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POSITIVITY AND NEGATIVITY OF SOLUTIONS TO $n \times n$ WEIGHTED SYSTEMS INVOLVING THE LAPLACE OPERATOR ON $\mathbb{R}^N$

BÉNÉDICTE ALZIARY, JACQUELINE FLECKINGER,
MARIE-HÉLÈNE LECUREUX, NA WEI

Abstract. We consider the sign of the solutions of a $n \times n$ system defined on the whole space $\mathbb{R}^N$, $N \geq 3$ and a weight function ρ with a positive part decreasing fast enough,

$$-\Delta U = \lambda \rho(x) MU + F,$$

where F is a vector of functions, M is a $n \times n$ matrix with constant coefficients, not necessarily cooperative, and the weight function ρ is allowed to change sign. We prove that the solutions of the $n \times n$ system exist and then we prove the local fundamental positivity and local fundamental negativity of the solutions when $|\lambda\sigma_1 - \lambda_\rho|$ is small enough, where σ_1 is the largest eigenvalue of the constant matrix M and λ_ρ is the "principal" eigenvalue of

$$-\Delta u = \lambda \rho(x) u, \quad \lim_{|x| \to \infty} u(x) = 0; \quad u(x) > 0, \quad x \in \mathbb{R}^N.$$

1. Introduction

Elliptic eigenvalue problems with indefinite weight arise naturally from linearization of many semilinear elliptic equations. A lot of literature in applied mathematics, engineering, physics, and biology treat such problems which occur in the study of transport theory, crystal coloration, laser theory, reaction-diffusion equations, fluid dynamics, etc. (see [10] and the references therein). Many researchers studied the indefinite weighted eigenvalue problems under various hypotheses (see [1, 2, 9, 10, 19]). Owing to the importance of such problems, we investigate some systems concerning the Laplace operator. Specially, we will obtain the sign of solutions of some weighted systems in this paper.

The positivity of solutions is usually shown with the maximum principles, which are also of great importance for the study of existence and uniqueness of solutions to some linear and nonlinear equations. Moreover, P. Clément and L. Peletier [11] proved an antimaximum principle for a linear equation $-\Delta u = \lambda u + f$ defined on a smooth bounded domain $\Omega \subset \mathbb{R}^N$ with Dirichlet boundary condition $\partial\Omega$. Let λ_1 be the principal eigenvalue of the Laplace operator $-\Delta$, which is endowed with

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homogeneous Dirichlet boundary condition. We recall here the maximum principle as well as the antimaximum principle for a bounded domain.

Proposition 1.1 (Hopf Maximum principle). *Assume $f \in L^p(\Omega)$, $p > N$ and $f \geq 0$, $f \not\equiv 0$. Let u be a solution of the equation $-\Delta u = \lambda u + f$, $u = 0$ on the boundary $\partial\Omega$. Then for $\lambda < \lambda_1$,*

$$u(x) > 0 \quad on\ \Omega; \quad \frac{\partial u}{\partial n}(x)\big|_{\partial\Omega} < 0,$$

where $\frac{\partial u}{\partial n}$ is the outward normal derivative on $\partial\Omega$.

Proposition 1.2 (Antimaximum principle). *Assume $f \in L^p(\Omega)$, $p > N$ and $f \geq 0$, $f \not\equiv 0$. u solves $-\Delta u = \lambda u + f$ in Ω, $u|_{\partial\Omega} = 0$. Moreover, $\partial\Omega$ is smooth enough. Then there exists $\delta > 0$, depending on f, such that for $\lambda_1 < \lambda < \lambda_1 + \delta$,*

$$u(x) < 0 \quad on\ \Omega; \quad \frac{\partial u}{\partial n}(x)\big|_{\partial\Omega} > 0.$$

These results are classical. However, many problems in mechanic, physic or biology lead to some more general problems as the following equation defined on the whole space $\mathbb{R}^N$, $N > 2$:

$$Lu = \lambda\rho(x)u + f, \quad x \in \mathbb{R}^N.$$

Here L is an elliptic operator as e.g. the Laplacian or the p-Laplacian. The weight ρ ensures the discreteness of the spectrum (e.g. $\rho^+ = max(\rho, 0)$ tends to zero fast enough). There are several results with regard to such kinds of problems, such as the maximum principle [20], the anti-maximum principle ([14] or [26]), the (local) fundamental positivity or negativity of solutions [21]. Fleckinger, Gossez and de Thélin also proved in [14] a local antimaximum principle for the p-Laplacian on the whole space $\mathbb{R}^N$, which follows the approach introduced in [13]. Maximum and antimaximum principles have been extended into notions of "fundamental positivity" and "fundamental negativity" by Alziary and Takáč [6] or [7], who introduced these definitions for the solutions of Schrödinger equation.

We say that a function u satisfies the "fundamental positivity" if there exists a constant $c > 0$ such that $u \geq c\varphi_\rho > 0$ a.e. in $\mathbb{R}^N$ and a function u satisfies the "fundamental negativity" if there exists a constant $c > 0$ such that $u \leq -c\varphi_\rho < 0$ a.e. in $\mathbb{R}^N$, where φ_ρ denotes a positive principal eigenfunction associated to the principal eigenvalue λ_ρ. We will explain it in detail in the next section. If these results are obtained on balls, we have the definitions of "local fundamental positivity" and "local fundamental negativity" respectively.

There are also some results for systems in [3, 12, 15, 16, 18, 20, 23], concerning the existence of principal eigenvalue and maximum principle.

Alziary, Takáč [6] investigate equations involving Schrödinger operators

$$-\Delta u + q(x)u = \lambda u + f(x) \tag{1.1}$$

in $L^2(\mathbb{R}^N)$. They proved a pointwise lower bound for the solution of (1.1) for a function $0 \leq f \not\equiv 0$. Alziary, Fleckinger, and Takáč showed in [4] and [5] the fundamental negativity of solutions to (1.1) when λ is slightly above the principal eigenvalue. Alziary, Fleckinger,Lécureux [3] obtained the fundamental positivity and fundamental negativity of a 2×2 systems of Schrödinger equations on $\mathbb{R}^N$. Then Lécureux [25] extended the results to the non cooperative $n \times n$ system

$$\mathcal{L}U = \lambda U + MU + F$$

on $\mathbb{R}^N$, where $\mathcal{L}$ is a diagonal matrix of Schrödinger operators of the form $\mathcal{L} := -\Delta + q$, q is a potential growing fast enough and M is a constant matrix.

