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TIME-DEPENDENT DOMAINS FOR NONLINEAR EVOLUTION OPERATORS AND PARTIAL DIFFERENTIAL EQUATIONS

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Dedicated to Professor Jerome A. Goldstein on his 70th birthday

Abstract. This article concerns the nonlinear evolution equation

$$\frac{du(t)}{dt} \in A(t)u(t), \quad 0 \le s < t < T,$$
$$u(s) = u_0$$

in a real Banach space X, where the nonlinear, time-dependent, and multi-valued operator $A(t) : D(A(t)) \subset X \to X$ has a time-dependent domain $D(A(t))$. It will be shown that, under certain assumptions on $A(t)$, the equation has a strong solution. Illustrations are given of solving quasi-linear partial differential equations of parabolic type with time-dependent boundary conditions. Those partial differential equations are studied to a large extent.

1. Introduction

Let $(X, \|\cdot\|)$ be a real Banach space with the norm $\|\cdot\|$, and let $T > 0$, ω be two real constants. Consider the nonlinear evolution equation

$$\begin{gathered}\frac{du(t)}{dt} \in A(t)u(t), \quad 0 \le s < t < T, \\ u(s) = u_0,\end{gathered} \tag{1.1}$$

where

$$A(t) : D(A(t)) \subset X \to X$$

is a nonlinear, time-dependent, and multi-valued operator. To solve (1.1), Crandall and Pazy [9] made the following hypotheses of (H1)–(H3) and the t-dependence hypothesis of either (H4) or (H5), for each $0 \le t \le T$.

(H1) $A(t)$ is dissipative in the sense that

$$\|u - v\| \le \|(u - v) - \lambda(g - h)\|$$

for all $u, v \in D(A(t))$, $g \in (A(t) - \omega)u, h \in (A(t) - \omega)v$, and for all $\lambda > 0$. Equivalently,

$$\Re(\eta(g - h)) \le 0$$

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for some $\eta \in G(u-v) \equiv \{\xi \in X^* : \|u-v\|^2 = \xi(u-v) = \|\xi\|_{X^*}^2\}$, the duality map of $(u-v)$ [27]. Here $(X^*, \|.\|_{X^*})$ is the dual space of X and $\Re(z)$ is the real part of a complex number z.

(H2) The range of $(I-\lambda A(t))$ contains the closure $\overline{D(A(t))}$ of $D(A(t))$ for small $0 < \lambda < \lambda_0$ with $\lambda_0\omega < 1$.

(H3) $\overline{D(A(t))} = \overline{D}$ is independent of t.

(H4) There are a continuous function $f : [0,T] \to X$ and a monotone increasing function $L : [0,\infty) \to [0,\infty)$, such that
$$\|J_\lambda(t)x - J_\lambda(\tau)x\| \le \lambda\|f(t)-f(\tau)\|L(\|x\|)$$
for $0 < \lambda < \lambda_0, 0 \le t, \tau \le T$, and $x \in \overline{D}$. Here $J_\lambda(t)x \equiv (I-\lambda A(t))^{-1}$ exists for $x \in \overline{D}$ by (H1) and (H2).

(H5) There is a continuous function $f : [0,T] \to X$, which is of bounded variation on $[0,T]$, and there is a monotone increasing function $L : [0,\infty) \to [0,\infty)$, such that
$$\|J_\lambda(t)x - J_\lambda(\tau)x\| \le \lambda\|f(t)-f(\tau)\|L(\|x\|)(1+|A(\tau)x|)$$
for $0 < \lambda < \lambda_0, 0 \le t, \tau \le T$, and $x \in \overline{D}$. Here
$$|A(\tau)x| \equiv \lim_{\lambda\to 0}\|\frac{(J_\lambda(\tau)-I)x}{\lambda}\|$$
by (H1) and (H2), which can equal ∞ [7, 9].

By defining the generalized domain $\hat{D}(A(t)) \equiv \{x \in \overline{D(A(t))} : |A(t)x| < \infty\}$ [7, 37], they [9] proved, among other things, that the limit
$$U(t,s)x \equiv \lim_{n\to\infty}\prod_{i=1}^{n} J_{\frac{t-s}{n}}(s+i\frac{t-s}{n})x \tag{1.2}$$
exists for $x \in \overline{D}$ and that $U(t,s)u_0$ is a unique solution, in a generalized sense, to the equation (1.1) for $u_0 \in \overline{D}$.

Because of the restriction in (H3) that $\overline{D(A(t))} = \overline{D}$ is independent of t, the boundary condition in the example in [9] does not depend on time. In this paper, in order to enlarge the scope of applications, we will consider a different set of hypotheses, the dissipativity condition (H1), the range condition (H2'), and the time-regulating condition (HA) below. Here a similar set of hypotheses was considered in [21] but the results were not satisfactory.

(H2') The range of $(I-\lambda A(t))$, denoted by E, is independent of t and contains $\overline{D(A(t))}$ for all $t \in [0,T]$ and for small $0 < \lambda < \lambda_0$ with $\lambda_0\omega < 1$.

(HA) There is a continuous function $f : [0,T] \to \mathbb{R}$, of bounded variation, and there is a nonnegative function L on $[0,\infty)$ with $L(s)$ bounded for bounded s, such that, for each $0 < \lambda < \lambda_0$, we have
$$\{J_\lambda(t)x - J_\lambda(\tau)y : 0 \le t, \tau \le T, x, y \in E\} = S_1(\lambda) \cup S_2(\lambda).$$
Here $S_1(\lambda)$ denotes the set
$$\big\{J_\lambda(t)x - J_\lambda(\tau)y : 0 \le t, \tau \le T, x, y \in E, \\ \|J_\lambda(t)x - J_\lambda(\tau)y\| \le L(\|J_\lambda(\tau)y\|)|t-\tau|\big\},$$
and $S_2(\lambda)$ denotes the set
$$\big\{J_\lambda(t)x - J_\lambda(\tau)y : 0 \le t, \tau \le T, x, y \in E, \|J_\lambda(t)x - J_\lambda(\tau)y\|$$

$$\le (1-\lambda\omega)^{-1}[\|x-y\| + \lambda|f(t)-f(\tau)|L(\|J_\lambda(\tau)y\|)(1+\frac{\|(J_\lambda(\tau)-I)y\|}{\lambda})]\}.$$

We will show that the limit in (1.2) for $x \in \overline{\hat{D}(A(s))} = \overline{D(A(s))}$ exists, and that this limit for $x = u_0 \in \hat{D}(A(s))$ is a strong solution to the equation (1.1), if $A(t)$ satisfies additionally an embedding property in [20] of embeddedly quasi-demi-closedness. We then apply the abstract theory to quasi-linear, parabolic partial differential equations with boundary conditions depending on time t. We finally show that, in those applications, each quantity

$$J_{\frac{t-s}{n}}(s+i\frac{t-s}{n})h = [I - \frac{t-s}{n}A(s+i\frac{t-s}{n})]^{-1}h, \quad i = 1,2,\dots,n$$

is the limit of a sequence where each term in the sequence is an explicit function $F(\phi)$ of the solution $\phi = \pounds_0^{-1}(h,\varphi)$ to the elliptic equation with $\varphi \equiv 0$:

$$\begin{gathered} -\Delta v(y) = h, \quad y \in \Omega, \\ \frac{\partial v}{\partial \nu} + v = \varphi, \quad y \in \partial\Omega. \end{gathered} \tag{1.3}$$

Here for the dimension of the space variable y equal to 2 or 3, the $\phi = \pounds_0^{-1}(h,0)$ and the solution $\pounds_0^{-1}(h,\varphi)$ to (1.3) can be computed numerically and efficiently by the boundary element methods [13, 34]. See Sections 4 and 5 for more details of these, including how $F(\phi)$ depends on ϕ, and for other aspects of the treated partial differential equations.

There are many related works, to cite a few, we mention [1, 2, 3, 4, 6, 8, 9, 10, 11, 15, 19, 16, 17, 18, 20, 22, 23, 21, 24, 27, 29, 30, 31, 32, 33, 36], especially the [24] for the recent development on nonlinear evolution equations where the hypothesis (H2) is relaxed.

The rest of this article will be organized as follows. Section 2 obtains some preliminary estimates, and Section 3 deals with the main results, where the nonlinear operator $A(t)$ is equipped with time-dependent domain $D(A(t))$. The Appendix in Section 6 examines the difference equations theory in our papers [22, 23, 24], whose results, together with those in Section 2, will be used to prove the main results in Section 3. Section 4 studies applications to linear or nonlinear partial differential equations of parabolic type, in which each corresponding elliptic solution $J_{\frac{t-s}{n}}(s+i\frac{t-s}{n})h$ will be derived theoretically. Finally, Section 5 follows Section 4 but derives each elliptic solution $J_{\frac{t-s}{n}}(s+i\frac{t-s}{n})h$ as the limit of a sequence where each term in the sequence is an explicit function of the solution ϕ to the elliptic equation (1.3) with $\varphi \equiv 0$. In either Section 4 or Section 5, other aspects of the treated partial differential equations are considered.

2. Some preliminary estimates

Within this section and the Sections 3 and 6, we can assume, without loss of generality, that $\omega \ge 0$ where ω is the ω in the hypothesis (H1). This is because the case $\omega < 0$ is the same as the case $\omega = 0$. This will be readily seen from the corresponding proofs.

To prove the main results Theorems 3.3 and 3.5 in Section 3, we need to make two preparations. One preparation is this section, and the other is the Appendix in Section 6.

Proposition 2.1. *Let $A(t)$ satisfy the dissipativity condition (H1), the range condition (H2') , and the time-regulating condition (HA), and let u_0 be in $D(A(s)) \subset E$ where $0 \leq s \leq T$. Let $0 < \epsilon < \lambda_0$ be so chosen that $0 < \epsilon\omega < 1$, and let $0 \leq t_i = s + i\epsilon \leq T$ where $i \in \mathbb{N}$. Then*

$$\|u_i - u_0\| \leq \eta^i L(\|u_0\|)(i\epsilon) + [\eta^{i-1}b_1 + \eta^{i-2}b_2 + \cdots + \eta b_{i-1} + b_i] \tag{2.1}$$

and

$$\begin{aligned} \|\frac{u_i - u_{i-1}}{\epsilon}\| \leq {} & [(c_i c_{i-1} \dots c_2)L(\|u_0\|) \quad \text{or } (c_i c_{i-1} \dots c_3)L(\|u_1\|) \text{ or } \dots \\ & \text{or } c_i L(\|u_{i-2}\|) \text{ or } L(\|u_{i-1}\|)] + [(c_i c_{i-1} \dots c_1)a_0 \\ & + (c_i c_{i-1} \dots c_2)d_1 + (c_i c_{i-1} \dots c_3)d_2 + \cdots + c_i d_{i-1} + d_i]. \end{aligned} \tag{2.2}$$

Here $u_i = \prod_{j=1}^{i} J_\epsilon(t_j)u_0$ exists uniquely by the hypotheses (H1) and (H2');
$\eta = (1 - \epsilon\omega)^{-1} > 1$;
$b_i = \eta\epsilon\|v_0\| + \eta\epsilon|f(t_i) - f(s)|L(\|u_0\|)(1 + \|v_0\|)$, where v_0 is any element in $A(s)u_0$;
$c_i = \eta[1 + L(\|u_{i-1}\|)|f(t_i) - f(t_{i-1})|]$;
$d_i = \eta L(\|u_{i-1}\|)|f(t_i) - f(t_{i-1})|$;
the right sides of (2.2) are interpreted as $[L(\|u_0\|)] + [c_1 a_0 + d_1]$ for $i = 1$; $[c_2 L(\|u_0\|)$ or $L(\|u_1\|)] + [c_2 c_1 a_0 + c_2 d_1 + d_2]$ for $i = 2$; ..., and so on; and

$$a_0 = \|\frac{u_0 - u_{-1}}{\epsilon}\|,$$

where u_{-1} is defined by $u_0 - \epsilon v_0 = u_{-1}$, with v_0 any element in $A(s)u_0$.

Proof. We will use the method of mathematical induction. Two cases will be considered, and for each case, we divide the proof into two steps.

Case 1. Here (2.1) is considered

Step 1. Claim that (2.1) is true for $i = 1$. This will follow from the arguments below. If $(u_1 - u_0) \in S_1(\epsilon)$ (defined in Section 1), then

$$\|u_1 - u_0\| = \|J_\epsilon(t_1)u_0 - J_\epsilon(s)(I - \epsilon A(s))u_0\| \leq L(\|u_0\|)|t_1 - s| \leq L(\|u_0\|)\epsilon,$$

which is less than or equal to the right-hand side of (2.1) with $i = 1$.

