Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 127, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

OPTIMAL CONTROL PROBLEM FOR A SIXTH-ORDER CAHN-HILLIARD EQUATION WITH NONLINEAR DIFFUSION

CHANGCHUN LIU, ZHAO WANG

Abstract. In this article, we study the initial-boundary-value problem for a sixth-order Cahn-Hilliard type equation

$$u_t = D^2\mu,$$
$$\mu = \gamma D^4 u - a(u)D^2u - \frac{a'(u)}{2}|Du|^2 + f(u) + ku_t,$$

which describes the separation properties of oil-water mixtures, when a substance enforcing the mixing of the phases is added. The optimal control of the sixth order Cahn-Hilliard type equation under boundary condition is given and the existence of optimal solution to the sixth order Cahn-Hilliard type equation is proved.

1. Introduction

We consider the equation

$$u_t = D^2\big[\gamma D^4 u - a(u)D^2u - \frac{a'(u)}{2}|Du|^2 + f(u) + ku_t\big], \tag{1.1}$$

in $\Omega \times (0,T)$, where Ω is a bounded subset in $\mathbb{R}$ and $\gamma > 0$ with the initial and boundary conditions

$$u(x,0) = u_0, \quad \text{in } \Omega, \tag{1.2}$$
$$u(x,t) = D^2u(x,t) = D^4u(x,t) = 0, \quad \text{on } \partial\Omega. \tag{1.3}$$

The function $f(u)$ stands for the derivative of a potential $F(u)$ with $F(u)$, $a(u)$ approximated, respectively, by a sixth and a second order polynomial

$$F(u) = \int_0^u f(s)ds = \gamma_1(u+1)^2(u^2+h_0)(u-1)^2, \tag{1.4}$$
$$a(u) = a_2u^2 + a_0, \tag{1.5}$$

where $\gamma_1 > 0$, $a_2 > 0$.

The model (1.1) describes the separation properties of oil-water mixtures, when a substance enforcing the mixing of the phases (a surfactant) is added. G. Schimperna

2000 *Mathematics Subject Classification.* 49J20, 35K35, 35K55.
Key words and phrases. Cahn-Hilliard equation; existence; optimal control; optimal solution.

Submitted March 1, 2012. Published August 14, 2012.

et al. [9] studied the equation (1.1) with logarithmic potential

$$F(r) = (1-r)\log(1-r) + (1+r)\log(1+r) - \frac{\lambda}{2}r^2, \quad \lambda > 0.$$

They investigated the behavior of the solutions to the sixth order system as the parameter γ tended to 0, the uniqueness and regularization properties of the solutions have been discussed.

When $k = 0$, equation (1.1) is the sixth order equation which describes dynamics of phase transitions in ternary oil-water-surfactant systems [2, 3, 4]. The surfactant which has a character that one part of it is hydrophilic and the other lipophilic is called amphiphile. In the system, almost pure oil, almost pure water and microemulsion which consists of a homogeneous, isotropic mixture of oil and water can coexist in equilibrium. Pawłow and Zajączkowski [8] proved that the problem (1.1)-(1.5) with $k = 0$ under consideration is well posed in the sense that it admits a unique global smooth solution which depends continuously on the initial datum. Liu [7] studied the equation

$$\frac{\partial u}{\partial t} - \operatorname{div}[m(u)(k\nabla\Delta^2 u + \nabla(-a(u)\Delta u - \frac{a'(u)}{2}|\nabla u|^2 + h(u))] = 0,$$

and proved the existence of classical solutions for two dimensions.

In past decades, the optimal control of distributed parameter system had been received much more attention in academic field. A wide spectrum of problems in applications can be solved by methods of optimal control, such as chemical engineering and vehicle dynamics. Modern optimal control theories and applied models are not only represented by ODE, but also by PDE. Kunisch and Volkwein [6] solved open-loop and closed-loop optimal control problems for the Burgers equation. Armaou and Christofides[1] studied the feedback control of Kuramto-Sivashing equation.

Recently, many authors studied the optimal control problem for the viscous PDE, such as Tian et al. [10]-[13], Zhao and Liu [15].

In this article, we consider the optimal control problem for the equation

$$\begin{aligned}&(u - kD^2u)_t + \frac{\gamma}{k}D^4(u - kD^2u) + D^2(a(u)D^2u + \frac{a'(u)}{2}|Du|^2)\\ =&\frac{\gamma}{k}D^4u + D^2f(u) + B^*\overline{\omega},\end{aligned}$$

with (1.2)-(1.5).