In this paper, we combine our methods with those of Lécureux [25] to study a $n \times n$ weighted systems defined on the whole space $\mathbb{R}^N$, $N \geq 3$:

$$-\Delta U = \lambda\rho(x)MU + F;$$

here Δ denotes the standard Laplace operator, λ is a real spectral parameter; $U = (u_1, u_2, \dots, u_n)^\top$ and $F = (f_1, f_2, \dots, f_n)^\top$. The functions f_i are measurable and bounded. The function ρ is allowed to change sign; i.e., ρ is an indefinite weight function. Moreover, $\rho^+ = \max(\rho, 0)$ tends to 0 fast enough as $|x| \to \infty$. The matrix M is a $n \times n$ matrix with real constant coefficients m_{ij} and real eigenvalues $\sigma_k \neq 0$; it is not necessarily cooperative, that is that the terms outside the diagonal may have any sign. Recall that a matrix is said to be cooperative if the coefficients outside the diagonal are positive. We derive our results from results for the case of one equation and of the 2×2 system which have been studied in [21].

This paper is organized as follows: in Section 2, we recall or prove results for the case of one equation. Then we give our main results for a positive weight in Section 3 and we prove them in Section 4; finally we extend our results to indefinite weight in Section 5.

2. Equations with positive weight

We introduce some notations and hypotheses in this section; then we recall and prove some results on local fundamental positivity and local fundamental negativity which we will use later.

Notation and Hypotheses for a positive weight. For a positive and bounded function r, we denote $L^2_r(\mathbb{R}^N)$ or shortly L^2_r the space

$$L^2_r(\mathbb{R}^N) = \big\{u : \int_{\mathbb{R}^N} ru^2 dx < \infty\big\} \quad \text{with norm } \|u\|_{L^2_r(\mathbb{R}^N)} = \Big(\int_{\mathbb{R}^N} ru^2 dx\Big)^{1/2}.$$

Let us consider the equation

$$-\Delta u = \lambda\rho(x)u + f, \quad \lim_{|x|\to\infty} u(x) = 0, \quad x \in \mathbb{R}^N, \quad N \geq 3, \tag{2.1}$$

where $f \in L^\infty(\mathbb{R}^N) \cap L^2_{1/\rho}$ and $\rho(x)$ is a positive smooth function decreasing faster than $1/|x|^2$ as $|x| \to \infty$. More precisely we assume that ρ satisfies the hypothesis:

(H1) $\rho(x) > 0$, is smooth, bounded and there exists some constants $K > 0$ and $\alpha > 1$ such that $\rho(x) \leq Kp_\alpha(x)$ where
$$p_\alpha(x) := (1 + |x|^2)^{-\alpha}. \tag{2.2}$$

Remark 2.1. It follows from (H1) that $\rho \in L^{N/2}(\mathbb{R}^N)$ but is not necessarily in L^1.

The space $\mathcal{D}^{1,2}$: We denote by $\mathcal{D}^{1,2}$ the closure of $\mathcal{C}_0^\infty$ with respect to the norm

$$\|u\|_{\mathcal{D}^{1,2}} = \Big(\int_{\mathbb{R}^N} |\nabla u|^2 dx\Big)^{1/2}.$$

It can be shown (see [24, Proposition 2.4]) that

$$\mathcal{D}^{1,2} = \{u \in L^{\frac{2N}{N-2}}(\mathbb{R}^N) : \nabla u \in (L^2(\mathbb{R}^N))^N\},$$

and that there exists $K > 0$ such that for all $u \in \mathcal{D}^{1,2}$,

$$\|u\|_{L^{\frac{2N}{N-2}}} \le K\|u\|_{\mathcal{D}^{1,2}}.$$

We also use Hardy's inequality and deduce that there exists two positive constants γ and γ' such that

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \ge \gamma \int_{\mathbb{R}^N} p_\alpha |u|^2 dx \ge \gamma' \int_{\mathbb{R}^N} \rho |u|^2 dx, \ \forall\, u \in \mathcal{D}^{1,2}. \tag{2.3}$$

Hence any $u \in \mathcal{D}^{1,2}$ is in L^2_ρ. Moreover the embedding $\mathcal{D}^{1,2}$ into L^2_ρ is compact [14, Lemma 2.3].

Remark 2.2. Note that by (2.3), $\mathcal{D}^{1,2}$ does not depend on ρ. Moreover note that the constant function is not in $\mathcal{D}^{1,2}$.

For any ρ satisfying (H1), we seek weak solutions to (2.1) in $\mathcal{D}^{1,2}$.
The principal eigenpair: Denote by $(\lambda_\rho, \varphi_\rho)$ the "principal" eigenpair which exists [9, 10] and satisfies

$$-\Delta\varphi_\rho = \lambda_\rho \rho(x)\varphi_\rho, \quad x \in \mathbb{R}^N; \quad \varphi_\rho(x) > 0, \quad \forall x \in \mathbb{R}^N; \quad \int_{\mathbb{R}^N} \rho\varphi_\rho^2 = 1, \tag{2.4}$$

with φ_ρ in L^2_ρ. Here λ_ρ is the smallest eigenvalue of (EV); it is simple and it is called the "principal eigenvalue". The eigenfunction φ_ρ corresponding to λ_ρ is called the "groundstate" or "principal eigenfunction". We define the subspaces X of $L^2_\rho(\mathbb{R}^N)$ and Y of $L^2_{1/\rho}(\mathbb{R}^N)$ as:

$$X := \{u \in L^2_\rho(\mathbb{R}^N) : u/\varphi_\rho \in L^\infty(\mathbb{R}^N)\}, \tag{2.5}$$

$$Y := \{u \in L^2_{1/\rho}(\mathbb{R}^N) : u/(\rho\varphi_\rho) \in L^\infty(\mathbb{R}^N)\}, \tag{2.6}$$

respectively, which are Banach spaces with the norms $\|u\|_X := \operatorname{ess\,sup}_{\mathbb{R}^N}(|u|/\varphi_\rho)$ and $\|u\|_Y := \operatorname{ess\,sup}_{\mathbb{R}^N}(|u/(\rho\varphi_\rho)|)$, respectively.