On the other and, if $(u_1 - u_0) \in S_2(\epsilon)$ (defined in Section 1), then

$$\|u_1 - u_0\| \leq \eta\|u_0 - u_0\| + \eta\epsilon\|v_0\| + \eta\epsilon|f(t_1) - f(s)|L(\|u_0\|)(1 + \|v_0\|),$$

which is less than or equal to the right-hand side of (2.1) with $i = 1$. Here v_0 is any element in $A(s)u_0$.

Step 2. By assuming that (2.1) is true for $i = i - 1$, we shall show that it is also true for $i = i$. If $(u_i - u_0) \in S_1(\epsilon)$, then

$$\|u_i - u_0\| = \|J_\epsilon(t_i)u_{i-1} - J_\epsilon(s)(I - \epsilon A(s))u_0\| \leq L(\|u_0\|)|t_i - s| = L(\|u_0\|)(i\epsilon),$$

which is less than or equal to the right side of (2.1) with $i = i$ because of $\eta^i > 1$.

On the other hand, if $(u_i - u_0) \in S_2(\epsilon)$, then

$$\|u_i - u_0\| \leq \eta\|u_{i-1} - u_0\| + b_i$$

where $\eta = (1 - \epsilon\omega)^{-1}$ and

$$b_i = \eta\epsilon\|v_0\| + \eta\epsilon|f(t_i) - f(s)|L(\|u_0\|)(1 + \|v_0\|).$$

This recursive inequality, combined with the induction assumption, readily gives

$$\|u_i - u_0\| \leq \eta\{\eta^{i-1}L(\|u_0\|)(i - 1)\epsilon + [\eta^{i-2}b_1 + \eta^{i-3}b_2 + \cdots + \eta b_{i-2} + b_{i-1}]\} + b_i$$

$$= \eta^i L(\|u_0\|)(i-1)\epsilon + [\eta^{i-1}b_1 + \eta^{i-2}b_2 + \dots + \eta b_{i-1} + b_i],$$

which is less than or equal to the right-hand side of (2.1) with $i = i$ because of $(i-1)\epsilon \le i\epsilon$.

Case 2. Here (2.2) is considered.

Step 1. Claim that (2.2) is true for $i = 1$. This follows from the Step 1 in Case 1, because there it was shown that

$$\|u_1 - u_0\| \le L(\|u_0\|)\epsilon \quad \text{or } b_1,$$

which, when divided by ϵ, is less than or equal to the right side of (2.2) with $i = 1$. Here $a_0 = \|v_0\|$, in which $a_0 = (u_0 - u_{-1})/\epsilon$ and $u_{-1} \equiv u_0 - \epsilon v_0$.

Step 2. By assuming that (2.2) is true for $i = i-1$, we will show that it is also true for $i = i$. If $(u_i - u_{i-1}) \in S_1(\epsilon)$, then

$$\|u_i - u_{i-1}\| \le L(\|u_{i-1}\|)|t_i - t_{i-1}| = L(\|u_{i-1}\|)\epsilon.$$

This, when divided by ϵ, has its right side less than or equal to one on the right-hand sides of (2.2) with $i = i$.

If $(u_i - u_{i-1}) \in S_2(\epsilon)$, then

$$\begin{aligned}\|u_i - u_{i-1}\| &\le (1-\epsilon\omega)^{-1}[\|u_{i-1} - u_{i-2}\| \\ &\quad + \epsilon|f(t_i) - f(t_{i-1})|L(\|u_{i-1}\|)(1 + \frac{\|u_{i-1} - u_{i-2}\|}{\epsilon})].\end{aligned}$$

By letting

$$\begin{gathered} a_i = \frac{\|u_i - u_{i-1}\|}{\epsilon}, \\ c_i = (1-\epsilon\omega)^{-1}[1 + L(\|u_{i-1}\|)|f(t_i) - f(t_{i-1})|], \quad \text{and} \\ d_i = L(\|u_{i-1}\|)(1-\epsilon\omega)^{-1}|f(t_i) - f(t_{i-1})|, \end{gathered}$$

it follows that $a_i \le c_i a_{i-1} + d_i$. Here notice that

$$u_0 - \epsilon v_0 = u_{-1}; \quad a_0 = \|\frac{u_0 - u_{-1}}{\epsilon}\| = \|v_0\|.$$

The above inequality, combined with the induction assumption, readily gives

$$\begin{aligned} a_i &\le c_i\big\{[(c_{i-1}c_{i-2}\dots c_2)L(\|u_0\|) \quad \text{or } (c_{i-1}c_{i-2}\dots c_3)L(\|u_1\|) \text{ or } \dots \\ &\quad \text{or } c_{i-1}L(\|u_{i-3}\|) \text{ or } L(\|u_{i-2}\|)] + [(c_{i-1}c_{i-2}\dots c_1)a_0 \\ &\quad + (c_{i-1}c_{i-2}\dots c_2)d_1 + (c_{i-1}c_{i-2}\dots c_3)d_2 + \dots \\ &\quad + c_{i-1}d_{i-2} + d_{i-1}]\big\} + d_i \\ &\le [(c_i c_{i-1}\dots c_2)L(\|u_0\|) \quad \text{or } (c_i c_{i-1}\dots c_3)L(\|u_1\|) \text{ or } \dots \\ &\quad \text{or } c_i L(\|u_{i-2}\|)] + [(c_i c_{i-1}\dots c_1)a_0 \\ &\quad + (c_i c_{i-1}\dots c_2)d_1 + (c_i c_{i-1}\dots c_3)d_2 + \dots + c_i d_{i-1} + d_i], \end{aligned}$$

each of which is less than or equal to one on the right sides of (2.2) with $i = i$.

The induction proof is now complete. □

Proposition 2.2. *Under the assumptions of Proposition 2.1, the following are true if u_0 is in $\hat{D}(A(s)) = \{y \in \overline{D(A(s))} : |A(s)y| < \infty\}$:*

$$\|u_i - u_0\| \le K_1(1-\epsilon\omega)^{-i}(2i+1)\epsilon \le K_1 e^{(T-s)\omega}(3)(T-s);$$

$$\|\frac{u_i - u_{i-1}}{\epsilon}\| \le K_3;$$

where the constants K_1 and K_3 depend on the quantities:

$$K_1 = K_1(L(\|u_0\|), (T-s), \omega, |A(s)u_0|, K_B);$$
$$K_2 = K_2(K_1, (T-s), \omega, \|u_0\|);$$
$$K_3 = K_3(L(K_2), (T-s), \omega, \|u_0\|, |A(s)u_0|, K_B);$$
$$K_B \textit{ is the total variation of } f \textit{ on } [0,T].$$

Proof. We divide the proof into two cases.

Case 1. Here $u_0 \in D(A(s))$. It follows immediately from Proposition 2.1 that

$$\|u_i - u_0\| \le N_1(1-\epsilon\omega)^{-i}(2i+1)\epsilon \le N_1 e^{(T-s)\omega}(3)(T-s);$$
$$\|\frac{u_i - u_{i-1}}{\epsilon}\| \le N_3;$$

where the constants N_1 and N_3 depend on the quantities:

$$N_1 = N_1(L(\|u_0\|), (T-s), \omega, \|v_0\|, K_B);$$
$$N_2 = N_2(N_1, (T-s), \omega, \|u_0\|);$$
$$N_3 = N_3(L(N_2), (T-s), \omega, \|u_0\|, \|v_0\|, K_B);$$
$$K_B \text{ is the total variation of } f \text{ on } [0,T].$$

We used here the estimate in [9, Page 65]

$$c_i \dots c_1 \le e^{i\epsilon\omega} e^{e_i + \dots + e_1},$$

where $e_i = L(\|u_{i-1}\|)|f(t_i) - f(t_{i-1})|$.

Case 2. Here $u_0 \in \hat{D}(A(s))$. This involves two steps.

Step 1. Let $u_0^\mu = (I - \mu A(s))^{-1} u_0$ where $\mu > 0$, and let

$$u_i = \prod_{j=1}^{i} J_\epsilon(t_j) u_0; \quad u_i^\mu = \prod_{j=1}^{i} J_\epsilon(t_j) u_0^\mu.$$

As in [31, Lemma 3.2, Page 9], we have, by letting $\mu \to 0$,

$$u_0^\mu \to u_0;$$

here notice that $D(A(s))$ is dense in $\hat{D}(A(s))$. Also it is readily seen that

$$u_i^\mu = \prod_{k=1}^{i} (I - \epsilon A(t_k))^{-1} u_0^\mu \to u_i = \prod_{k=1}^{i} (I - \epsilon A(t_k))^{-1} u_0$$

as $\mu \to 0$, since $(A(t) - \omega)$ is dissipative for each $0 \le t \le T$.

Step 2. Since $u_0^\mu \in D(A(s))$, Case 1 gives

$$\begin{gathered} \|u_i^\mu - u_0^\mu\| \le N_1(L(\|u_0^\mu\|), (T-s), \omega, \|v_0^\mu\|, K_B)(1-\epsilon\omega)^{-i}(2i+1)\epsilon \\ \frac{\|u_i^\mu - u_{i-1}^\mu\|}{\epsilon} \le N_3(L(N_2), (T-s), \omega, \|u_0^\mu\|, \|v_0^\mu\|, K_B), \end{gathered} \tag{2.3}$$

where

$$N_2 = N_2(N_1, (T-s), \omega, \|u_0^\mu\|),$$

and v_0^μ is any element in $A(s)(I-\mu A(s))^{-1}u_0$. We can take

$$v_0^\mu = w_0^\mu \equiv \frac{(J_\mu(s)-I)u_0}{\mu},$$

since $w_0^\mu \in A(s)(I-\mu A(s))^{-1}u_0$.

On account of $u_0 \in \hat{D}(A(s))$, we have

$$\lim_{\mu\to 0}\|\frac{(J_\mu(s)-I)u_0}{\mu}\| = |A(s)u_0| < \infty.$$

Thus, by letting $\mu \to 0$ in (2.3) and using Step 1, the results in the Proposition 2.2 follow. The proof is complete. □

3. Main results

Using the estimates in Section 2, together with the difference equations theory, the following result will be shown in in Section 6.

Proposition 3.1. *Under the assumptions of Proposition 3.2, the following inequality is true*

$$a_{m,n} \le \begin{cases} L(K_2)|n\mu - m\lambda|, & \text{if } S_2(\mu)=\emptyset;\\ c_{m,n}+s_{m,n}+d_{m,n}+f_{m,n}+g_{m,n}, & \text{if } S_1(\mu)=\emptyset; \end{cases}$$

where $a_{m,n}, c_{m,n}, s_{m,n}, f_{m,n}, g_{m,n}$ and $L(K_2)$ are defined in Proposition 3.2.

In view of this and Proposition 2.1, we are led to the following claim.

Proposition 3.2. *Let $x \in \hat{D}(A(s))$ where $0 \le s \le T$, and let $\lambda, \mu > 0$, $n, m \in \mathbb{N}$, be such that $0 \le (s+m\lambda), (s+n\mu) \le T$, and such that $\lambda_0 > \lambda \ge \mu > 0$ for which $\mu\omega, \lambda\omega < 1$. If $A(t)$ satisfies the dissipativity condition (H1), the range condition (H2'), and the time-regulating condition (HA), then the inequality is true:*

$$a_{m,n} \le c_{m,n}+s_{m,n}+d_{m,n}+e_{m,n}+f_{m,n}+g_{m,n}. \tag{3.1}$$

Here

$$a_{m,n} \equiv \|\prod_{i=1}^{n} J_\mu(s+i\mu)x - \prod_{i=1}^{m} J_\lambda(s+i\lambda)x\|;$$

$$\gamma \equiv (1-\mu\omega)^{-1} > 1; \quad \alpha \equiv \frac{\mu}{\lambda}; \quad \beta \equiv 1-\alpha;$$

$$c_{m,n} = 2K_1\gamma^n[(n\mu - m\lambda) + \sqrt{(n\mu-m\lambda)^2+(n\mu)(\lambda-\mu)}];$$

$$s_{m,n} = 2K_1\gamma^n(1-\lambda\omega)^{-m}\sqrt{(n\mu-m\lambda)^2+(n\mu)(\lambda-\mu)};$$

$$d_{m,n} = [K_4\rho(\delta)\gamma^n(m\lambda)] + \{K_4\frac{\rho(T)}{\delta^2}\gamma^n[(m\lambda)(n\mu-m\lambda)^2+(\lambda-\mu)\frac{m(m+1)}{2}\lambda^2]\};$$

$$e_{m,n} = L(K_2)\gamma^n\sqrt{(n\mu-m\lambda)^2+(n\mu)(\lambda-\mu)};$$

$$f_{m,n} = K_1[\gamma^n\mu + \gamma^n(1-\lambda\omega)^{-m}\lambda];$$

$$g_{m,n} = K_4\rho(|\lambda-\mu|)\gamma^n(m\lambda);$$

$$K_4 = \gamma L(K_2)(1+K_3); \quad \delta > 0 \quad \textit{is arbitrary};$$

$$\rho(r) \equiv \sup\{|f(t)-f(\tau)| : 0 \le t,\tau \le T, |t-\tau| \le r\}$$

where $\rho(r)$ is the modulus of continuity of f on $[0,T]$; and K_1, K_2, and K_3 are defined in Proposition 2.2.