When $y = u - D^2u$, we take the distributed optimal control problem

$$\begin{gathered}\min \mathcal{J}(y,\overline{\omega}) = \frac{1}{2}\|Cy - z\|_S^2 + \frac{\delta}{2}\|\overline{\omega}\|^2_{L^2(0,T;Q_0)},\\ \text{s. t. } y_t + \frac{\gamma}{k}D^4y - \frac{\gamma}{k}D^4u + D^2(a(u)D^2u + \frac{a'(u)}{2}|Du|^2) - D^2f(u) = B^*\overline{\omega},\\ y(x,0) = y_0 = u_0 - D^2u(x,0),\\ u(x,t) = D^2u(x,t) = D^4u(x,t) = 0.\end{gathered} \tag{1.6}$$

For a fixed $T > 0$, we set $\Omega = (0,1)$ and $Q = \Omega \times (0,T)$. Let $Q_0 \subset Q$ be an open set with positive measure.

Let $V = H_0^2(0,1), H = L^2(0,1), V^* = H^{-2}(0,1)$ and $H^* = L^2(0,1)$ are dual spaces respectively, we have $V \hookrightarrow H = H^* \hookrightarrow V^*$. The extension operator $B^* \in$

$L(L^2(0,T;Q_0),L^2(0,T;V^*))$ is given by

$$B^*q=\begin{cases} q, & q\in Q_0,\\ 0, & q\in Q/Q_0.\end{cases} \tag{1.7}$$

The space $W(0,T;V)$ is defined by

$$W(0,T;V)=\{y,y\in L^2(0,T;V),y_t\in L^2(0,T;V^*)\}$$

which is a Hilbert space endowed with common inner product.

The plan of the paper is as follows. In section 2, we prove the existence of the weak solution in a special space. The optimal control is discussed in section 3, and the existence of an optimal solution is proved.

2. Existence of weak solutions

Consider the the sixth-order Cahn-Hilliard equation

$$\begin{aligned}&(u-kD^2u)_t+\frac{\gamma}{k}D^4(u-kD^2u)+D^2(a(u)D^2u+\frac{a'(u)}{2}|Du|^2)\\&=\frac{\gamma}{k}D^4u+D^2f(u)+B^*\overline{\omega},\end{aligned} \tag{2.1}$$

under the initial condition

$$u(x,0)=u_0,$$

and boundary condition

$$u(x,t)=D^2u(x,t)=D^4u(x,t)=0,$$

where $B^*\overline{\omega}\in L^2(0,T;V^*)$ and the control item $\overline{\omega}\in L^2(0,T;Q_0)$.

Let $y=u-kD^2u$. Then the above problem is rewritten as

$$\begin{gathered}y_t+\frac{\gamma}{k}D^4y-\frac{\gamma}{k}D^4u+D^2(a(u)D^2u+\frac{a'(u)}{2}|Du|^2)-D^2f(u)=B^*\overline{\omega},\\ y(x,0)=y_0=u_0-D^2u_0,\\ u(x,t)=D^2u(x,t)=D^4u(x,t)=0,\end{gathered} \tag{2.2}$$

with (1.4)-(1.5).

Now, we give the definition of the weak solution for problem (2.2) in the space $W(0,T;V)$.

Definition 2.1. A function $y(x,t)\in W(0,T;V)$ is called a weak solution to problem (2.2), if

$$\begin{aligned}&\frac{d}{dt}(y,\phi)+\frac{\gamma}{k}(D^2y,D^2\phi)-(\frac{\gamma}{k}D^2u,D^2\phi)\\&+(a(u)D^2u+\frac{a'(u)}{2}|Du|^2,D^2\phi)-(f(u),D^2\phi)=(B^*\overline{\omega},\phi)_{V^*,V},\end{aligned}$$

for all $\phi\in V$, a.e. $t\in[0,T]$ and $y_0\in H$ are valid.

Theorem 2.2. *Problem* (2.2) *admits a weak solution* $y(x,t)\in W(0,T;V)$ *in the interval* $[0,T]$, *if* $B^*\overline{\omega}\in L^2(0,T;V^*)$, $y_0\in H$ *and* $u_0\in H^2(\Omega)\cap H_0^1(\Omega)$.

Proof. Employ the standard Galerkin method. The fourth-order differential operator $A = \partial_x^4$ is a linear unbounded self-adjoint operator in H with $D(A) = \{u | u \in H^4(\Omega), u|_{\partial\Omega} = D^2 u|_{\partial\Omega} = 0\}$ dense in H, where H is a Hilbert space with a scalar product $(\cdot,\cdot)$ and norm $\|\cdot\|$.