The function $u \in \mathcal{D}^{1,2}$ is a weak solution of Equation (2.1) if

$$\int_{\mathbb{R}^N} (\nabla u \cdot \nabla \eta) = \lambda \int_{\mathbb{R}^N} \rho(x) u\eta + \int_{\mathbb{R}^N} f\eta, \ \ \forall \eta \in \mathcal{D}^{1,2}.$$

This solution is also classical by regularity properties.

Theorem 2.3 ([9, Sec. 4], [20, Lemma 4])**.** *Assume* (H1). *There exists a positive principal eigenvalue λ_ρ which is given by*

$$0 < \lambda_\rho = \inf_{\{u \in \mathcal{D}^{1,2}, \int_{\mathbb{R}^N} \rho u^2 = 1\}} \int |\nabla u|^2. \tag{2.7}$$

The equality in (2.7) *holds if and only if u is proportional to φ_ρ.*

Remark 2.4. The existence of a principal eigenvalue and Equation (2.7) are still valid, under some conditions, for a non negative weight and for an indefinite weight (see e.g. [9, 10]). We recall the proof later (Lemma 5.1).

2.1. **Results for one equation with positive weight.** First, we recall a maximum principle and a (local) antimaximum principle for the case of one equation.

Proposition 2.5 (Maximum Principle, [20]). *We suppose that ρ satisfies Hypothesis* (H1)*, $f \in L^\infty$, $f \geq 0$, $f \not\equiv 0$. A necessary and sufficient condition for having a (strong) maximum principle, (that is f bounded, $f \geq 0$, $f \not\equiv 0$ implies any solution u to* (2.1) *is positive), is $0 < \lambda < \lambda_\rho$. Moreover, if $0 < \lambda < \lambda_\rho$, then there exists a solution to* (2.1) *and this solution is unique and positive.*

Corollary 2.6. *Assume that $f \in Y$. For $0 < \lambda < \lambda_\rho$, u exists and*

$$0 \leq u(x) \leq \frac{\|f\|_Y}{\lambda_\rho - \lambda}\varphi_\rho(x), \quad \forall x \in \mathbb{R}^N.$$

.

Proposition 2.7 (Local antimaximum principle, [26, Theorem 5.2] or [14, Theorem 3.3]). *, Assume* (H1) *and $f \in L^\infty$, $f \geq 0$, $f \not\equiv 0$. Let $R > 0$ be given; there exists $\delta > 0$ depending on R, f and ρ such that for $\lambda_\rho < \lambda < \lambda_\rho + \delta$, then $u < 0$ on the ball $B_R = \{x \in \mathbb{R}^N / |x| < R\}$.*

Now we improve maximum and antimaximum principle and also the results on fundamental positivity and fundamental negativity shown in [21]. For that purpose, we assume that

(H2) $f = \rho h$ and $h \in L^\infty_{\rm loc} \cap L^2_\rho$.

Theorem 2.8 (Local fundamental positivity and negativity). *Assume that* (H1) *and* (H2) *are satisfied; moreover assume that*

$$\int_{\mathbb{R}^N} f\varphi_\rho > 0. \tag{2.8}$$

Then, for any given $R > 0$, there exist positive numbers $k', k'', K', K'', \delta, \delta'$ (depending on R, ρ and f) such that on $B_R = \{x \in \mathbb{R}^N : |x| < R\}$:

- *for $\lambda_\rho - \delta < \lambda < \lambda_\rho$,*

$$0 < \frac{k'}{\lambda_\rho - \lambda}\varphi_\rho \leq u \leq \frac{k''}{\lambda_\rho - \lambda}\varphi_\rho. \tag{2.9}$$

  (Local fundamental positivity)
- *for $\lambda_\rho < \lambda < \lambda_\rho + \delta'$,*

$$-\frac{K''}{\lambda - \lambda_\rho}\varphi_\rho \leq u \leq -\frac{K'}{\lambda - \lambda_\rho}\varphi_\rho < 0. \tag{2.10}$$

  (Local fundamental negativity)

Corollary 2.9 ([21]). *Assume* (H1), (H2) *and* (2.8) *are satisfied; then for any given $R > 0$, on B_R, $\frac{|u|}{\varphi_\rho} \to +\infty$ as $\lambda \to \lambda_\rho$.*

This corollary plays an important role when studying systems of equations.

3. Main results for a system with positive weight

We consider now a $n \times n$ system:

$$-\Delta U = \lambda\rho(x)MU + F, \quad x \in \mathbb{R}^N, \quad U \to 0 \quad \text{as } |x| \to +\infty, \tag{3.1}$$

where ρ satisfies Hypothesis (H1). We assume the hypothesis

(H3) M is a $n \times n$ non-degenerate matrix which has constant coefficients and has only real eigenvalues. Moreover, the largest one which is denoted by σ_1 is positive and algebraically and geometrically simple.

Of course some of the other eigenvalues can be equal. Therefore we write them in decreasing order $\sigma_1 > \sigma_2 \geq \dots \geq \sigma_n$.

The eigenvalues of $M = (m_{ij})_{1 \leq i,j \leq n}$, σ_1, σ_2, $\dots$, σ_n are the roots of the associated characteristic polynomial

$$p_M(\sigma) = \det(\sigma I_n - M) = \prod(\sigma - \sigma_j), \tag{3.2}$$

where I_n is the $n \times n$ identity matrix.

Remark 3.1. By (3.2), $\sigma > \sigma_1 \Rightarrow p_M(\sigma) > 0$.

In fact, Matrix M can be expressed as $M = PJP^{-1}$, where $P = (p_{ij})$ is the change of basis matrix of M and J is the Jordan canonical form (upper triangular matrix) associated with M. The diagonal entries of J are the ordered eigenvalues of M and $p_M(\sigma) = p_J(\sigma)$.

In the following, we denote by Q the eigenspace associated with σ_1 ($\dim Q = 1$) and by T the hyperplain spanned by the other column vectors of matrix P. By Hypothesis (H3), $\mathbb{R}^N = Q \bigoplus T$. Now we define another hypothesis.

(H4) Assume that $F = (f_i) = (\rho h_i) \in Y$, $1 \leq i \leq n$, and $h_i \in L^\infty_{\rm loc} \cap L^2_\rho$ for any $i = 1, \dots, n$.