Proof. We will use the method of mathematical induction and divide the proof into two steps. Step 2 will involve six cases.

Step 1. (3.1) is clearly true by Proposition 2.2, if $(m,n) = (0,n)$ or $(m,n) = (m,0)$.

Step 2. By assuming that (3.1) is true for $(m,n) = (m-1,n-1)$ or $(m,n) = (m,n-1)$, we will show that it is also true for $(m,n) = (m,n)$. This is done by the arguments below.

Using the nonlinear resolvent identity in [6], we have

$$\begin{aligned} a_{m,n} = \|J_u(s+n\mu)\prod_{i=1}^{n-1} J_\mu(s+i\mu)x \\ - J_\mu(s+m\lambda)[\alpha\prod_{i=1}^{m-1} J_\lambda(s+i\lambda)x + \beta\prod_{i=1}^{m} J_\lambda(s+i\lambda)x)]\|. \end{aligned}$$

Here $\alpha = \mu/\lambda$ and $\beta = (\lambda-\mu)/\lambda$.

Under the time-regulating condition (HA), it follows that, if the element inside the norm of the right side of the above equality is in $S_1(\mu)$, then, by Proposition 2.2 with $\epsilon = \mu$,

$$a_{m,n} \le L(\|\prod_{i=1}^{n} J_\mu(s+i\mu)x\|)|m\lambda - n\mu| \le L(K_2)|m\lambda - n\mu|, \tag{3.2}$$

which is less than or equal to the right-hand side of (3.1) with $(m,n) = (m,n)$, where $\gamma^n > 1$.

If that element instead lies in $S_2(\mu)$, then, by Proposition 2.2 with $\epsilon = \mu$,

$$\begin{aligned} a_{m,n} &\le \gamma(\alpha a_{m-1,n-1} + \beta a_{m,n-1}) + \gamma\mu|f(s+m\lambda) - f(s+n\mu)| \\ &\quad \times L(\|\prod_{i=1}^{n} J_\mu(s+i\mu)x\|)[1 + \|\frac{\prod_{i=1}^{n} J_\mu(s+i\mu)x - \prod_{i=1}^{n-1} J_\mu(s+i\mu)x}{\mu}\|] \\ &\le [\gamma\alpha a_{m-1,n-1} + \gamma\beta a_{m,n-1}] + K_4\mu\rho(|n\mu - m\lambda|), \end{aligned} \tag{3.3}$$

where $K_4 = \gamma L(K_2)(1+K_3)$ and $\rho(r)$ is the modulus of continuity of f on $[0,T]$. From this, it follows that proving the relations is sufficient under the induction assumption:

$$\gamma\alpha p_{m-1,n-1} + \gamma\beta p_{m,n-1} \le p_{m,n}; \tag{3.4}$$

$$\gamma\alpha q_{m-1,n-1} + \gamma\beta q_{m,n-1} + K_4\mu\rho(|n\mu - m\lambda|) \le q_{m,n}; \tag{3.5}$$

where $q_{m,n} = d_{m,n}$, and $p_{m,n} = c_{m,n}$ or $s_{m,n}$ or $e_{m,n}$ or $f_{m,n}$ or $g_{m,n}$.

Now we consider five cases.

Case 1. Here $p_{m,n} = c_{m,n}$. Under this case, (3.4) is true because of the calculations, where

$$b_{m,n} = \sqrt{(n\mu - m\lambda)^2 + (n\mu)(\lambda - \mu)}$$

was defined and the Schwartz inequality was used:

$$\alpha[(n-1)\mu - (m-1)\lambda] + \beta[(n-1)\mu - m\lambda] = (n\mu - m\lambda);$$

$$
\begin{aligned}
\alpha b_{m-1,n-1}+\beta b_{m,n-1} &= \sqrt{\alpha}\sqrt{\alpha}b_{m-1,n-1}+\sqrt{\beta}\sqrt{\beta}b_{m,n-1}\\
&\le (\alpha+\beta)^{1/2}(\alpha b_{m-1,n-1}^2+\beta b_{m,n-1}^2)^{1/2}\\
&\le \{(\alpha+\beta)(n\mu-m\lambda)^2+2(n\mu-m\lambda)[\alpha(\lambda-\mu)-\beta\mu]\\
&\quad +[\alpha(\lambda-\mu)^2+\beta\mu^2]+(n-1)\mu(\lambda-\mu)\}^{1/2}\\
&= b_{m,n}.
\end{aligned}
$$

Here

$$
\alpha+\beta=1;\quad \alpha(\lambda-\mu)-\beta\mu=0;\quad \alpha(\lambda-\mu)^2+\beta\mu^2=\mu(\lambda-\mu).
$$

Case 2. Here $p_{m,n}=s_{m,n}$. Under this case, (3.4) is true, as is with the Case 1, by noting that

$$
(1-\lambda\omega)^{-(m-1)}\le(1-\lambda\omega)^{-m}.
$$

Case 3. Here $q_{m,n}=d_{m,n}$. Under this case, (3.5) is true because of the calculations:

$$
\begin{aligned}
&\gamma\alpha d_{m-1,n-1}+\gamma\beta d_{m,n-1}+K_4\mu\rho(|n\mu-m\lambda|)\\
&\le \{\gamma\alpha[K_4\rho(\delta)\gamma^{n-1}(m-1)\lambda]+\gamma\beta[K_4\rho(\delta)\gamma^{n-1}(m\lambda)]\}\\
&\quad +\gamma\alpha\{K_4\frac{\rho(T)}{\delta^2}\gamma^{n-1}[(m-1)\lambda\,((n-1)\mu-(m-1)\lambda)^2+(\lambda-\mu)\frac{(m-1)m}{2}\lambda^2]\}\\
&\quad +\gamma\beta\{K_4\frac{\rho(T)}{\delta^2}\gamma^{n-1}[(m\lambda)\,((n-1)\mu-m\lambda)^2+(\lambda-\mu)\frac{m(m+1)}{2}\lambda^2]\}\\
&\quad +K_4\mu\rho(|n\mu-m\lambda|)\\
&= K_4\rho(\delta)\gamma^n[(\alpha+\beta)(m\lambda)-\alpha\lambda]\\
&\quad +K_4\frac{\rho(T)}{\delta^2}\gamma^n\{\alpha[(n\mu-m\lambda)^2+2(n\mu-m\lambda)(\lambda-\mu)+(\lambda-\mu)^2](m\lambda-\lambda)\\
&\quad +[\alpha(\lambda-\mu)\frac{m(m+1)}{2}\lambda^2-\alpha(\lambda-\mu)m\lambda^2]\\
&\quad +\beta[(n\mu-m\lambda)^2-2(n\mu-m\lambda)\mu+\mu^2](m\lambda)\\
&\quad +[\beta(\lambda-\mu)\frac{m(m+1)}{2}\lambda^2]\}+K_4\mu\rho(|n\mu-m\lambda|)\\
&\quad \le K_4\rho(\delta)\gamma^n[(m\lambda)-\mu]+K_4\mu\rho(|n\mu-m\lambda|)\\
&\quad +K_4\frac{\rho(T)}{\delta^2}\gamma^n[(m\lambda)(n\mu-m\lambda)^2+(\lambda-\mu)\frac{m(m+1)}{2}\lambda^2-\mu(n\mu-m\lambda)^2]\\
&\equiv r_{m,n},
\end{aligned}
$$

where the negative terms $[2(n\mu-m\lambda)(\lambda-\mu)+(\lambda-\mu)^2](-\lambda)$ were dropped,

$$
\alpha 2(n\mu-m\lambda)(\lambda-\mu)-\beta 2(n\mu-m\lambda)\mu=0,
$$

and

$$
[\alpha(\lambda-\mu)^2+\beta\mu^2](m\lambda)=(m\lambda)\mu(\lambda-\mu),
$$

which cancelled

$$
-\alpha(\lambda-\mu)m\lambda^2=-(m\lambda)\mu(\lambda-\mu);
$$

it follows that $r_{m,n}\le d_{m,n}$, since

$$
K_4\mu\rho(|n\mu-m\lambda|)
$$

$$\le \begin{cases} K_4\mu\rho(\delta) \le K_4\mu\rho(\delta)\gamma^n, & \text{if } |n\mu - m\lambda| \le \delta; \\ K_4\mu\rho(T)\frac{(n\mu-m\lambda)^2}{\delta^2} \le K_4\mu\rho(T)\gamma^n\frac{(n\mu-m\lambda)^2}{\delta^2}, & \text{if } |n\mu - m\lambda| > \delta. \end{cases}$$

Case 4. Here $p_{m,n} = e_{m,n}$. Under this case, (3.4) is true, as is with the Case 1.

Case 5. Here $p_{m,n} = f_{m,n}$. Under this case, (3.4) is true because of the calculations:

$$\begin{aligned} \gamma\alpha f_{m-1,n-1} + \gamma\beta f_{m,n-1} &= \gamma\alpha K_1[\gamma^{n-1}\mu + \gamma^{n-1}(1-\lambda\omega)^{-(m-1)}\lambda] \\ &\quad + \gamma\beta K_1[\gamma^{n-1}\mu + \gamma^{n-1}(1-\lambda\omega)^{-m}\lambda] \\ &\le K_1[(\alpha+\beta)\gamma^n\mu + (\alpha+\beta)\gamma^n(1-\lambda\omega)^{-m}\lambda], \\ &= f_{m,n}. \end{aligned}$$

Case 6. Here $p_{m,n} = g_{m,n}$. Under this case, (3.4) is true because of the calculations:

$$\begin{aligned} \gamma\alpha g_{m-1,n-1} + \gamma\beta g_{m,n-1} &\le K_4\gamma^n\rho(|\lambda-\mu|)\alpha(m-1)\lambda + K_4\gamma^n\rho(|\lambda-\mu|)\beta(m\lambda) \\ &\le K_4\gamma^n\rho(|\lambda-\mu|)(\alpha+\beta)(m\lambda) \\ &= g_{m,n}. \end{aligned}$$

Now the proof is complete. □

Here is one of our two main results:

Theorem 3.3. *If the nonlinear operator $A(t)$ satisfies the dissipativity condition* (H1)*, the range condition* (H2')*, and the time-regulating condition* (HA) *, then*

$$U(s+t,s)u_0 \equiv \lim_{n\to\infty}\prod_{i=1}^{n} J_{\frac{t}{n}}(s+i\frac{t}{n})u_0$$

exists for $u_0 \in \overline{\hat{D}(A(s))} = \overline{D(A(s))}$ where $s,t \ge 0$ and $0 \le (s+t) \le T$, and is the so-called a limit solution to the equation (1.1) *. Furthermore, this limit $U(s+t,s)u_0$ has the Lipschitz property*

$$\|U(s+t,s)u_0 - U(s+\tau,s)u_0\| \le k|t-\tau|$$

for $0 \le s+t, s+\tau \le T$ and for $u_0 \in \hat{D}(A(s))$.

Proof. For $x \in \hat{D}(A(s))$, it follows from Proposition 3.2, by setting $\mu = \frac{t}{n}, \lambda = \frac{t}{m}$, and $\delta^2 = \sqrt{\lambda-\mu}$ that, as $n,m \to \infty$, $a_{m,n}$ converges to 0, uniformly for $0 \le (s+t) \le T$. Thus

$$\lim_{n\to\infty}\prod_{i=1}^{n} J_{\frac{t}{n}}(s+i\frac{t}{n})x$$

exists for $x \in \hat{D}(A(s))$. This limit also exits for $x \in \overline{\hat{D}(A(s))} = \overline{D(A(s))}$), on following the limiting arguments in Crandall-Pazy [9].