There exists orthogonal basis $\{\psi_i\}$ of H. Let $\{\psi_i\}_{i=1}^{\infty}$ be the eigenfunctions of the operator $A = \partial_x^4$ with

$$A\psi_j = \lambda_j \psi_j, \quad 0 < \lambda_1 \leq \lambda_2 \leq \dots, \quad \text{as } j \to \infty.$$

For $n \in \mathbb{N}$, we define the discrete ansatz space by

$$V_n = \operatorname{span}\{\psi_1, \psi_2, \dots, \psi_n\} \subset V.$$

Set $y_n(t) = y_n(x,t) = \sum_{i=1}^{n} y_i^n(t)\psi_i(x)$ require $y_n(0,\cdot) \mapsto y_0$ in H holds true.

To prove the existence of a unique weak solution to the problem (2.2), we are going to analyze the limiting behavior of sequences of smooth functions $\{y_n\}$ and $\{u_n\}$.

Performing the Galerkin process for the problem (2.2), we have

$$\begin{gathered}
y_{n,t} + \frac{\gamma}{k} D^4 y_n - \frac{\gamma}{k} D^4 u_n + D^2\big(a(u_n) D^2 u_n + \frac{a'(u_n)}{2} |Du_n|^2\big) - D^2 f(u_n) = B^* \overline{\omega}, \\
y_n(x,0) = y_{n,0} = u_{n,0} - D^2 u_n(x,0), \\
u_n(x,t) = D^2 u_n(x,t) = D^4 u_n(x,t) = 0.
\end{gathered} \tag{2.3}$$

According to ODE theory, there is a unique solution to (2.3) in the interval $[0, t_n]$. We should show that the solution is uniformly bounded when $t_n \to T$.

First step, multiplying the first equation of (2.3) by

$$\mu_n = \gamma D^4 u_n - a(u_n) D^2 u_n - \frac{a'(u_n)}{2} |Du_n|^2 + f(u_n) + k u_{n,t},$$

and integrating with respect to x, we obtain

$$\frac{d}{dt} E(u_n) + \|D\mu_n\|^2 + k\|u_{n,t}\|^2 = (B^*\overline{\omega}, \mu_n)_{V^*,V}, \tag{2.4}$$

where

$$E(u_n) = \int_0^1 \Big(\frac{\gamma}{2}|D^2 u_n|^2 + \frac{a(u_n)}{2}|Du_n|^2 + F(u_n)\Big) dx, \tag{2.5}$$

$$F(u_n) = \gamma_1 (u_n^6 + (h_0 - 2) u_n^4 + (1 - 2h_0) u_n^2 + h_0). \tag{2.6}$$

Applying a simple calculation, we have

$$F(u_n) \geq C_1 u_n^6 - C_0, \tag{2.7}$$

where $C_1 > 0$ and $C_0 \geq 0$.

Since $B^*\overline{\omega} \in L^2(0,T;V^*)$ is a control item, we assume

$$\|B^*\overline{\omega}\|_{V^*} \leq M. \tag{2.8}$$

Taking account (2.4), (2.7), (2.8) and (1.4) and integrating (2.4) with respect to time from 0 to t, we know

$$\begin{aligned}
&\int_0^1 \Big(\frac{\gamma}{2}|D^2 u_n|^2 + \frac{a_2}{2} u_n^2 |Du_n|^2 + C_1 u_n^6\Big) dx + \int_0^t \|D\mu_n\|^2 dt + k \int_0^t \|u_{n,t}\|^2 dt \\
&\leq \int_0^1 \frac{|a_0|}{2} |Du_n|^2 dx + E(u_{n,0}) + C_0 + \int_0^t |(B^*\overline{\omega}, \mu_n)_{V^*,V}| dt
\end{aligned}$$

$$
\begin{aligned}
&\le \varepsilon_1 \int_0^1 \frac{|a_0|}{2} |D^2 u_n|^2 dx + C(\varepsilon_1) \int_0^1 u_n^2 dx \\
&\quad + E(u_{n,0}) + C_0 + \int_0^t \|B^*\overline{\omega}\|_{V^*} \|\mu_n\|_V dt \\
&\le \varepsilon_1 \int_0^1 \frac{|a_0|}{2} |D^2 u_n|^2 dx + C(\varepsilon_1)\varepsilon_2 \int_0^1 u_n^6 dx + C(\varepsilon_2) \\
&\quad + E(u_{n,0}) + C_0 + C(\varepsilon) \int_0^t \|B^*\overline{\omega}\|_{V^*}^2 dt + \varepsilon \int_0^t \|D^2 \mu_n\|^2 dt \\
&= \varepsilon_1 \int_0^1 \frac{|a_0|}{2} |D^2 u_n|^2 dx + C(\varepsilon_1)\varepsilon_2 \int_0^1 u_n^6 dx + C(\varepsilon_2) \\
&\quad + E(u_{n,0}) + C_0 + C(\varepsilon) \int_0^t \|B^*\overline{\omega}\|_{V^*}^2 dt + \varepsilon \int_0^t \|u_{n,t}\|^2 dt,
\end{aligned}
$$

where

$$
E(u_{n,0}) = \int_0^1 \Big(\frac{\gamma}{2}|D^2 u_{n,0}|^2 + \frac{a(u_{n,0})}{2}|Du_{n,0}|^2 + F(u_{n,0})\Big) dx.
$$