Theorem 3.2. *Assume that* (H1), (H3), (H4) *are satisfied. We also assume that there exists an eigenvector $\Theta \in Q$ associated with σ_1 such that $F(x) = F_Q(x) + F_T(x)$ with*

$$F_Q(x) = \widetilde{f_1}(x)\Theta \not\equiv 0 \quad \text{and} \quad \int \widetilde{f_1}\varphi_\rho > 0;$$

also $F_Q(x) \in Q$ and $F_T(x) \in T$. Then, for any $R > 0$, there exist two positive real numbers δ and δ', depending on F, M, ρ, R such that:

(I) If $\lambda_\rho - \delta < \lambda\sigma_1 < \lambda_\rho$, System (3.1) *has a unique solution $U = (u_i)$. Moreover, on B_R, for each integer $i \in [1, n]$, u_i has the sign of p_{i1}, the i^{th} item of the first column vector of the matrix $P = (p_{ij})$.*

(II) If $\lambda_\rho < \lambda\sigma_1 < \lambda_\rho + \delta'$, System (3.1) *has a unique solution $U = (u_i)$. Moreover, on B_R, for each integer $i \in [1, n]$, u_i has the sign of $-p_{i1}$, where p_{i1} is as above.*

Remark 3.3. From (3.2) , we can also derive (always for $\lambda > 0$) that $\lambda\sigma_1 < \lambda_\rho$ implies $\det(B) > 0$, where

$$B := (\lambda_\rho I_n - \lambda M) \tag{3.3}$$

Cooperative system: We make comments on the case of a cooperative system; that is, a system where the coefficients outside the diagonal of M are positive: $m_{ij} > 0, i \neq j$.

Remark 3.4. For the case of a cooperative system and $\lambda = 1$, Theorem 3.2 extends earlier results in [20, Thm. 7].

Remark 3.5. Always for a cooperative system, the first column p_{i1}, which consists in the components of the first eigenvector, by Perron-Frobinius theorem ([8, p. 27], [22]), is of one sign hence all components u_i have the same sign for $1 \leq i \leq n$.

From [20], we derive the following result.

Corollary 3.6. *Assume* (H1), (H3) *are satisfied and M is a cooperative matrix. $F = (f_i) \in L^\infty \cap L^2_{1/\rho}$. The cooperative system satisfies the maximum principle if and only if $B := (\lambda_\rho I_n - \lambda M)$ is a nonsingular M-matrix.*

A nonsingular square matrix $\mathcal{B} = (b_{ij})$ is a M-matrix if $b_{ij} \le 0$ for $i \ne j$, $b_{ii} > 0$ and if all principal minors extracted from $\mathcal{B}$ are positive (see [8, 15]).

Remark 3.7. For a cooperative system, the condition $0 < \lambda\sigma_1 < \lambda_\rho$ is equivalent to Condition (3.3); that is,

(a) $b_{ij} = -\lambda m_{ij} \le 0$ for $i \ne j$ by $m_{ij} \ge 0$ (M is cooperative).
(b) $\lambda_\rho - \lambda\sigma_1, \dots, \lambda_\rho - \lambda\sigma_n$ are eigenvalues of B. They are real and positive

$$\Leftrightarrow 0 < \lambda\sigma_1 < \lambda_\rho \Leftrightarrow \lambda\sigma_n \le \dots \le \lambda\sigma_2 < \lambda\sigma_1 < \lambda_\rho.$$

4. Proofs for a positive weight

Proof of Corollary 2.6. It follows simply from the maximum principle. Consider the two equations

$$-\Delta(c\varphi_\rho) = \lambda_\rho \rho (c\varphi_\rho),$$
$$-\Delta u = \lambda \rho u + f,$$

where c is a constant. By substraction

$$-\Delta(c\varphi_\rho - u) = \lambda\rho(c\varphi_\rho - u) + (\lambda_\rho - \lambda)\rho(c\varphi_\rho) - f;$$

Choosing $c = \|f\|_Y/(\lambda_\rho - \lambda)$ we have $(\lambda_\rho - \lambda)\rho(c\varphi_\rho) - f \ge 0, \not\equiv 0$ and we derive the result by the classical maximum principle. $\square$

Proof of Theorem 2.8. This is the same proof as in [21], but our hypothesis here are more general. We study Equation (2.1):

$$-\Delta u = \lambda\rho(x)u + f, \quad \lim_{|x|\to\infty} u(x) = 0, \quad x \in \mathbb{R}^N, \quad N \ge 3,$$

when (H1) and (H2) are satisfied as well as (2.8): $\int f\varphi_\rho > 0$.

We decompose u. Set $u = u^*\varphi_\rho + u^\perp$ where $\int u^\perp \rho\varphi_\rho = 0$. Also $u^* = \int \rho u \varphi_\rho$. Analogously set $f = \rho h$ and decompose h: $h = h^*\varphi_\rho + h^\perp$. Equation (2.1) becomes

$$(\lambda_\rho - \lambda)\rho u^* \varphi_\rho - \Delta u^\perp = \lambda\rho u^\perp + h^*\rho\varphi_\rho + \rho h^\perp.$$

Multiply by φ_ρ and integrate. We obtain $(\lambda_\rho - \lambda)u^* = h^*$. Hence

$$u = \frac{h^*}{\lambda_\rho - \lambda}\varphi_\rho + u^\perp.$$

Also $u^\perp$ is solution to

$$-\Delta u^\perp = \lambda\rho u^\perp + \rho h^\perp. \tag{4.1}$$

Since $\int |\nabla u^\perp|^2 \ge \lambda_2 \int \rho(u^\perp)^2$, λ_2 being the second eigenvalue of our problem, we compute:

$$(\lambda_2 - \lambda)\int_{\mathbb{R}^N} \rho(x)(u^\perp)^2 \le \int_{\mathbb{R}^N} \rho(x)h^\perp u^\perp \le \Big[\int_{\mathbb{R}^N} \rho(x)(h^\perp)^2 \int_{\mathbb{R}^N} \rho(x)(u^\perp)^2\Big]^{1/2}.$$

Hence

$$\int_{\mathbb{R}^N} \rho(x)(u^\perp)^2 \le \Big(\frac{1}{\lambda_2 - \lambda}\Big)^2 \int_{\mathbb{R}^N} \rho(x)(h^\perp)^2.$$

When $\lambda < \lambda_\rho < \lambda_2$, $\lambda_2 - \lambda > \lambda_2 - \lambda_\rho$. Hence there exists $K > 0$, depending on h, such that

$$\int_{\mathbb{R}^N} \rho(x)(u^\perp)^2 \le K(\lambda_2 - \lambda_\rho)^{-2};$$

this upper bound is independent of λ. When $\lambda_\rho < \lambda < \lambda_2$, we choose $\lambda_2 - \lambda > \frac{1}{4}(\lambda_2 - \lambda_\rho)$. Again there exists $K' > 0$ such that

$$\int_{\mathbb{R}^N} \rho(x)(u^\perp)^2 \le K'(\lambda_2 - \lambda_\rho)^{-2};$$

and again the upper bound is independent of λ.