On the other hand, setting $\mu = \lambda = t/n$, $m = [\frac{t}{\mu}]$ and setting $\delta^2 = \sqrt{\lambda-\mu}$, it follows that

$$\lim_{n\to\infty}\prod_{i=1}^{n} J_{\frac{t}{n}}(s+i\frac{t}{n})u_0 = \lim_{\mu\to 0}\prod_{i=1}^{[\frac{t}{\mu}]} J_{\mu}(s+i\mu)u_0. \tag{3.6}$$

Now, to show the Lipschitz property, (3.6) and Crandall-Pazy [9, Page 71] will be used. From Proposition 2.2, it is derived that

$$\begin{aligned}
\|u_n - u_m\| &\le \|u_n - u_{n-1}\| + \|u_{n-1} - u_{n-2}\| + \dots + \|u_{m+1} - u_m\| \\
&\le K_3\mu(n-m) \quad \text{for } x \in \hat{D}(A(s)), \\
u_n &= \prod_{i=1}^{n} J_\mu(s+i\mu)x, \quad u_m = \prod_{i=1}^{m} J_\mu(s+i\mu)x,
\end{aligned}$$

where $n = [t/\mu]$, $m = [\tau/\mu]$, $t > \tau$ and $0 < \mu < \lambda_0$. The proof is completed by making $\mu \to 0$ and using (3.6). □

Now discretize (1.1) as

$$\begin{gathered}
u_i - \epsilon A(t_i)u_i \ni u_{i-1}, \\
u_i \in D(A(t_i)),
\end{gathered} \tag{3.7}$$

where $n \in \mathbb{N}$ is large, and ϵ is such that $s \le t_i = s + i\epsilon \le T$ for each $i = 1, 2, \dots, n$. Here notice that, for $u_0 \in E$, u_i exists uniquely by the hypotheses (H1) and (H2').

Let $u_0 \in \hat{D}(A(s))$, and construct the Rothe functions [12, 32]. Let

$$\begin{gathered}
\chi^n(s) = u_0, \quad C^n(s) = A(s), \\
\chi^n(t) = u_i, \quad C^n(t) = A(t_i) \quad \text{for } t \in (t_{i-1}, t_i],
\end{gathered}$$

and let

$$\begin{gathered}
u^n(s) = u_0, \\
u^n(t) = u_{i-1} + (u_i - u_{i-1})\frac{t - t_{i-1}}{\epsilon} \quad \text{for } t \in (t_{i-1}, t_i] \subset [s, T].
\end{gathered}$$

Since $\|\frac{u_i - u_{i-1}}{\epsilon}\| \le K_3$ for $u_0 \in \hat{D}(A(s))$ by Proposition 2.1, it follows that, for $u_0 \in \hat{D}(A(s))$,

$$\begin{gathered}
\lim_{n\to\infty} \sup_{t\in[0,T]} \|u^n(t) - \chi^n(t)\| = 0, \\
\|u^n(t) - u^n(\tau)\| \le K_3|t - \tau|,
\end{gathered} \tag{3.8}$$

where $t, \tau \in (t_{i-1}, t_i]$, and that, for $u_0 \in \hat{D}A(s))$,

$$\begin{gathered}
\frac{du^n(t)}{dt} \in C^n(t)\chi^n(t), \\
u^n(s) = u_0,
\end{gathered} \tag{3.9}$$

where $t \in (t_{i-1}, t_i]$. Here the last equation has values in $B([s, T]; X)$, which is the real Banach space of all bounded functions from $[s, T]$ to X.

Proposition 3.4. *If $A(t)$ satisfies the assumptions in Theorem 3.3, then*

$$\lim_{n\to\infty} u^n(t) = \lim_{n\to\infty} \prod_{i=1}^{n} J_{\frac{t-s}{n}}(s + i\frac{t}{n})u_0$$

uniformly for finite $0 \le (s+t) \le T$ and for $u_0 \in \hat{D}(A(s))$.

Proof. The asserted uniform convergence will be proved by using the Ascoli-Arzela Theorem [33].

Pointwise convergence will be proved first. For each $t \in [s,T)$, we have $t \in [t_i, t_{i+1})$ for some i, and so $i = [\frac{t-s}{\epsilon}]$, the greatest integer that is less than or equal to $\frac{t-s}{\epsilon}$. That u_i converges is because, for each above t,

$$\begin{aligned} \lim_{\epsilon\to 0} u_i &= \lim_{\epsilon\to 0} \prod_{k=1}^{i} (I - \epsilon A(t_k))^{-1} u_0 \\ &= \lim_{n\to\infty} \prod_{k=1}^{n} [I - \frac{t-s}{n} A(s + k\frac{t-s}{n})]^{-1} u_0 \end{aligned} \tag{3.10}$$

by (3.6), which has the right side convergent by Theorem 3.3. Since

$$\|\frac{u_i - u_{i-1}}{\epsilon}\| \le K_3$$

for $u_0 \in \hat{D}(A(s))$, we see from the definition of $u^n(t)$ that

$$\lim_{n\to\infty} u^n(t) = \lim_{\epsilon\to 0} u_i = \lim_{n\to\infty} \prod_{i=1}^{n} J_{\frac{t-s}{n}}(s + i\frac{t-s}{n}) u_0$$

for each t.

On the other hand, due to

$$\|\frac{u_i - u_{i-1}}{\epsilon}\| \le K_3$$

again, we see that $u^n(t)$ is equi-continuous in $C([s,T];X)$, the real Banach space of all continuous functions from $[s,T]$ to X. Thus it follows from the Ascoli-Arzela theorem [33] that, for $u_0 \in \hat{D}(A(s))$, some subsequence of $u^n(t)$ (and then itself) converges uniformly to some

$$u(t) = \lim_{n\to\infty} \prod_{i=1}^{n} J_{\frac{t-s}{n}}(s + i\frac{t-s}{n}) u_0 \in C([s,T];X).$$

This completes the proof. □

Now consider a strong solution. Let $(Y, \|\cdot\|_Y)$ be a real Banach space, into which the real Banach space $(X, \|\cdot\|)$ is continuously embedded. Assume additionally that $A(t)$ satisfies the embedding property of embeddedly quasi-demi-closedness:

(HB) If $t_n \in [0,T] \to t$, if $x_n \in D(A(t_n)) \to x$, and if $\|y_n\| \le k$ for some $y_n \in A(t_n)x_n$, then $\eta(A(t)x)$ exists and

$$|\eta(y_{n_l}) - z| \to 0$$

for some subsequence y_{n_l} of y_n, for some $z \in \eta(A(t)x)$, and for each $\eta \in Y^* \subset X^*$, the real dual space of Y.

Here is the other main result.

Theorem 3.5. *Let $A(t)$ satisfy the dissipativity condition* (H1)*, the range condition* (H2')*, the time-regulating condition* (HA)*, and the embedding property* (HB)*. Then equation* (1.1)*, for $u_0 \in \hat{D}(A(s))$, has a strong solution*

$$u(t) = \lim_{n\to\infty} \prod_{i=1}^{n} J_{\frac{t-s}{n}}(s + i\frac{t}{n}) u_0$$

in Y, in the sense that

$$\frac{d}{dt}u(t) \in A(t)u(t) \quad \text{in } Y \text{ for almost every } t \in (0,T);$$
$$u(s) = u_0.$$

The solution is unique if $Y \equiv X$. Furthermore,

$$\|u(t) - u(\tau)\|_X \le K_3|t-\tau|$$

for $0 \le s \le t$, $\tau \le T$, a result from Theorem 3.3.

The results in the above theorem follow from Theorem 3.3 and the proof in [20, page 364], [21, pages 262-263].

Remark 3.6. The results in Sections 2 and 3 are still true if the range condition (H2') is replaced by the weaker condition (H2") below, provided that the initial conditions $u_0 \in \hat{D}(A(s))(\supset D(A(s)))$ and $u_0 \in \overline{\hat{D}(A(s))} = \overline{D(A(s))}(\supset D(A(s)))$ are changed to the condition $u_0 \in D(A(s))$. This is readily seen from the corresponding proofs. Here

- (H2") The range of $(I - \lambda A(t))$, denoted by E, is independent t and contains $D(A(t))$ for all $t \in [0,T]$ and for small $0 < \lambda < \lambda_0$ with $\lambda_0\omega < 1$.

4. Applications to partial differential equations (I)

Within this section, K will denote a constant that can vary with different occasions. Now we make the following assumptions:

- (A1) Ω is a bounded smooth domain in $\mathbb{R}^n, n \ge 2$, and $\partial\Omega$ is the boundary of Ω.
- (A2) $\nu(x)$ is the unit outer normal to $x \in \partial\Omega$, and μ is a real number such that $0 < \mu < 1$.
- (A3) $\alpha(x,t,p) \in C^2(\overline{\Omega} \times \mathbb{R}^n)$ is true for each $t \in [0,T]$, and is continuous in all its arguments. Furthermore, $\alpha(x,t,p) \ge \delta_0 > 0$ is true for all x, z, and all $t \in [0,T]$, and for some constant $\delta_0 > 0$.
- (A4) $g(x,t,z,p) \in C^2(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ is true for each $t \in [0,T]$, is continuous in all its arguments, and is monotone non-increasing in z for each t, x, and p.
- (A5) $\frac{g(x,t,z,p)}{\alpha(x,t,p)}$ is of at most linear growth in p, that is ,

$$|\frac{g(x,t,z,p)}{\alpha(x,t,p)}| \le M(x,t,z)(1+|p|)$$

  for some continuous function M and for all $t \in [0,T]$ when $|p|$ is large enough.
- (A6) $\beta(x,t,z) \in C^3(\Omega \times \mathbb{R})$ is true for each $t \in [0,T]$, is continuous in all its arguments, and is strictly monotone increasing in z so that $\beta_z \ge \delta_0 > 0$ for the constant $\delta_0 > 0$ in (A3).
- (A7)

$$\begin{gathered}
|\alpha(x,t,p) - \alpha(x,\tau,p)| \le |\zeta(t) - \zeta(\tau)|N_1(x,|p|), \\
|g(x,t,z,p) - g(x,\tau,z,p)| \le |\zeta(t) - \zeta(\tau)|N_2(x,|z|,|p|), \\
|\beta(x,t,z) - \beta(x,\tau,z)| \le |t-\tau|N_3(x,|z|)
\end{gathered}$$

  are true for some continuous positive functions N_1, N_2, N_3 and for some continuous function ζ of bounded variation.

Define the t-dependent nonlinear operator $A(t) : D(A(t)) \subset C(\overline{\Omega}) \to C(\overline{\Omega})$ by

$$D(A(t)) = \{u \in C^{2+\mu}(\overline{\Omega}) : \frac{\partial u}{\partial \nu} + \beta(x,t,u) = 0 \quad \text{on } \partial\Omega\} \quad \text{and}$$
$$A(t)u = \alpha(x,t,Du)\Delta u + g(x,t,u,Du) \quad \text{for } u \in D(A(t)).$$

Example 4.1. Consider the equation

$$\begin{gathered} \frac{\partial}{\partial t}u(x,t) = \alpha(x,t,Du)\Delta u + g(x,t,u,Du), \quad (x,t) \in \Omega \times (0,T), \\ \frac{\partial}{\partial \nu}u + \beta(x,t,u) = 0, \quad x \in \partial\Omega, \\ u(x,0) = u_0, \end{gathered} \tag{4.1}$$

for $u_0 \in D(A(0))$. The above equation has a strong solution

$$u(t) = \lim_{n\to\infty} \prod_{i=1}^{n} J_{\frac{t}{n}}(i\frac{t}{n})u_0$$

in $L^2(\Omega)$ with

$$\frac{\partial}{\partial \nu}u(t) + \beta(x,t,u(t)) = 0, \quad x \in \partial\Omega,$$

and the solution $u(t)$ satisfies the property

$$\sup_{t\in[0,T]} \|u(t)\|_{C^{1+\mu}(\overline{\Omega})} \le K$$

for some constant K.

Proof. It was shown in [21, Pages 264-268] that $A(t)$ satisfies the dissipativity condition (H1), the range condition (H2") with $E = C^{\mu}(\overline{\Omega})$ for any $0 < \mu < 1$, and satisfies the time-regulating condition (HA) and the embedding property (HB). Here the third line on [21, Page 268]:

$$\times[\|N_2(z, \|v\|_\infty, \|Dv\|_\infty)\|_\infty + \frac{\|N_1(z, \|Dv\|_\infty)\|_\infty}{\delta_1}\|A(\tau)v\|_\infty)]$$

should have $\|A(\tau)v\|_\infty$ replaced by

$$[\|A(\tau)v\|_\infty + \|g(z,\tau,v,Dv)\|_\infty].$$

Hence Remark 3.6 and Theorems 3.3 and 3.5 are applicable.

It remains to prove that $u(t)$ satisfies the mentioned property and the middle equation in (4.1) in $C(\overline{\Omega})$. This basically follows from [21, pages 264-268]. To this end, the u_i in (3.7) will be used.

