphi}). \tag{3.1}$$

Therefore, using hypothesis (H2) and Proposition 2.4 (ii) we deduce by Lebesgue's theorem that $V(pf(v_k))$ and $V(qg(u_k))$ converge respectively to $V(pf(v))$ and $V(qg(u))$ as k tends to infinity. Then, on D, (u,v) satisfies

$$\begin{gathered} u = \widetilde{\varphi} - \lambda V(pf(v)) \\ v = \widetilde{\psi} - \mu V(qg(u)). \end{gathered} \tag{3.2}$$

Moreover, by (3.2) and the monotonicity of the functions f and g we obtain $pf(v) \le \widetilde{p}$ and $qg(u) \le \widetilde{q}$. So $pf(v), qg(u) \in K(D)$ and consequently by Proposition 2.4 (ii) we have $V(pf(v)), V(qg(u)) \in \mathcal{C}(\overline{D} \cup \{\infty\})$. Now using the fact that the functions $\widetilde{\varphi}$ and $\widetilde{\psi}$ are continuous we conclude that u and v are continuous and satisfy in the distributional sense $\Delta u = \lambda pf(v)$ and $\Delta v = \mu qg(u)$ in D. Now, since $H_D\varphi = \varphi$ on $\partial D, \lim_{x\to z\in\partial D} h(x) = 0$, and $\lim_{x\to z\in\partial D} V(\widetilde{p})(x) = 0$, we conclude that $\lim_{x\to z\in\partial D} u(x) = a\varphi(z)$. By similar arguments we have $\lim_{x\to z\in\partial D} v(x) = b\psi(z)$. Furthermore, by Proposition 2.4 (ii) and Proposition 2.3, we have $\lim_{|x|\to+\infty} \frac{1}{h(x)}V(pf(v)) = 0$ and $\lim_{|x|\to+\infty} \frac{1}{h(x)}V(qg(u)) = 0$. Hence (u, v) is a continuous positive solution of the problem (1.7), which completes the proof. $\square$

4. Proof of Theorem 1.2

In the sequel, we recall that $h_0 = H_D\Phi$ is a fixed positive harmonic function in D and h is the function defined by (2.2).

Proof. Let α_{p_0} and α_{q_0} be the constants defined by (2.4) associated respectively to the functions p_0 and q_0 given in the hypothesis (H5). Put $c = 1 + \alpha_{p_0} + \alpha_{q_0}$. Suppose that

$$\varphi(x) \geq c\Phi(x) \quad \text{and} \quad \psi(x) \geq c\Phi(x), \quad \forall x \in \partial D.$$

Then by the maximum principle it follows that for each $x \in D$

$$H_D\varphi(x) \geq ch_0(x), \tag{4.1}$$

$$H_D\psi(x) \geq ch_0(x). \tag{4.2}$$

Consider the nonempty convex set Ω given by

$$\Omega := \{w \in \mathcal{C}(\overline{D} \cup \{\infty\}) : h_0 \leq w \leq H_D\varphi\}.$$

Let T be the operator defined on Ω by

$$Tw := H_D\varphi - V(pf[\beta h + H_D\psi - V(qg(w + \alpha h))]).$$

We shall prove that the operator T has a fixed point. First, let us prove that the operator T maps Ω to its self. Let $w \in \Omega$. Since $w + \alpha h \geq h_0$, then from hypothesis (H4) we deduce that

$$V(qg(w + \alpha h)) \leq V(qg(h_0)). \tag{4.3}$$

Therefore, using (4.2) and (4.3) we obtain

$$\begin{aligned} v &:= \beta h + H_D\psi - V(qg(w + \alpha h) \geq \beta h + H_D\psi - V(q_0h_0) \\ &\geq \beta h + H_D\psi - \alpha_{q_0}h_0 \geq \beta h + (c - \alpha_{q_0})h_0. \end{aligned}$$