Now choose $R > 0$. In both cases, $u^\perp$ being bounded in $L^2_\rho(\mathbb{R}^N)$, is also bounded on $L^2_\rho(B_R)$ and then in $L^2(B_R)$. Therefore, since $\lambda < \lambda_2$ the right-hand side term in (4.1): $\lambda\rho u^\perp + \rho h^\perp$ is also bounded in $L^2(B_R)$.
By a classical bootstrap method we derive (e.g. as in [17]) that $u^\perp$ is bounded in $L^q(B_R)$ and finally in B_R with respect to the sup-norm. On B_R, the groundstate φ_ρ is bounded below.

For $\lambda_\rho - \delta < \lambda < \lambda_\rho < \lambda_2$, it is possible to choose $\lambda_\rho - \lambda > 0$ small enough so that

$$|u^\perp| \le K' \frac{h^*}{\lambda_2 - \lambda_\rho}\varphi_\rho = K'' \frac{h^*}{\lambda_\rho - \lambda}\varphi_\rho < \frac{h^*}{\lambda_\rho - \lambda}\varphi_\rho.$$

Hence we obtain

$$u = \frac{h^*}{\lambda_\rho - \lambda}\varphi_\rho + u^\perp \le \frac{h^*}{\lambda_\rho - \lambda}\varphi_\rho + K'' \frac{h^*}{\lambda_\rho - \lambda}\varphi_\rho = (1 + K'')\frac{h^*}{\lambda_\rho - \lambda}\varphi_\rho,$$

and

$$u = \frac{h^*}{\lambda_\rho - \lambda}\varphi_\rho + u^\perp \ \ge \ \frac{h^*}{\lambda_\rho - \lambda}\varphi_\rho - K'' \frac{h^*}{\lambda_\rho - \lambda}\varphi_\rho = (1 - K'')\frac{h^*}{\lambda_\rho - \lambda}\varphi_\rho,$$

then (2.9) is obtained.

For $\lambda_\rho < \lambda < \lambda_2$, it is possible to choose $\lambda - \lambda_\rho > 0$ small enough so that $|u^\perp| \le K' \frac{h^*}{\lambda_2 - \lambda_\rho}\varphi_\rho \le K'' \frac{h^*}{\lambda - \lambda_\rho}\varphi_\rho$. Then (2.10) can be obtained similarly to the discussion above. $\square$

Proof of Theorem 3.2. We study $-\Delta U = \lambda\rho(x)MU + F$ when M is a $n \times n$ matrix and ρ a non negative weight. The case of a 2×2 system is studied in [21].

We set

$$U = P\widetilde{U} \Leftrightarrow \widetilde{U} = P^{-1}U, \quad F = P\widetilde{F} \Leftrightarrow \widetilde{F} = P^{-1}F,$$

here $\widetilde{U}$ and $\widetilde{F}$ are column vectors with components $\widetilde{u}_i$ and $\widetilde{f}_i$, respectively, $1 \le i \le n$. P has constant coefficients. System (3.1) can be written as

$$-\Delta\widetilde{U} = \lambda\rho(x)J\widetilde{U} + \widetilde{F}, \tag{4.2}$$

where J is a Jordan canonical form which has p Jordan blocks J_i $(1 \le j \le p \le n)$. Every Jordan block is a square $k_i \times k_i$ matrix of the form:

$$J_i = \begin{pmatrix} \sigma_i & 1 & 0 & \\ & \ddots & \ddots & \\ & 0 & \ddots & 1 \\ & & & \sigma_i \end{pmatrix}.$$

By Hypothesis (H3), the first block is 1×1: $J_1 = (\sigma_1)$.

We will adapt the method in [25]. By Hypothesis (H3), we obtain the first equation
$$-\Delta\widetilde{u}_1=\lambda\rho(x)\sigma_1\widetilde{u}_1+\tilde{f}_1, \tag{4.3}$$
where $\int\tilde{f}_1\varphi_\rho>0$, and $\tilde{f}_1\in Y$.
(**I**) If $\lambda\sigma_1<\lambda_\rho$. If $\lambda_\rho-\delta<\lambda\sigma_1<\lambda_\rho$, by Theorem 2.8, and (4.3), we get the local fundamental positivity for $\widetilde{u}_1$:
$$0<\frac{k'}{(\lambda_\rho-\lambda\sigma_1)}\varphi_\rho\le\widetilde{u}_1\le\frac{k''}{(\lambda_\rho-\lambda\sigma_1)}\varphi_\rho, \tag{4.4}$$
on the ball $B_R=\{x\in\mathbb{R}^N:|x|<R\}$. As indicated in Corollary 2.9, (4.4) implies $\widetilde{u}_1/\varphi_\rho\to+\infty$ as $\lambda\sigma_1\to\lambda_\rho$.

Then we consider the Jordan blocks J_i with $2\le i\le n$. J_i is a $k_i\times k_i$ matrix. For $i=1$, $k_1=1$; for $i\ge2$, we denote $s_i=\sum_{j=1}^{i-1}k_j$. All this gives equations from line s_i+1 to line s_i+k_i-1, we obtain k_i-1 equations:
$$-\Delta\widetilde{u}_j=\lambda\rho(x)\sigma_i\widetilde{u}_j+\lambda\rho(x)\widetilde{u}_{j+1}+\tilde{f}_j,\quad \text{if } s_i+1\le j<s_i+k_i-1. \tag{4.5}$$
The last equation in this block is
$$-\Delta\widetilde{u}_j=\lambda\rho(x)\sigma_i\widetilde{u}_j+\tilde{f}_j,\quad \text{for } j=s_i+k_i=s_{i+1}\ (i\ge2). \tag{4.6}$$
From $\lambda\sigma_i<\lambda\sigma_1<\lambda_\rho$ and Corollary 2.6, we know that the solution to (4.6); that is, $\widetilde{u}_{s_{i+1}}$ exists and
$$|\widetilde{u}_{s_{i+1}}|\le\frac{\|\tilde{f}_{s_{i+1}}\|_Y}{\lambda_\rho-\lambda\sigma_i}\varphi_\rho.$$
Take e.g. $\lambda\sigma_1>\frac{\lambda_\rho}{2}$, hence let $\lambda_{\min}=\frac{\lambda_\rho}{2\sigma_1}$. Then we obtain
$$|\widetilde{u}_{s_{i+1}}|\le\frac{\|\tilde{f}_{s_{i+1}}\|_Y}{\lambda(\sigma_1-\sigma_i)}\varphi_\rho\le\frac{\|\tilde{f}_{s_{i+1}}\|_Y}{\lambda_{\min}(\sigma_1-\sigma_i)}\varphi_\rho.$$
So we know that $\widetilde{u}_{s_{i+1}}\in X$ and $\|\widetilde{u}_{s_{i+1}}\|_X\le c_{s_{i+1}}$, where $c_{s_{i+1}}$ depends only on F,M.