Since $A(t)$ satisfies (H1), (H2"), and (HA), it follows from Proposition 2.2 and Remark 3.6 that

$$\|\frac{u_i - u_{i-1}}{\epsilon}\| = \|A(t_i)u_i\|_\infty \le K_3 \quad \text{and} \quad \|u_i\|_\infty \le K_2.$$

Thus, from linear L^p elliptic theory [35, 14], it follows that $\|u_i\|_{W^{2,p}} \le K$ for some constant K, whence

$$\|u_i\|_{C^{1+\eta}} \le K \tag{4.2}$$

for any $0 < \eta < 1$ by the Sobolev embedding theorem [14]. This, together with the interpolation inequality [14] and the Ascoli-Arzela theorem [14, 33], implies that a convergent subsequence of u_i converges in $C^{1+\mu}(\overline{\Omega})$ for any $0<\lambda<\eta<1$. Therefore, on account of (3.10) and Proposition 3.4,

$$\sup_{t\in[0,T]}\|u(t)\|_{C^{1+\mu}}\le K$$

results for $u_0\in D(A(0))$, and $u(t)$ satisfies the middle equation in (4.1) in $C(\overline{\Omega})$. The proof is complete □

Consider the linear equation

$$\begin{gathered}\frac{\partial u(x,t)}{\partial t}=\sum_{i,j=1}^{n}a_{ij}(x,t)D_{ij}u(x,t)+\sum_{i=1}^{n}b_i(x,t)D_iu(x,t)+c(x,t)u(x,t)\\ \text{for } (x,t)\in\Omega\times(0,T),\\ \frac{\partial}{\partial\nu}u+\beta(x,t)u=0,\quad x\in\partial\Omega,\\ u(x,0)=u_0,\end{gathered}\tag{4.3}$$

in which the following are assumed. Let $a_{ij}(x,t)=a_{ji}(x,t)$, and let

$$\lambda_{\min}|\xi|^2\le\sum_{i,j}^{n}a_{ij}(x,t)\xi_i\xi_j\le\lambda_{\max}|\xi|^2$$

for some positive constants $\lambda_{\min}$, $\lambda_{\max}$, for all $\xi\in\mathbb{R}^n$, and for all x,t. Let

$$a_{ij}(x,t),\ b_i(x,t),\ c(x,t)\in C^{\mu}(\overline{\Omega})$$

uniformly for all t, be continuous in all their arguments, and be of bounded variation in t uniformly for x. Let $c(x,t)\le 0$ for all x,t,

$$\beta(x,t)\in C^{1+\mu}(\overline{\Omega}),\quad 0<\mu<1$$

for all t, and $\beta(x,t)\ge\delta>0$ for some constant $\delta>0$. Finally, let $\beta(x,t)$ and $c(x,t)$ be continuous in all its arguments, and let $\beta(x,t)$ be Lipschitz continuous in t uniformly for x.

Example 4.2. If $\sum_{i,j}a_{ij}(x,t)D_{ij}u(x,t)=a_0(x,t)\Delta u(x,t)$ for some $a_0(x,t)$, then the equation (4.3), for $u_0\in D(A(0))$, has a strong solution

$$u(t)=\lim_{n\to\infty}\prod_{i=1}^{n}J_{\frac{t}{n}}(i\frac{t}{n})u_0$$

in $L^2(\Omega)$ with

$$\frac{\partial}{\partial\nu}u(t)+\beta(x,t)u(t)=0,\quad x\in\partial\Omega,$$

and $u(t)$ satisfies the property

$$\sup_{t\in[0,T]}\|u(t)\|_{C^{1+\mu}(\overline{\Omega})}\le K.$$

Proof. Linear elliptic equation theory [14, Pages 128-130] shows that the corresponding operator $A(t)$ satisfies the range condition (H2”) with $E=C^{\mu}(\overline{\Omega})$. The arguments in [21, Pages 267-268] shows that $A(t)$ satisfies the dissipativity condition (H1), the time-regulating condition (HA), and the embedding property (HB). The proof is complete, after applying Remark 3.6, Theorems 3.3 and 3.5, and the proof for Theorem 4.1. □

Example 4.3. Suppose that

$$a_{ij}(x),\ b_i(x),\ c(x) \in C^{1+\mu}(\overline{\Omega}),\ \beta(x) \in C^{2+\mu}(\overline{\Omega})$$

are independent of t, where $0 < \mu < 1$. Then equation (4.3) has a unique classical solution

$$u(t) = \lim_{n\to\infty} \prod_{i=1}^{n} J_{\frac{t}{n}}(i\frac{t}{n})u_0 = \lim_{n\to\infty}(I - \frac{t}{n}A)^{-n}u_0$$

for $u_0 \in D(A)$ with $Au_0 \in D(A)$, and the solution has the properties that $\frac{du(t)}{dt}$ is Lipschitz continuous in t, and that

$$\|\frac{du}{dt}\|_{C^{1+\mu}(\overline{\Omega})} \le K.$$

Furthermore, $\frac{d}{dt}u$ is differentiable in t and $\frac{d^2}{dt^2}u(t)$ is Lipschitz continuous in t, if u_0 is in $D(A^3)$ such that $A^3u_0 \in D(A)$. More regularity of $\frac{du}{dt}$ in t can be obtained iteratively.

Remark 4.4. In order for u_0 to be in $D(A^2)$, more smoothness assumptions should be imposed on the coefficient functions $a_{ij}(x), b_i(x), c(x)$ and $\beta(x)$.

Proof. Here observe that the operator A is not closed, and so [20, Theorem 1 Page 363] does not apply directly.

The u_i in (3.7) will be used, and $u_0 \in D(A)$ with $Au_0 \in D(A)$ will be assumed for a moment. It follows that

$$Au_i = \frac{u_i - u_{i-1}}{\epsilon} = (I - \epsilon A)^{-i}(Au_0),$$

and hence, by (4.2) which is for the proof of Theorem 4.1,

$$\|Au_i\|_{C^{1+\eta}(\overline{\Omega})} = \|(I - \epsilon A)^{-i}(Au_0)\|_{C^{1+\eta}(\overline{\Omega})} \le K$$

for $Au_0 \in D(A)$ and for any $0 < \eta < 1$. This implies

$$\|u_i\|_{C^{3+\eta}(\overline{\Omega})} \le K$$

by the Schauder global estimate with more smoothness in the linear elliptic theory [14]. Consequently, on using the interpolation inequality [14] and the Ascoli-Arzela theorem [14, 33], we have

$$Au_i \to Au(t) = U(t)(Au_0)$$

through some subsequence with respect to the topology in $C^{1+\lambda}(\overline{\Omega})$ for any $0 < \lambda < \eta < 1$. Here

$$U(t)u_0 \equiv \lim_{n\to\infty}(I - \frac{t}{n}A)^{-n}u_0.$$

The rest follows from [20, Page 363], where the Lipschitz property in Theorem 3.3 and Remark 3.6 will be used. □

Now consider the linear equation with the space dimension 1:

$$\begin{gathered}\frac{\partial u}{\partial t} = a(x,t)u_{xx} + b(x,t)u_x + c(x,t)u, \quad (x,t) \in (0,1) \times (0,T),\\ u'(j,t) = (-1)^j \beta_j(j,t)u(j,t), \quad j = 0, 1,\\ u(x,0) = u_0(x).\end{gathered} \tag{4.4}$$

Here we assume that a, b, c are jointly continuous in $x \in [0,1]$, $t \in [0,T]$, and are of bounded variation in t uniformly for all x, that $c(x,t) \le 0$ and $a(x,t) \ge \delta_0$ for some constant $\delta_0 > 0$, and finally that $\beta_j \geq \delta_0 > 0, j = 0, 1$ are jointly continuous in x, t, and are Lipschitz continuous in t, uniformly over x.

Let $A(t)$: $D(A(t)) \subset C[0,1] \to C[0,1]$ be the operator defined by

$$\begin{gathered} A(t)u \equiv a(x,t)u'' + b(x,t)u' + c(c,t)u \quad \text{for } u \in D(A(t)) \text{ where} \\ D(A(t)) \equiv \{v \in C^2[0,1] : v'(j) = (-1)^j \beta_j(j,t)v(j), j = 0, 1\}. \end{gathered}$$

Following [20] and the proof for the previous case of higher space dimensions, and applying linear ordinary differential equation theory [5, 25] and Theorem 3.5, the next example is readily proven. Here the range condition (H2') is satisfied with $E = C[0,1] \supset \overline{D(A(t))}$ for all t.

Example 4.5. Equation (4.4) has a strong solution

$$u(t) = \lim_{n\to\infty} (I - \frac{t}{n}A)^{-n} u_0$$

in $L^2(0,1)$ for $u_0 \in \hat{D}(A(0))$, and $u(t)$ satisfies the middle equation in (4.4) and the Lipschitz property

$$\|u(t) - u(\tau)\|_\infty \leq k|t - \tau|$$

for $u_0 \in \hat{D}(A(0))$ and for $0 \leq t, \tau \leq T$.

In the case that a, b, c, β_j, for $j = 0, 1$, are independent of t, the Theorem 1 in [20, Page 363], together with the Lipschitz property in the Theorem 3.3 in this paper, will readily deliver the following example. Here it is to be observed that the corresponding operator A is closed.

Example 4.6. If the coefficient functions $a, b, c, \beta_j, j = 0, 1$ are independent of t, then the equation (4.4) has a unique classical solution

$$u(t) = \lim_{n\to\infty} (I - \frac{t}{n}A)^{-n} u_0$$

for $u_0 \in D(A)$ with $Au_0 \in \overline{D(A)}$. This $u(t)$ has this property that the function $\frac{du}{dt}$ is continuous in t.

Furthermore, $u(t)$ is Lipschitz continuous in t for $u_0 \in \hat{D}(A)$, and $\frac{du}{dt}$ is Lipschitz continuous in t for $u_0 \in D(A)$ with $Au_0 \in \hat{D}(A)$, and is differentiable in t for $u_0 \in D(A^2)$ with $A^2 u_0 \in \overline{D(A)}$. More regularity of $\frac{du}{dt}$ can be obtained iteratively.

Remark 4.7. In order for u_0 to be in $D(A^2)$, more smoothness assumptions should be imposed on the coefficient functions $a(x), b(x), c(x)$, and $\beta_j, j = 0, 1$.

5. Applications to partial differential equations (II)

In this section, it will be further shown that, for each concrete $A(t)$ in Section 4, the corresponding quantity

$$J_{\frac{t}{n}}(i\frac{t}{n})h = [I - \frac{t}{n}A(i\frac{t}{n})]^{-1}h, \quad i = 1, 2, \ldots, n$$

is the limit of a sequence where each term in the sequence is an explicit function of the solution ϕ to the elliptic equation (1.3) with $\varphi \equiv 0$.

We start with the case of linear $A(t)$ and consider the parabolic equation (4.3).

Proposition 5.1. *For $h \in C^{\mu}(\overline{\Omega})$, the solution u to the equation*
$$[I - \epsilon A(t)]u = h \tag{5.1}$$
where $0 \leq t \leq T$ and $\epsilon > 0$, is the limit of a sequence where each term in the sequence is an explicit function of the solution ϕ to the elliptic equation (1.3) *with $\varphi \equiv 0$. Here $A(t)$ is the linear operator corresponding to the parabolic equation* (4.3)*.*

Proof. The linear operator $A(t) : D(A(t)) \subset C(\overline{\Omega}) \to C(\overline{\Omega})$ is defined by
$$A(t)u \equiv \sum_{i,j} a_{ij}(x,t)D_{ij}u + \sum_{i} b_i(x,t)D_i u + c(x,t)u$$
$$\text{for } u \in D(A(t)) \equiv \{u \in C^{2+\mu}(\overline{\Omega}) : \frac{\partial u}{\partial \nu} + \beta(x,t)u = 0 \quad \text{on } \partial\Omega\}.$$

Solvability of (5.1) follows from [14, Pages 128-130], where the method of continuity [14, Page 75] is used. By writing out fully how the method of continuity is used, it will be seen that the solution u is the limit of a sequence where each term in the sequence is an explicit function of the solution ϕ to the elliptic equation (1.3) with $\varphi \equiv 0$. To this end, set
$$U_1 = C^{2+\mu}(\overline{\Omega}), \quad U_2 = C^{\mu}(\overline{\Omega}) \times C^{1+\mu}(\partial\Omega),$$
$$L_\tau u = \tau[u - \epsilon A(t)u] + (1-\tau)(-\Delta u) \quad \text{in} \quad \Omega,$$
$$N_\tau u = \tau[\frac{\partial u}{\partial \nu} + \beta(x,t)u] + (1-\tau)(\frac{\partial u}{\partial \nu} + u) \quad \text{on} \quad \partial\Omega,$$
where $0 \leq \tau \leq 1$. Define the linear operator $\pounds_\tau : U_1 \to U_2$ by
$$\pounds_\tau u = (L_\tau u, N_\tau u)$$
for $u \in U_1$, and assume that $\pounds_s$ is onto for some $s \in [0,1]$.