This yields

$$v \geq h_0 > 0. \tag{4.4}$$

So, $Tw \leq H_D\varphi$. On the other hand, by (4.4), the monotonicity of f and Proposition 2.4 (i), we obtain

$$V(pf(v)) \leq V(pf(h_0)) = V(p_0h_0) \leq \alpha_{p_0}h_0. \tag{4.5}$$

Then, by (4.1) and (4.5), we have

$$Tw \geq H_D\varphi - \alpha_{p_0}h_0 \geq (1 + \alpha_{q_0})h_0 \geq h_0.$$

Hence $T\Omega \subseteq \Omega$. Next, let us prove that the set $T\Omega$ is relatively compact in $\mathcal{C}(\overline{D} \cup \{\infty\})$. Let $w \in \Omega$, then by (H4), (H5) and using Proposition 2.4 (iii) it follows that the family of functions

$$\Big\{ \int_D G(.,y)p(y)f[\beta h + H_D\psi - V(qg(w+\alpha h))](y)dy : w \in \Omega \Big\}$$

is relatively compact in $\mathcal{C}(\overline{D} \cup \{\infty\})$. Since $H_D\varphi \in \mathcal{C}(\overline{D} \cup \{\infty\})$ we deduce that $T\Omega$ is relatively compact in $\mathcal{C}(\overline{D} \cup \{\infty\})$.

Now we prove the continuity of the operator T in Ω in the supremum norm. Let (w_k) be a sequence in Ω which converges uniformly to a function w in Ω. Then using (4.4) and the monotonocity of f we have, for each x in D,

$$\begin{aligned}
&p(x)|f(\beta h + H_D\psi - V(qg(w_k+\alpha h)))(x) - f(\beta h + H_D\psi - V(qg(w+\alpha h)))(x)| \\
&\le 2f(h_0)p(x) \le 2\|h_0\|_\infty p_0(x)
\end{aligned}$$

Using the fact that Vp_0 is bounded, we conclude by the continuity of f and the dominated convergence theorem that for all $x \in D$, $Tw_k(x) \to Tw(x)$ as $k \to +\infty$. Consequently, as $T\Omega$ is relatively compact in $\mathcal{C}(\overline{D} \cup \{\infty\})$, we deduce that the pointwise convergence implies the uniform convergence; that is,

$$\|Tw_k - Tw\|_\infty \quad \text{as } k \to +\infty$$

Therefore, T is a continuous mapping of Ω to itself. So, since $T\Omega$ is relatively compact in $\mathcal{C}(\overline{D} \cup \{\infty\})$, it follows that T is compact mapping on Ω. Thus, the Schauder fixed-point theorem yields the existence of $w \in \Omega$ such that

$$w = H_D\varphi - V(pf[\beta h + H_D\psi - V(qg(w+\alpha h))].$$

Put $u(x) = w(x) + \alpha h(x)$ and $v(x) = \beta h(x) + H_D\psi(x) - V(qg(u))(x)$ for $x \in D$. Then (u,v) is a positive continuous solution of (1.7) with $a = 1, b = 1$, for the same arguments as in the proof of Theorem 1.1. $\square$

Example 4.1. Let $D = \overline{B(0,1)}^c$ be the exterior of the unit closed disk, $0 < \theta < 1$ and $0 < \gamma < 1$. Let p, q be two nonnegative functions such that the functions $(\frac{|x|}{|x|-1})^{1+\theta}p(x)$ and $(\frac{|x|}{|x|-1})^{1+\gamma}q(x)$ are in $K(D)$. Suppose that the functions φ and ψ are nonnegative continuous on ∂D. Then for a fixed nontrivial nonnegative continuous function Φ in ∂D, there exists a constant $c > 1$ such that if $\varphi \ge c\Phi$ and $\psi \ge c\Phi$ on ∂D, the problem

$$\begin{gathered}
\Delta u = p(x)v^{-\gamma}, \quad \text{in } D \\
\Delta v = q(x)u^{-\theta}, \quad \text{in } D \\
u\big|_{\partial D} = \varphi, \quad v\big|_{\partial D} = \psi, \quad \lim_{|x|\to+\infty} \frac{u(x)}{\ln|x|} = \alpha \ge 0, \quad \lim_{|x|\to+\infty} \frac{v(x)}{\ln|x|} = \beta \ge 0,
\end{gathered}$$

has a positive continuous solution (u,v) satisfying

$$\begin{gathered}
H_D\Phi(x) + \alpha h(x) \le u(x) \le H_D\varphi(x) + \alpha h(x), \\
H_D\Phi(x) + \beta h(x) \le v(x) \le H_D\psi(x) + \beta h(x),
\end{gathered}$$

for each $x \in D$. Indeed, from [1] there exists $c_0 > 0$ such that for each $x \in D$,

$$c_0 \frac{|x|-1}{|x|} \le H_D\Phi(x).$$

It follows that $p_0 := p\frac{(H_D\Phi(x))^{-\theta}}{H_D\Phi(x)} \in K(D)$. In a similar way we have $q_0 \in K(D)$. Thus the hypothesis (H5) is satisfied.

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