In line $j=s_{i+1}-1$, we have
$$-\Delta\widetilde{u}_j=\lambda\rho(x)\sigma_i\widetilde{u}_j+\lambda\rho(x)\widetilde{u}_{s_{i+1}}+\tilde{f}_j\ (i\ge2)$$
from (4.5). Using Corollary 2.6, we obtain the existence of $\widetilde{u}_j$ and
$$\begin{aligned}|\widetilde{u}_j|&\le\frac{\|\lambda\rho(x)\widetilde{u}_{s_{i+1}}+\tilde{f}_j\|_Y}{\lambda_\rho-\lambda\sigma_i}\varphi_\rho\le\frac{\lambda\|\rho(x)\widetilde{u}_{s_{i+1}}\|_Y+\|\tilde{f}_j\|_Y}{\lambda(\sigma_1-\sigma_i)}\varphi_\rho\\&\le\frac{\lambda c_{s_{i+1}}+m_j}{\lambda(\sigma_1-\sigma_i)}\varphi_\rho\le\Big(\frac{c_{s_{i+1}}}{\sigma_1-\sigma_i}+\frac{m_j}{\lambda_{\min}(\sigma_1-\sigma_i)}\Big)\varphi_\rho\end{aligned}$$
where $\|\rho(x)\widetilde{u}_{s_{i+1}}\|_Y\le c_{s_{i+1}}$, $\|\tilde{f}_j\|_Y\le m_j$. So for $j=s_{i+1}-1$, we have $\widetilde{u}_j\in X$ and $\|\widetilde{u}_j\|_X\le c_j$, where c_j depends only on $F,\ M,\ \rho$. Similar discussion from line s_{i+1} to line s_i+1, we obtain that, for each integer j with $s_i+1\le j\le s_{i+1}$, $\|\widetilde{u}_j\|_X\le c_j$ in each Jordan's block J_i, where the real number c_j depends only on F,M,ρ.

In conclusion, we have that for $2\le j\le n$,
$$|\widetilde{u}_j|\le c_j\varphi_\rho, \tag{4.7}$$
where c_j depends only on F,M,ρ. For $j=1$,
$$\widetilde{u}_j\ge c\varphi_\rho, \tag{4.8}$$
where c depends on F,M,λ; moreover $\widetilde{u}_1/\varphi_\rho\to+\infty$ as $\lambda\sigma_1\to\lambda_\rho$.

Now we go back to the functions u_i. For any given $R > 0$, $U = P\tilde{U} = (u_i)$ implies that for each u_i, $1 \le i \le n$, we have

$$u_i = p_{i1}\widetilde{u}_1 + \sum_{j=2}^{n} p_{ij}\widetilde{u}_j. \tag{4.9}$$

So $\widetilde{u}_1/\varphi_\rho \to +\infty$ as $\lambda\sigma_1 \to \lambda_\rho$; by (4.7), $\sum_{j=2}^n p_{ij}\widetilde{u}_j$ is bounded by a constant times φ_ρ. Therefore, on any ball B_R, for any $i \in \{1, 2, \dots, n\}$, there exists $\delta = \inf_i \delta_i > 0$ such that for $\lambda_\rho - \delta < \lambda\sigma_1 < \lambda_\rho$, u_j has the sign of p_{ij} .
(II) If $\lambda\sigma_1 > \lambda_\rho$ and $\lambda\sigma_1 - \lambda_\rho$ is small enough, then there exists $\delta' > 0$ such that $\lambda\sigma_k < \lambda_\rho < \lambda\sigma_1 < \lambda_\rho + \delta'$. In this case, we also have Equations (4.3), (4.5) and (4.6).

By Theorem 2.8 and (4.3), we can get the local fundamental negativity: On any ball B_R, there exists $\delta_1 > 0$ such that if $\lambda_\rho < \lambda\sigma_1 < \lambda_\rho + \delta_1 < \lambda_\rho + \delta'$, then

$$-\frac{K''}{\lambda\sigma_1 - \lambda_\rho}\varphi_\rho \le \widetilde{u}_1 \le -\frac{K'}{\lambda\sigma_1 - \lambda_\rho}\varphi_\rho < 0,$$

and $\widetilde{u}_1/\varphi_\rho \to -\infty$ as $\lambda\sigma_1 \to \lambda_\rho$. In the equations $-\Delta\widetilde{u}_i = \lambda\rho(x)\sigma_k\widetilde{u}_i + \widetilde{f}_i$, we have $\lambda\sigma_k < \lambda_\rho$, so we can use the local fundamental positivity results and by (4.7), $\sum_{j=2}^n p_{ij}\widetilde{u}_j$ is bounded by a constant times φ_ρ. In $u_i = p_{i1}\widetilde{u}_1 + \sum_{j=2}^n p_{ij}\widetilde{u}_j$, we have $\sum_{j=2}^n p_{ij}\widetilde{u}_j$ bounded by a constant times φ_ρ and $\widetilde{u}_1/\varphi_\rho \to -\infty$ as $\lambda\sigma_1 \to \lambda_\rho$. So, on any B_R, there exists $\delta' > 0$ such that for $\lambda_\rho < \lambda\sigma_1 < \lambda_\rho + \delta'$, we obtain that u_i has the sign of $-p_{i1}$. $\square$

5. The case of an indefinite weight

We suppose now that ρ changes sign and we set $\rho^+(x) = max\{\rho(x), 0\}$ and $\rho = \rho^+ - \rho^-$. We assume the hypothesis

(H1') – ρ is smooth, bounded and positive somewhere; that is, $\Omega_+ := \{x \in \mathbb{R}^N/\rho(x) > 0\}$ is with positive Lebesgue measure.
– ρ^+ satisfies (H1) with some $\alpha > 1$.