It follows from [14, Pages 128-130] that
$$\|u\|_{U_1} \leq C\|\pounds_\tau u\|_{U_2}, \tag{5.2}$$
where the constant C is independent of τ. This implies that $\pounds_s$ is one to one, and so $\pounds_s^{-1}$ exists. By making use of $\pounds_s^{-1}$, the equation, for $w_0 \in U_2$ given,
$$\pounds_\tau u = w_0$$
is equivalent to the equation
$$u = \pounds_s^{-1}w_0 + (\tau - s)\pounds_s^{-1}(\pounds_0 - \pounds_1)u,$$
from which a linear map $S : U_1 \to U_1$,
$$Su = S_s u \equiv \pounds_s^{-1}w_0 + (\tau - s)\pounds_s^{-1}(\pounds_0 - \pounds_1)u$$
is defined. The unique fixed point u of $S = S_s$ will be related to the solution of (5.1).

By choosing $\tau \in [0,1]$ such that
$$|s - \tau| < \delta \equiv [C(\|\pounds_0\|_{U_1 \to U_2} + \|\pounds_1\|_{U_1 \to U_2})]^{-1}, \tag{5.3}$$
it follows that $S = S_s$ is a strict contraction map. Therefore S has a unique fixed point w, and the w can be represented by
$$\lim_{n\to\infty} S^n 0 = \lim_{n\to\infty} (S_s)^n 0$$
because of $0 \in U_1$. Thus $\pounds_\tau$ is onto for $|\tau - s| < \delta$.

It follows that, by dividing $[0,1]$ into subintervals of length less than δ and repeating the above arguments in a finite number of times, $\pounds_\tau$ becomes onto for all $\tau \in [0,1]$, provided that it is onto for some $\tau \in [0,1]$. Since $\pounds_0$ is onto by the potential theory [14, Page 130], we have that $\pounds_1$ is also onto. Therefore, for $w_0 = (h,0)$, the equation

$$\pounds_1 u = w_0$$

has a unique solution u, and the u is the seeked solution to (5.1). Here it is to be observed that $\phi \equiv \pounds_0^{-1}(h,0)$ is the unique solution $\pounds_0^{-1}(h,\varphi)$ to the elliptic equation (1.3) with $\varphi \equiv 0$:

$$\begin{gathered} -\Delta v = h, \quad x \in \Omega, \\ \frac{\partial v}{\partial \nu} + v(x) = 0 \quad \text{on} \quad \partial\Omega, \end{gathered}$$

and that

$$\begin{gathered} S0 = S_0 0 = \pounds_0^{-1}(h,0), \\ S^2 0 = (S_0)^2 0 = \pounds_0^{-1}(h,0) + \pounds_0^{-1}[|\tau - 0|(\pounds_0 - \pounds_1)\pounds_0^{-1}(h,0)], \\ \dots . \end{gathered}$$

The proof is complete. □

Remark 5.2. • The solution u is eventually represented by

$$u(x) = \pounds_0^{-1} H((h,0)),$$

where $H((h,0))$ is a convergent series in which each term is basically obtained by, repeatedly, applying the linear operator $(\pounds_0 - \pounds_1)\pounds_0^{-1}$ to $(h,0)$ for a certain number of times.

• The quantity $\pounds_0^{-1}(h,\varphi)$, for each $(h,\varphi) \in U_2$ given, can be computed numerically and efficiently by the boundary element methods [13, 34], if the dimension of the space variable x equals 2 or 3.

• The constant C above in (5.2) and (5.3) depends on $n, \mu, \lambda_{\min}, \Omega$, and on the coefficient functions $a_{ij}(x,t), b_i(x,t), c(x,t), \beta(x,t)$, and is not known explicitly [14]. Therefore, the corresponding δ cannot be determined in advance, and so, when dealing with the elliptic equation (5.1) in Proposition 5.1 numerically, it is more possible, by choosing $\tau \in [0,1]$ such that $|s-\tau|$ is smaller, that the sequence $S^n 0$ will converge, for which $|s-\tau| < \delta$ occurs.

Next, we extend the above techniques to the case of nonlinear $A(t)$, and consider the nonlinear parabolic equation (4.1); more work is required in this case.

Proposition 5.3. *For $h \in C^\mu(\overline{\Omega})$, the solution u to the equation* (5.1)

$$[I - \epsilon A(t)]u = h$$

where $0 \le t \le T$ and $\epsilon > 0$, is the limit of a sequence where each term in the sequence is an explicit function of the solution ϕ to the elliptic equation (1.3) *with $\varphi \equiv 0$. Here $A(t)$ is the nonlinear operator corresponding to the parabolic equation* (4.1)*, and $\beta(x,t,0) \equiv 0$ is assumed additionally.*

Proof. The nonlinear operator $A(t) : D(A(t)) \subset C(\overline{\Omega}) \to C(\overline{\Omega})$ is defined by

$$D(A(t)) = \{u \in C^{2+\mu}(\overline{\Omega}) : \frac{\partial u}{\partial \nu} + \beta(x,t,u) = 0 \quad \text{on} \quad \partial\Omega\},$$

$$A(t)u = \alpha(x,t,Du)\Delta u + g(x,t,u,Du), \quad u \in D(A(t)).$$

Equation (5.1) with the nonlinear $A(t)$ has been solved in [21], but here the proof will be based on the contraction mapping theorem as in the proof of Proposition 5.1. To this end, set

$$\begin{gathered}
U_1 = C^{2+\mu}(\overline{\Omega}), \\
U_2 = C^{\mu}(\overline{\Omega}) \times C^{1+\mu}(\partial\Omega), \\
L_\tau u = \tau[u - \epsilon A(t)u] + (1-\tau)(u - \Delta u), \quad x \in \Omega, \\
N_\tau u = \tau[\frac{\partial u}{\partial \nu} + \beta(x,t,u)] + (1-\tau)(\frac{\partial u}{\partial \nu} + u) \quad \text{on } \partial\Omega,
\end{gathered}$$

where $0 \le \tau \le 1$. Define the nonlinear operator $\pounds_\tau : U_1 \to U_2$ by

$$\pounds_\tau u = (L_\tau u, N_\tau u)$$

for $u \in U_1$, and assume that $\pounds_s$ is onto for some $s \in [0,1]$.

As in proving that $A(t)$ satisfies the dissipativity (H1) where the maximum principle was used, $\pounds_s$ is one to one, and so $\pounds_s^{-1}$ exists. By making use of $\pounds_s^{-1}$, the equation, for $w_0 \in U_2$ given, $\pounds_\tau u = w_0$ is equivalent to the equation

$$u = \pounds_s^{-1}[w_0 + (\tau - s)(\pounds_0 - \pounds_1)u],$$

from which a nonlinear map

$$\begin{gathered}
S : U_1 \to U_1, \\
Su = S_s u \equiv \pounds_s^{-1}[w_0 + (\tau - s)(\pounds_0 - \pounds_1)u] \quad \text{for } u \in U_1
\end{gathered}$$

is defined. The unique fixed point of $S = S_s$ will be related to the solution of (5.1) with nonlinear $A(t)$.

By restricting $S = S_s$ to the closed ball of the Banach space U_1,

$$B_{s,r,w_0} \equiv \{u \in U_1 : \|u - \pounds_s^{-1} w_0\|_{C^{2+\mu}} \le r > 0\},$$

and choosing small enough $|\tau - s|$, we will show that $S = S_s$ leaves B_{s,r,w_0} invariant. This will be done by the following steps 1 to 4.

Step 1. It follows as in [21, Pages 265-266] that, for $\pounds_\tau v = (f, \chi)$,

$$\begin{gathered}
\|v\|_\infty \le k_{\{\|f\|_\infty, \|\chi\|_{C(\partial\Omega)}\}}, \\
\|Dv\|_{C^\mu} \le k_{\{\|v\|_\infty\}} \|Dv\|_\infty + k_{\{\|v\|_\infty, \|f\|_\infty, \|\chi\|_{C(\partial\Omega)}\}}, \\
\|v\|_{C^{1+\mu}} \le k_{\{\|\chi\|_{C(\partial\Omega)}, \|f\|_\infty\}}, \\
\|v\|_{C^{2+\mu}} \le K \|\pounds_\tau v\|_{U_2} = K \|\pounds_\tau v\|_{C^\mu(\overline{\Omega}) \times C^{1+\mu}(\partial\Omega)}
\end{gathered} \tag{5.4}$$

where $k_{\{\|f\|_\infty\}}$ is a constant depending on $\|f\|_\infty$, and similar meaning is defined for other constants k's; further, K is independent of τ, but depends on n, δ_0, μ, Ω, and on the coefficient functions $\alpha(x,t,Dv), g(x,t,v,Dv), \beta(x,t,v)$, which have incorporated the dependence of v, Dv into $\|\pounds_\tau v\|_{U_2}$.

Step 2. It is readily seen that, for $v \in C^{2+\mu}(\overline{\Omega})$ with $\|v\|_{C^{2+\mu}} \le R > 0$, we have

$$\|\pounds_\tau v\|_{U_2} \le k_{\{R\}} \|v\|_{C^{2+\mu}}, \tag{5.5}$$

where $k_{\{R\}}$ is independent of τ.

Step 3. It will be shown that, if

$$\|u\|_{C^{2+\mu}} \le R, \quad \|v\|_{C^{2+\mu}} \le R > 0,$$

then

$$\|\pounds_\tau u - \pounds_\tau v\|_{U_2} \le k_{\{R\}} \|u - v\|_{C^{2+\mu}}. \tag{5.6}$$

It will be also shown that, if

$$\pounds_\tau u = (f, \chi_1), \quad \pounds_\tau v = (w, \chi_2),$$

then

$$\begin{aligned} \|u - v\|_{C^{2+\mu}} &\le k_{\{\|\pounds_\tau u\|_{U_2}, \|\pounds_\tau v\|_{U_2}\}} [\|f - w\|_{C^\mu} + \|\chi_1 - \chi_2\|_{C^{1+\mu}}] \\ &= k_{\{\|\pounds_\tau u\|_{U_2}, \|\pounds_\tau v\|_{U_2}\}} \|\pounds_\tau u - \pounds_\tau v\|_{U_2}. \end{aligned} \tag{5.7}$$

Here $K_{\{R\}}$ and $K_{\{\|\pounds_\tau u\|_{U_2}, \|\pounds_\tau v\|_{U_2}\}}$ are independent of τ.

Using the mean value theorem, we have that

$$\begin{aligned} f - w &= L_\tau u - L_\tau v \\ &= (u - v) - (1 - \tau)\Delta(u - v) - \tau\epsilon[\alpha\Delta(u - v) \\ &\quad + \alpha_p(x, t, p_1)(Du - Dv)\Delta v + g_p(x, t, u, p_2)(Du - Dv) \\ &\quad + g_z(x, t, z_1, Dv)(u - v)], \quad x \in \Omega, \\ \frac{\partial(u - v)}{\partial\nu} &+ [\beta(x, t, u) - \beta(x, t, v)] = \chi_1 - \chi_2 \quad \text{on } \partial\Omega, \end{aligned}$$

were p_1, p_2 are some functions between Du and Dv, and z_1 is some function between u and v.

It follows as in (5.5) that

$$\|\pounds_\tau u - \pounds_\tau v\|_{U_2} \le k_{\{R\}} \|u - v\|_{C^{2+\mu}},$$

which is the desired estimate.

On the other hand, the maximum principle yields

$$\|u - v\|_\infty \le k_{\{\|f - w\|_\infty, \|\chi_1 - \chi_2\|_\infty\}}$$

and (5.4) yields

$$\|u\|_{C^{2+\mu}} \le K\|\pounds_\tau u\|_{U_2}, \quad \|v\|_{C^{2+\mu}} \le K\|\pounds_\tau v\|_{U_2}.$$

Thus, it follows from the Schauder global estimate [14] that

$$\|u - v\|_{C^{2+\mu}} \le k_{\{\|\pounds_\tau u\|_{U_2}, \|\pounds_\tau\|_{U_2}\}} \|\pounds_\tau u - \pounds_\tau v\|_{U_2},$$

which is the other desired estimate.

Step 4. Consequently, for $u \in B_{s,r,w_0}$, we have that, by (5.4),

$$\|u\|_{C^{2+\mu}} \le r + \|\pounds_s^{-1} w_0\|_{C^{2+\mu}} \le r + K\|w_0\|_{U_2} \equiv R_{\{r, \|w_0\|_{U_2}\}}, \tag{5.8}$$

and that

$$\begin{aligned} &\|Su - \pounds_s^{-1} w_0\|_{C^{2+\mu}} \\ &\le k_{\{\|w_0\|_{U_2}, \|w_0 + (\tau - s)(\pounds_0 - \pounds_1)u\|_{U_2}\}} \|(\tau - s)(\pounds_0 - \pounds_1)u\|_{U_2} \quad \text{by (5.7)} \\ &\le |\tau - s| k_{\{\|w_0\|_{U_2}, R_{\{r, \|w_0\|_{U_2}\}}\}} \quad \text{by (5.5) and (5.8).} \end{aligned}$$

Here the constant $k_{\{\|w_0\|_{U_2}, R_{\{r, \|w_0\|_{U_2}\}}\}}$ when w_0 given and r chosen, is independent of τ and s. Hence, by choosing some sufficiently small $\delta_1 > 0$, there results

$$S = S_s : B_{s,r,w_0} \subset U_1 \to B_{s,r,w_0} \subset U_1$$

for $|\tau - s| < \delta_1$; that is, B_{s,r,w_0} is left invariant by $S = S_s$.

Next, it will be shown that, for small $|\tau - s|$, $S = S_s$ is a strict contraction on B_{s,r,w_0}, from which $S = S_s$ has a unique fixed point. Because, for $u, v \in B_{s,r,w_0}$,

$$\|u\|_{C^{2+\mu}} \leq R_{\{r,\|w_0\|_{U_2}\}}, \quad \|v\|_{C^{2+\mu}} \leq R_{\{r,\|w_0\|_{U_2}\}} \quad \text{by (5.8)},$$

it follows that, by (5.5),

$$\begin{aligned} &\|w_0 + (\tau - s)(\pounds_0 - \pounds_1)u\|_{U_2} \leq k_{\{\|w_0\|_{U_2}, R_{\{r,\|w_0\|_{U_2}\}}\}}, \\ &\|w_0 + (\tau - s)(\pounds_0 - \pounds_1)v\|_{U_2} \leq k_{\{\|w_0\|_{U_2}, R_{\{r,\|w_0\|_{U_2}\}}\}}, \end{aligned} \tag{5.9}$$

and that, by (5.6),

$$\|(\tau - s)[(\pounds_0 - \pounds_1)u - (\pounds_0 - \pounds_1)v]\|_{U_2} \leq |\tau - s| k_{\{R_{\{r,\|w_0\|_{U_2}\}}\}} \|u - v\|_{C^{2+\mu}}. \tag{5.10}$$

Therefore, on account of (5.7), (5.9), and (5.10), we obtain

$$\|Su - Sv\|_{C^{2+\mu}} \leq |\tau - s| k_{\{R_{\{r,\|w_0\|_{U_2}\}}, \|w_0\|_{U_2}\}} k_{\{R_{\{r,\|w_0\|_{U_2}\}}\}} \|u - v\|_{C^{2+\mu}}.$$

Here the constant $k_{\{R_{\{r,\|w_0\|_{U_2}\}}, \|w_0\|_{U_2}\}} k_{\{R_{\{r,\|w_0\|_{U_2}\}}\}}$ when w_0 given and r chosen, is independent of τ and s. Hence, by choosing some sufficiently small $\delta_2 > 0$, it follows that

$$S = S_s : B_{s,r,w_0} \to B_{s,r,w_0}$$

ia a strict contraction for

$$|\tau - s| < \delta_2 \leq \delta_1.$$

Furthermore, the unique fixed point w of $S = S_s$ can be represented by

$$\lim_{n\to\infty} S^n 0 = \lim_{n\to\infty} (S_s)^n 0$$

if $\beta(x, t, 0) \equiv 0$ and if $r = r_{\{K\|w_0\|_{U_2}\}}$ is chosen such that

$$r = r_{\{K\|w_0\|_{U_2}\}} \geq K\|w_0\|_{U_2} \geq \|\pounds_s^{-1} w_0\|_{C^{2+\mu}} \tag{5.11}$$

(by (5.8)); this is because 0 belongs to B_{s,r,w_0} in this case. Thus $\pounds_\tau$ is onto for $|\tau - s| < \delta_2$.

It follows that, by dividing $[0, 1]$ into subintervals of length less than δ_2 and repeating the above arguments in a finite number of times, $\pounds_\tau$ becomes onto for all $\tau \in [0, 1]$, provided that it is onto for some $\tau \in [0, 1]$. Since $\pounds_0$ is onto by linear elliptic theory [14], we have that $\pounds_1$ is also onto. Therefore, the equation, for $w_0 = (h, 0)$,

$$\pounds_1 u = w_0$$

has a unique solution u, and the u is the sought solution to (5.1).

Here it is to be observed that $\psi \equiv \pounds_0^{-1}(h, 0)$ is the unique solution to the elliptic equation

$$\begin{gathered} v - \Delta v = h, \quad x \in \Omega, \\ \frac{\partial v}{\partial \nu} + v(x) = 0 \quad \text{on} \quad \partial\Omega, \end{gathered}$$

and that, by Proposition 5.1, ψ is the limit of a sequence where each term in the sequence is an explicit function of the solution ϕ to the elliptic equation (1.3) with $\varphi \equiv 0$.

It is also to be observed that

$$\begin{gathered} S0 = S_0 0 = \pounds_0^{-1}(h, 0), \\ S^2 0 = (S_0)^2 0 = \pounds_0^{-1}[(h, 0) + |\tau - 0|(\pounds_0 - \pounds_1)\pounds_0^{-1}(h, 0)], \end{gathered}$$

$$\dots,$$

where $(\pounds_0 - \pounds_1)\pounds_0^{-1}$ is a nonlinear operator. The proof is complete. $\square$

Remark 5.4. The constants $k_{\{R_{\{r,\|w_0\|_{U_2}\}}\}}$ and $k_{\{R_{\{r,\|w_0\|_{U_2}\}},\|w_0\|_{U_2}\}}k_{\{R_{\{r,\|w_0\|_{U_2}\}}\}}$, when w_0 is given and when r is chosen and conditioned by (5.11), is not known explicitly, and so the corresponding δ_2 cannot be determined in advance. Hence, when dealing with the elliptic equation (5.1) in Proposition 5.3 numerically, it is more possible, by choosing $\tau \in [0,1]$ such that $|\tau - s|$ is smaller, that the sequence $S^n 0$ will converge, for which $|\tau - s| < \delta_2 \le \delta_1$ occurs.

Finally, what will be considered is the linear equation (4.4) of space dimension 1.

Proposition 5.5. *For $h \in C[0,1]$, the solution u to the equation* (5.1)

$$[I - \epsilon A(t)]u = h$$

where $0 \le t \le T$ and $\epsilon > 0$, is the limit of a sequence where each term in the sequence is an explicit function of the solution ϕ to the ordinary differential equation

$$\begin{gathered} v - v'' = h \quad x \in (0,1), \\ v'(j) = (-1)^j v(j), \quad j = 0, 1. \end{gathered} \tag{5.12}$$

Here $A(t)$ is the linear operator corresponding to the parabolic equation (4.4).

Proof. The linear operator $A(t) : D(A(t)) \subset C[0,1] \to C[0,1]$ is defined by

$$\begin{gathered} A(t)u \equiv a(x,t)u'' + b(x,t)u' + c(x,t)u \quad \text{for } u \in D(A(t)) \text{ where} \\ D(A(t)) \equiv \{v \in C^2[0,1] : v'(j) = (-1)^j \beta_j(j,t)v(j), \quad j = 0, 1\}. \end{gathered}$$

The contraction mapping theorem in the proof of Proposition 5.1 will be used in order to solve the equation (5.1). To this end, set, for $0 \le \tau \le 1$,

$$\begin{aligned} U_1 &= C^2[0,1], \quad U_2 = C[0,1] \times \mathbb{R}^2, \\ L_\tau u &= \tau[u - \epsilon A(t)u] + (1-\tau)(u - u''), \\ N_\tau u &= \Big(\tau[u'(0) - \beta_0(0,t)u(0)] + (1-\tau)[u'(0) - u(0)], \\ &\quad \tau[u'(1) + \beta_1(1,t)u(1)] + (1-\tau)[u'(1) + u(1)]\Big). \end{aligned}$$

Define the linear operator $\pounds_\tau : U_1 \to U_2$ by

$$\pounds_\tau u = (L_\tau u, N_\tau u)$$

for $u \in U_1$, and assume that $\pounds_s$ is onto for some $s \in [0,1]$.

The following will be readily derived.

- For $u \in C^2[0,1]$, we have

$$\|\pounds_\tau u\|_{U_2} = \|\pounds_\tau u\|_{C[0,1]\times\mathbb{R}^2} \le k_{\{a,b,c,\beta_0,\beta_1\}}\|u\|_{C^2}, \tag{5.13}$$

where $k_{\{a,b,c,\beta_0,\beta_1\}}$ is independent of τ, and can be computed, depending on the given $a(x,t), b(x,t), c(x,t), \beta_0(0,t)$, and $\beta_1(1,t)$.

- For $\pounds_\tau u = (h,(r,s))$, the maximum principle shows

$$\|u\|_\infty \le \|h\|_\infty + |\frac{r}{\beta_0(0,t)}| + |\frac{s}{\beta_1(1,t)}|.$$

This, together with the known interpolation inequality [15, Page 65] or [27, Pages 7-8]

$$\|u'\|_\infty \leq \frac{2}{\lambda}\|u\|_\infty + \frac{\lambda}{2}\|u''\|_\infty$$

for any $\lambda > 0$, applied to $\pounds_\tau u = (h, (r, s))$, it follows that, by choosing small enough $\lambda = \lambda_1$,

$$\|u\|_{C^2} \leq k_{\{\lambda_1,a,b,c,\beta_0,\beta_1\}}(\|h\|_\infty + |r| + |s|) = k_{\{\lambda_1,a,b,c,\beta_0,\beta_1\}}\|\pounds_\tau u\|_{U_2}, \tag{5.14}$$

where $k_{\{\lambda_1,a,b,c,\beta_0,\beta_1\}}$ is independent of τ and can be computed explicitly.

On account of the estimate (5.14), $\pounds_s$ is one to one, and so $\pounds_s^{-1}$ exists. Thus, making use of $\pounds_s^{-1}$, the equation, for $w_0 \in U_2$ given, $\pounds_\tau u = w_0$ is equivalent to the equation

$$u = \pounds_s^{-1}w_0 + (\tau - s)\pounds_s^{-1}(\pounds_0 - \pounds_1)u,$$

from which a linear map

$$\begin{gathered} S : U_1 = C^2[0,1] \to U_1 = C^2[0,1], \\ Su = S_s u \equiv \pounds_s^{-1}w_0 + (\tau - s)\pounds_s^{-1}(\pounds_0 - \pounds_1)u, \quad u \in U_1 \end{gathered}$$

is defined. Because of (5.14) and (5.13), it follows that this S is a strict contraction if

$$|\tau - s| < \delta = [k_{\{\lambda_1,a,b,c,\beta_0,\beta_1\}}2k_{\{a,b,c,\beta_0,\beta_1\}}]^{-1}.$$

The rest of the proof will be the same as that for Proposition 5.1, in which the equation, for $w_0 = (h, (0, 0))$,

$$\pounds_1 u = w_0$$

has a unique solution u, and the u is the sought solution. $\square$

Remark 5.6. • The $\delta = [k_{\{\lambda_1,a,b,c,\beta_0,\beta_1\}}2k_{\{a,b,c,\beta_0,\beta_1\}}]^{-1}$ in the above proof of Proposition 5.5 can be computed explicitly.

• The quantity $\pounds_0^{-1}(h, (0, 0))$ is represented by the integral

$$\pounds_0^{-1}(h, (0, 0)) = \int_0^1 g_0(x, y)h(y)\, dy,$$

where $g_0(x, y)$ is the Green function associated with the boundary value problem

$$\begin{gathered} u - u'' = h \quad \text{in} \quad (0, 1), \\ u'(j) = (-1)^j u(j), \quad j = 0, 1. \end{gathered}$$

This $g_0(x, y)$ is known explicitly by a standard formula.

• As before, we have

$$\begin{gathered} S0 = S_0 0 = \pounds_0^{-1}(h, (0, 0)), \\ S^2 0 = S_0^2 0 = \pounds_0^{-1}(h, (0, 0)) + \pounds_0^{-1}[|\tau - 0|(\pounds_0 - \pounds_1)\pounds_0^{-1}(h, (0, 0))], \\ \dots . \end{gathered}$$

6. Appendix

In this section, the Proposition 3.1 in Section 3 will be proved, using the theory of difference equations. We now introduce its basic theory [26]. Let

$$\{b_n\} = \{b_n\}_{n\in\{0\}\cup\mathbb{N}} = \{b_n\}_{n=0}^{\infty}$$

be a sequence of real numbers. For such a sequence $\{b_n\}$, we further extend it by defining

$$b_n = 0 \quad \text{if } n = -1, -2, \dots .$$

The set of all such sequences $\{b_n\}$'s will be denoted by S. Thus, if $\{a_n\} \in S$, then $0 = a_{-1} = a_{-2} = \dots$.

Define a right shift operator $E : S \to S$ by

$$E\{b_n\} = \{b_{n+1}\} \quad \text{for } \{b_n\} \in S.$$

For $c \in \mathbb{R}$ and $c \neq 0$, define the operator $(E-c)^* : S \to S$ by

$$(E-c)^*\{b_n\} = \{c^n \sum_{i=0}^{n-1} \frac{b_i}{c^{i+1}}\}$$

for $\{b_n\} \in S$. Here the first term on the right side of the equality, corresponding to $n = 0$, is zero.