We seek for solutions in the space $\mathcal{D}^{1,2}_{\rho^-} := \{u \in \mathcal{D}^{1,2} \text{ s.t. } \int_{\mathbb{R}^N} \rho^- u^2 < \infty\}$.
The principal eigenpair We denote by $(\lambda_\rho > 0, \varphi_\rho > 0)$ the associated principal eigenpair. From the hypothesis on ρ, we know that a non-positive eigenvalue may exist. Since $\lambda_\rho = (-\lambda)(-\rho)$, we only consider the case of positive $\lambda > 0$; also (2.7) adapted to this case is stated as:

Lemma 5.1. *Assume* (H1'). *Then*

$$0 < \lambda_\rho = \inf_{\{u \in \mathcal{D}^{1,2}_{\rho^-} : \int_{\mathbb{R}^N} \rho u^2 = 1\}} \int |\nabla u|^2. \tag{5.1}$$

The equality holds if and only if u is proportional to φ_ρ.

Now define the hypothesis

(H2') There exist $\alpha > 1$, $\epsilon > 0$ with $f = (\rho^+ + \epsilon p_\alpha)h$ such that $h \in L^\infty_{\text{loc}} \cap L^2_{p_\alpha}$.
Results for a weight changing sign: We state our results first for one equation, then for an $n \times n$ system. It is easy to prove that Proposition 2.7 and Corollary 2.6 are still valid for this indefinite weight.

Theorem 5.2. *Assume* (H1'), (H2') *and* (2.8)*: $\int_{\mathbb{R}^N} f\varphi_\rho > 0$. Then, for any $R > 0$, there exists $\delta > 0$ (depending on R, f and ρ) such that*

- *for $\lambda < \lambda_\rho$, and $\lambda_\rho - \lambda < \delta$, there exist positive constants k' and k'', depending on h and R, such that on $B_R = \{x \in \mathbb{R}^N : |x| < R\}$,*
$$0 < \frac{k'}{\lambda_\rho - \lambda}\varphi_\rho \le u \le \frac{k''}{\lambda_\rho - \lambda}\varphi_\rho. \tag{5.2}$$
("Local Fundamental Positivity")
- *for $\lambda_\rho < \lambda < \lambda_\rho + \delta$, there are positive constants K', K'', depending on f, φ_ρ and R such that on the ball $B_R = \{x \in \mathbb{R}^N : |x| < R\}$:*
$$-\frac{K''}{\lambda - \lambda_\rho}\varphi_\rho \le u \le -\frac{K'}{\lambda - \lambda_\rho}\varphi_\rho < 0. \tag{5.3}$$
("Local fundamental negativity")

We consider finally System (3.1) and use the same notation as above, and following hypothesis.

(H'4) We assume that $F = (f_i) = (\rho h_i) \in Y$ with $h_i \in L^\infty_{\rm loc} \cap L^2_{p_\alpha}$ for any $i = 1, \dots, n$.

Theorem 5.3. *Assume that* (H1'), (H4'), (H3) *are satisfied. We also assume that there exists an eigenvector $\Theta \in Q$ associated with σ_1 such that $F(x) = \widetilde{f_1}(x)\Theta + F_T(x)$ with $\int \widetilde{f_1}\varphi_\rho > 0$ where $F_Q(x) \in Q$, $F_T(x) \in T$. Then, for any $R > 0$, there exist two positive real numbers δ and δ', depending on F, M, R, ρ such that*

(I) If $\lambda_\rho - \delta < \lambda\sigma_1 < \lambda_\rho$, System (3.1) *has a unique solution $U = (u_i)$. Moreover, on any B_R, for each integer $i \in [1, n]$, we have $u_i \in X$ and u_i has the sign of p_{i1}, the i^{th} item of the first column vector of the matrix $P = (p_{ij})$.*

(II) If $\lambda_\rho < \lambda\sigma_1 < \lambda_\rho + \delta'$, System (3.1) *has a unique solution $U = (u_i)$. Moreover, on any B_R, for each integer $i \in [1, n]$, we have $u_i \in X$ and u_i has the sign of $-p_{i1}$, where p_{i1} is given as above.*

Proof of Lemma 5.1. First, we recall for completeness the proof of the existence of a positive principal eigenvalue, denoted as in Section 1, λ_ρ associated with a principal eigenfunction φ_ρ. We follow e.g. [16]. The equation
$$-\Delta u = \lambda\rho(x)u \quad \text{on } \mathbb{R}^N$$
can be rewritten as
$$\begin{gathered}-\Delta u + \lambda\rho^-(x)u = \lambda\rho^+(x)u \quad \text{on } \mathbb{R}^N \iff \\ -\Delta u + \lambda\rho^-_\alpha(x)u = \lambda\rho^+_\alpha(x)u \quad \text{on } \mathbb{R}^N,\end{gathered}$$
where $\rho^\pm_\alpha := \rho^\pm + \epsilon p_\alpha$. This shift is introduced since ρ^+ is not necessarily positive but only ≥ 0. We study now, for $\mu > 0$,
$$-\Delta u + \mu\rho^-_\alpha(x)u = \lambda\rho^+_\alpha(x)u \quad \text{on } \mathbb{R}^N. \tag{5.4}$$
For the rest of this article, we set $L_\mu u := -\Delta u + \mu\rho^-_\alpha(x)u$.

Since ρ^+_α decreases fast enough, we have an analogous to Theorem 2.3: there exists a principal eigenpair $(\lambda^*(\mu), \psi_\mu)$ such that
$$\begin{gathered}L_\mu\psi_\mu = \lambda^*(\mu)\rho^+_\alpha(x)\psi_\mu \quad \text{on } \mathbb{R}^N, \\ \lambda^*(\mu) = \inf_{\{u \in \mathcal{D}^{1,2}_{\rho^-_\alpha} : \int_{\mathbb{R}^N} \rho^+_\alpha u^2 = 1\}} \int (|\nabla u|^2 + \mu\rho^-_\alpha(x)u^2)dx.\end{gathered}$$
Note that, by (2.3), $\mathcal{D}^{1,2}_{\rho^-_\alpha} = \mathcal{D}_{\rho^-}$.