Define, for $\{b_n\} \in S$,

$$(E-c)^{i*}\{b_n\} = [(E-c)^*]^i\{b_n\}, \quad i = 1, 2, \dots;$$
$$(E-c)^0\{b_n\} = \{b_n\}.$$

It follows that $(E-c)^*$ acts approximately as the inverse of $(E-c)$ in this sense

$$(E-c)^*(E-c)\{b_n\} = \{b_n - c^n b_0\}.$$

Next we extend the above definitions to doubly indexed sequences. For a doubly indexed sequence $\{\rho_{m,n}\} = \{\rho_{m,n}\}_{m,n=0}^{\infty}$ of real numbers, let

$$E_1\{\rho_{m,n}\} = \{\rho_{m+1,n}\}; \quad E_2\{\rho_{m,n}\} = \{\rho_{m,n+1}\}.$$

Thus, E_1 and E_2 are the right shift operators, which acts on the first index and the second index, respectively. It is easy to see that

$$E_1E_2\{\rho_{m,n}\} = E_2E_1\{\rho_{m,n}\}\,.$$

Before we prove the Proposition 3.1, we need the following four lemmas, which are proved in [23, 22, 23, 24], respectively.

Lemma 6.1. *If* (3.3) *is true, then*

$$\begin{aligned}\{a_{m,n}\} \le{}& (\alpha\gamma(E_2-\beta\gamma)^*)^m\{a_{0,n}\} + \sum_{i=0}^{m-1}(\gamma\alpha(E_2-\gamma\beta)^*)^i\{(\gamma\beta)^n a_{m-i,0}\} \\ &+ \sum_{j=1}^{m}(\gamma\alpha)^{j-1}((E_2-\gamma\beta)^*)^j\{r_{m+1-j,n+1}\},\end{aligned} \tag{6.1}$$

where $r_{m,n} = K_4\mu\rho(|n\mu - m\lambda|)$.

Lemma 6.2. *The following equality holds:*

$$((E_2-\beta\gamma)^*)^m\{n\gamma^n\} = \{\frac{n\gamma^n}{\alpha^m}\frac{1}{\gamma^m} - \frac{m\gamma^n}{\alpha^{m+1}}\frac{1}{\gamma^m} + \Big(\sum_{i=0}^{m-1}\binom{n}{i}\frac{\beta^{n-i}}{\alpha^{m+1-i}}(m-i)\frac{1}{\gamma^m}\Big)\gamma^n\}.$$

Here γ, α and β are defined in Proposition 3.2.

Lemma 6.3. *The following equality holds:*

$$((E-\beta\gamma)^*)^j\{\gamma^n\} = \{\Big(\frac{1}{\alpha^j} - \frac{1}{\alpha^j}\sum_{i=0}^{j-1}\binom{n}{i}\beta^{n-i}\alpha^i\Big)\gamma^{n-j}\} = \{\Big(\frac{1}{\alpha^j}\sum_{i=j}^{n}\beta^{n-i}\alpha^i\Big)\gamma^{n-j}\}$$

for $j \in \mathbb{N}$. Here γ, α and β are defined in Proposition 3.2

Lemma 6.4. *The following equality holds:*

$$\begin{aligned}
(E-\beta\gamma)^{m*}\{n^2\gamma^n\} &= \gamma^{n-m}\{\frac{n^2}{\alpha^m} - \frac{(2m)n}{\alpha^{m+1}} + (\frac{m(m-1)}{\alpha^{m+2}} + \frac{m(1+\beta)}{\alpha^{m+2}}) \\
&\quad - \sum_{j=0}^{m-1}(\frac{(m-j)(m-j-1)}{\alpha^{m-j+2}} + \frac{(m-j)(1+\beta)}{\alpha^{m-j+2}})\binom{n}{j}\beta^{n-j}\}.
\end{aligned}$$

Here γ, α, and β are defined in Proposition 3.2.

Proof of Proposition 3.1. If $S_2(\mu) = \emptyset$, then (3.2) is true, and so

$$a_{m,n} \le L(K_2)|n\mu - m\lambda|.$$

If $S_1(\mu) = \emptyset$, then (3.3) is true, and so the inequality (6.1) follows by Lemma 6.1. Since, by Proposition 2.2,

$$\begin{gathered}
a_{0,n} \le K_1\gamma^n(2n+1)\mu; \\
a_{m-i,0} \le K_1(1-\lambda\omega)^{-m}[2(m-i)+1]\lambda;
\end{gathered}$$

it follows from Lemma 6.3 and from the Proposition 3 and its proof of [22, Pages 115-116] that the first two terms of the right side of the inequality (6.1) is less than or equal to

$$c_{m,n} + s_{m,n} + f_{m,n}.$$

We finally estimate the third term, denoted by $\{t_{m,n}\}$, of the right-hand side of (6.1). Observe that, using the subadditivity of ρ, we have

$$\begin{aligned}
\{t_{m,n}\} &\le \sum_{j=1}^{m}(\gamma\alpha)^{j-1}(E_2-\gamma\beta)^{j*}K_4\mu\{\rho(|\lambda-\mu|) + \rho(|n\mu - m\lambda + j\lambda|)\} \\
&\le \sum_{j=1}^{m}(\gamma\alpha)^{j-1}(E_2-\gamma\beta)^{j*}K_4\mu\{\gamma^n\rho(|\lambda-\mu|) + \gamma^n\rho(|n\mu-(m-j)\lambda|)\} \\
&\equiv \{u_{m,n}\} + \{v_{m,n}\},
\end{aligned}$$

where $\gamma = (1-\mu\omega)^{-1} > 1$. It follows from Lemma 6.3 that

$$\begin{aligned}
\{u_{m,n}\} &\le \{K_4\mu\gamma^n\rho(|\lambda-\mu|)\sum_{j=1}^{m}\alpha^{j-1}\frac{1}{\alpha^j}\sum_{i=1}^{n}\binom{n}{i}\beta^{n-i}\alpha^i\} \\
&\le \{K_4\gamma^n\rho(|\lambda-\mu|)\mu\frac{1}{\alpha}m\} = \{K_4\rho(|\lambda-\mu|)\gamma^n(m\lambda)\}.
\end{aligned}$$

To estimate $\{v_{m,n}\}$, as in Crandall-Pazy [9, page 68], let $\delta > 0$ be given and write

$$\{v_{m,n}\} = \{I^{(1)}_{m,n}\} + \{I^{(2)}_{m,n}\},$$

where $\{I^{(1)}_{m,n}\}$ is the sum over indices with $|n\mu - (m-j)\lambda| < \delta$, and $\{I^{(2)}_{m,n}\}$ is the sum over indices with $|n\mu - (m-j)\lambda| \geq \delta$. As a consequence of Lemma 6.3, we have

$$\begin{aligned}\{I^{(1)}_{m,n}\} &\leq \{K_4\mu\gamma^n\rho(\delta)\sum_{j=1}^{m}\alpha^{j-1}\frac{1}{\alpha^j}\sum_{i=j}^{n}\binom{n}{i}\beta^{n-i}\alpha^i\}\\ &\leq \{K_4\rho(\delta)\mu\gamma^n m\frac{1}{\alpha}\} = \{K_4\rho(\delta)\gamma^n m\lambda\}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\{I^{(2)}_{m,n}\} &\leq K_4\mu\rho(T)\sum_{j=1}^{m}(\gamma\alpha)^{j-1}(E_2-\gamma\beta)^{j*}\{\gamma^n\}\\ &\leq K_4\mu\rho(T)\sum_{j=1}^{m}(\gamma\alpha)^{j-1}(E_2-\gamma\beta)^{j*}\{\gamma^n\frac{[n\mu-(m-j)\lambda]^2}{\delta^2}\},\end{aligned}$$

which will be less than or equal to

$$\{K_4\frac{\rho(T)}{\delta^2}\gamma^n[(m\lambda)(n\mu-m\lambda)^2+(\lambda-\mu)\frac{m(m+1)}{2}\lambda^2]\}$$

and so the proof is complete. This is because of the calculations, where Lemmas 6.2, 6.3, and 6.4 were used:

$$\begin{aligned}&[n\mu-(m-j)\lambda]^2 = n^2\mu^2 - 2(n\mu)(m-j)\lambda + (m-j)^2\lambda^2;\\ &\sum_{j=1}^{m}(\gamma\alpha)^{j-1}(E_2-\gamma\beta)^{j*}\{\gamma^n n^2\}\mu^2\\ &= \gamma^{n-1}\sum_{j=1}^{m}\alpha^{j-1}\{\frac{n^2}{\alpha^j}-\frac{2jn}{\alpha^{j+1}}+[\frac{j(j-1)}{\alpha^{j+2}}+\frac{j(1+\beta)}{\alpha^{j+2}}]\\ &\quad -\sum_{i=0}^{j-1}[\frac{(j-i)(j-i-1)}{\alpha^{j-i+2}}+\frac{(j-i)(1+\beta)}{\alpha^{j-i+2}}]\binom{n}{i}\beta^{n-i}\}\mu^2\\ &\leq \gamma^n\sum_{j=1}^{m}\{\frac{n^2}{\alpha}-\frac{2jn}{\alpha^2}+[\frac{j(j-1)}{\alpha^3}+\frac{j(1+\beta)}{\alpha^3}]\}\mu^2,\end{aligned}$$

where the negative terms associated with $\sum_{i=0}^{j-1}$ were dropped;

$$\begin{aligned}&\sum_{j=1}^{m}(\gamma\alpha)^{j-1}(E_2-\gamma\beta)^{j*}\{\gamma^n n\}[2\mu(m-j)\lambda](-1)\\ &= \sum_{j=1}^{m}(\gamma\alpha)^{j-1}\{\gamma^{n-j}[\frac{n}{\alpha^j}-\frac{j}{\alpha^{j+1}}\\ &\quad +\sum_{i=0}^{j-1}\binom{n}{i}\beta^{n-i}\alpha^{i-j-1}(j-i)]\}[2\mu(m-j)\lambda](-1)\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{j=1}^{m} \gamma^n \{\frac{n}{\alpha} - \frac{j}{\alpha^2}\}[2\mu(m-j)\lambda](-1),\\
&= \sum_{j=1}^{m} \gamma^n \alpha^{-1} \{-2(n\mu)(m\lambda) + j[2n\mu\lambda + \frac{2\mu}{\alpha}(m\lambda)] - j^2(\frac{2\mu\lambda}{\alpha})\};
\end{aligned}
$$

where the negative terms associated with $\sum_{i=0}^{j-1}$ were dropped;

$$
\begin{aligned}
&\sum_{j=1}^{m} (\gamma\alpha)^{j-1}(E_2 - \gamma\beta)^{j*}\{\gamma^n\}(m-j)^2\lambda^2\\
&= \sum_{j=1}^{m} (\gamma\alpha)^{j-1}\{\gamma^{n-j}[\frac{1}{\alpha^j} - \frac{1}{\alpha^j}\sum_{i=0}^{j-1}\binom{n}{i}\beta^{n-i}\alpha^i]\}(m-j)^2\lambda^2\\
&\le \sum_{j=1}^{m} \gamma^n \alpha^{-1}(m^2 - 2mj + j^2)\lambda^2,
\end{aligned}
$$

where the negative terms associated with $\sum_{i=0}^{j-1}$ were dropped.

Adding up the right sides of the above three inequalities and grouping them as a polynomial in j of degree two, we have the following: The term involving $j^0 = 1$ has the factor

$$
\mu\frac{1}{\alpha}\sum_{j=1}^{m}[n^2\mu^2 - 2(n\mu)(m\lambda) + (m\lambda)^2] = (m\lambda)(n\mu - m\lambda)^2;
$$

the term involving j^2 has the factor

$$
\frac{\mu^2}{\alpha^3} - \frac{2\mu\lambda}{\alpha^2} + \frac{\lambda^2}{\alpha} = 0;
$$

the term involving j has two parts, one of which has the factor

$$
\frac{2n\mu\lambda}{\alpha} + \frac{2\mu m\lambda}{\alpha^2} - \frac{2m\lambda^2}{\alpha} - \frac{2n\mu^2}{\alpha^2} = 0,
$$

and the other of which has the factor

$$
\mu\sum_{j=1}^{m}(\frac{1+\beta}{\alpha^3} - \frac{1}{\alpha^3})j\mu^2 = (\lambda - \mu)\frac{m(m+1)}{2}\lambda^2.
$$

The proof is complete. $\square$

Remark 6.5. The results in Proposition 3.1 are true for $n, m \ge 0$, but a similar result in the [23, Proposition 4, page 236] has the restriction $n\mu - m\lambda \ge 0$ which is not suitable for a mathematical induction proof.

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