$\lambda^*(\mu)$ varies continuously and monotonically with respect to μ; it increases. Moreover $\lambda^*(0)$ is the principal eigenvalue satisfying

$$-\Delta u = \lambda \rho^+_\alpha(x) u \quad \text{on } \mathbb{R}^N$$

where the weight is positive. Hence we know that $\lambda^*(0)$ exists and is positive. Also, by properties of monotonicity of eigenvalues w.r.t. the domain (for Dirichlet boundary conditions) , since $\Omega^+ \subset \mathbb{R}^N$, $\lambda^*(\mu)$ is always less than τ^*_μ the positive principal eigenvalue of $L_\mu u = \tau \rho^+_\alpha u$ defined on Ω_+ with Dirichlet boundary conditions. This curve cuts the bisectrix at τ^0, the positive principal eigenvalue of $-\Delta u = \tau^0 \rho^+ u$ on Ω^+. Hence , the (continuous) curve $(\mu, \lambda^*(\mu))$ intersects the bisectrix at λ^0 such that, for some u:

$$-\Delta u + \lambda^0 \rho^-_\alpha(x) u = L_{\lambda^0} u = \lambda^0 \rho^+_\alpha(x) u \iff -\Delta u = \lambda^0 \rho(x) u \quad \text{on } \mathbb{R}^N.$$

λ^0 is the (unique) positive principal eigenvalue λ_ρ of

$$-\Delta u = \lambda \rho(x) u, \ x \in \mathbb{R}^N,$$

and λ_ρ defined by (5.1) and $u = k\varphi_\rho$. It does not depend on α. □

Remark 5.4. $(\lambda^*(\mu) - \lambda_\rho)(\mu - \lambda_\rho) > 0$ *for* μ *varying around* λ_ρ.

Proof of Theorem 5.2. Consider the equation

$$-\Delta u = \lambda \rho(x) u + f,$$

rewritten as

$$-\Delta u + \lambda \rho^-_\alpha(x) u = \lambda \rho^+_\alpha(x) u + f.$$

This equation with positive weight can be treated as (2.1) except that the Laplacian $-\Delta$ is replaced by $L^* := -\Delta + \lambda_\rho \rho^-_\alpha$. Obviously, its principal eigenpair is $(\lambda_\rho, \varphi_\rho)$: $L^* \varphi_\rho = \lambda_\rho \rho^+_\alpha(x) \varphi_\rho$. We denote by σ_2 the second eigenvalue of this problem. As previously (proof of Theorem 2.8), we decompose $u = u^* \varphi_\rho + u^\perp$ where $\int \rho^+_\alpha u^\perp \varphi_\rho = 0$. Also $h = h^* \varphi_\rho + h^\perp$. We derive analogous inequalities

$$u = \frac{h^*}{\lambda_\rho - \lambda} \varphi_\rho + u^\perp$$

and

$$\int \rho^+_\alpha (u^\perp)^2 \le \Big(\frac{1}{\sigma_2 - \lambda}\Big)^2 \int \rho^+_\alpha (h^\perp)^2.$$

For $\lambda < \lambda_\rho$, $\sigma_2 - \lambda > \sigma_2 - \lambda_\rho$.

For $\lambda > \lambda_\rho$, choose λ close enough to λ_ρ so that $\sigma_2 - \lambda > \frac{1}{4}(\sigma_2 - \lambda_\rho)$.

We derive as for positive weight that $|u^\perp_\lambda|$ is locally bounded and, for $|\lambda - \lambda_\rho|$, small enough we derive the results for one equation as for a positive weight. We conclude as in the proof of Theorems 2.8 and 3.2. □

Proof of Theorem 5.3. For a positive weight, we set

$$U = P\widetilde{U} \Leftrightarrow \widetilde{U} = P^{-1} U, \quad F = P\widetilde{F} \Leftrightarrow \widetilde{F} = P^{-1} F.$$

System (3.1) can be written as

$$-\Delta \widetilde{U} = \lambda \rho(x) J \widetilde{U} + \widetilde{F}, \tag{5.5}$$

where J is a Jordan canonical form which has p Jordan blocks J_i $(1 \le j \le p \le n)$. We are also lead to Equations (4.3), (4.5) and (4.6) in $\tilde{u}_j$ with indefinite weight that we can study using Theorem 5.2. Going back to $U = P\tilde{U}$, we have (4.9):

$$u_i = p_{i1}\widetilde{u}_1 + \sum_{j=2}^{n} p_{ij}\widetilde{u}_j, \quad 1 \le i \le n.$$

Finally on a ball B_R, we have
(**I**) If $\lambda_\rho - \delta < \lambda\sigma_1 < \lambda_\rho$ and $\lambda_\rho - \lambda\sigma_1$ small enough, we have $\widetilde{u}_1/\varphi_\rho \to +\infty$ as $\lambda\sigma_1 \to \lambda_\rho$ and $\sum_{j=2}^n p_{ij}\widetilde{u}_j$ is bounded by a constant times φ_ρ.
(**II**) If $\lambda_\rho < \lambda\sigma_1 < \lambda_\rho + \delta'$ and $\lambda\sigma_1 - \lambda_\rho$ small enough, we have $\widetilde{u}_1/\varphi_\rho \to -\infty$ as $\lambda\sigma_1 \to \lambda_\rho$ and $\sum_{j=2}^n p_{ij}\widetilde{u}_j$ is bounded by a constant times φ_ρ. $\square$

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Bénédicte Alziary
Institut de Mathématique -UMR CNRS 5219- et Ceremath-UT1, Université de Toulouse, 31042 Toulouse Cedex, France
E-mail address: alziary@univ-tlse1.fr

Jacqueline Fleckinger
Institut de Mathématique -UMR CNRS 5219- et Ceremath-UT1, Université de Toulouse, 31042 Toulouse Cedex, France
E-mail address: jfleckinger@gmail.com

Marie-Hélène Lecureux
Institut de Mathématique -UMR CNRS 5219- et Ceremath-UT1, Université de Toulouse, 31042 Toulouse Cedex, France
E-mail address: mhlecureux@gmail.com

Na Wei
Dept. Appl. Math., Northwestern Polytechnical Univ. 710072 Xi'an, China
School of Stat. & Math., Zhongnan Univ. Eco. & Law, 430073, Wuhan, China
E-mail address: nawei8382@gmail.com