Choosing $\varepsilon_1, \varepsilon_2$ and ε sufficiently small, from the above inequality and the Poincaré inequality, we have

$$
\int_0^1 |D^2 u_n|^2 dx \le C, \tag{2.9}
$$

$$
\int_0^1 |Du_n|^2 dx \le C, \tag{2.10}
$$

$$
\int_0^1 u_n^6 dx \le C, \tag{2.11}
$$

$$
\iint_{Q_T} |u_{n,t}|^2 \, dx \, dt \le C. \tag{2.12}
$$

From (2.11), we know that

$$
\int_0^1 u_n^2 dx \le C. \tag{2.13}
$$

By (2.9), (2.10) and (2.13), we obtain

$$
\|u_n\|_{H^2} \le C. \tag{2.14}
$$

By Sobolev's imbedding theorem it follows from (2.14) that

$$
\|u_n\|_{L^\infty} \le C, \quad \|Du_n\|_{L^\infty} \le C. \tag{2.15}
$$

Second step, multiplying (1.1) by $D^2 u_n$ and integrating with respect to x, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\int_0^1 |Du_n|^2 dx + k\int_0^1 |D^2 u_n|^2 dx\Big) + \gamma \int_0^1 |D^4 u_n|^2 dx \\
&= -\int_0^1 D^2 f(u_n) D^2 u_n dx + \int_0^1 a(u_n) D^2 u_n D^4 u_n dx \\
&\quad + \int_0^1 \frac{a'(u_n)}{2} |Du_n|^2 D^4 u_n dx - (B^*\overline{\omega}, D^2 u_n)_{V^*,V}.
\end{aligned} \tag{2.16}
$$

From a simple calculation, we have

$$a'(u_n) = 2a_2 u_n, \tag{2.17}$$

$$D^2 f(u_n) = f'(u_n) D^2 u_n + f''(u_n)(Du_n)^2, \tag{2.18}$$

where

$$f'(u_n) = \gamma_1(30u_n^4 + 12(h_0 - 2)u_n^2 + 2(1 - 2h_0)) \geq -C_2, \quad C_2 > 0, \tag{2.19}$$

$$f''(u) = \gamma_1 120u_n^3 + 24(h_0 - 2)u_n. \tag{2.20}$$

Thus it follows from (2.14), (2.18) and(2.19) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\int_0^1 |Du_n|^2 dx + k\int_0^1 |D^2 u_n|^2 dx\Big) + \gamma\int_0^1 |D^4 u_n|^2 dx \\
&\leq -\int_0^1 (f'(u_n)D^2 u_n + f''(u_n)|Du_n|^2) D^2 u_n dx + \int_0^1 (a_2 u_n^2 + a_0) D^2 u_n D^4 u_n dx \\
&\quad + \int_0^1 \frac{a'(u_n)}{2} |Du_n|^2 D^4 u_n dx + \|B^*\overline{\omega}\|_{V^*}\|D^2 u_n\|_V \\
&\leq C_2\int_0^1 |D^2 u_n|^2 dx + C(\|u_n\|_{L^\infty}^3 + \|u_n\|_{L^\infty})\|Du_n\|_{L^\infty}\int_0^1 |Du_n||D^2 u_n| dx \\
&\quad + |a_2|\|u_n\|_{L^\infty}^2 \int_0^1 |D^2 u_n||D^4 u_n| dx + |a_0|\int_0^1 |D^2 u_n||D^4 u_n| dx \\
&\quad + |a_2|\int_0^1 |u_n||Du_n|^2 |D^4 u_n| dx + C(\varepsilon)\|B^*\overline{\omega}\|_{V^*}^2 + \varepsilon\int_0^1 |D^4 u_n|^2 dx \\
&\leq C_2\int_0^1 |D^2 u_n|^2 dx + C(\|u_n\|_{L^\infty}^3 + \|u_n\|_{L^\infty})\|Du_n\|_{L^\infty}\Big(\varepsilon\int_0^1 |Du_n|^2\Big) \\
&\quad + C(\|u_n\|_{L^\infty}^3 + \|u_n\|_{L^\infty})\|Du_n\|_{L^\infty}\Big(C(\varepsilon)\int_0^1 |D^2 u_n|^2 dx\Big) \\
&\quad + |a_2|\|u_n\|_{L^\infty}^2\Big(C(\varepsilon)\int_0^1 |D^2 u_n|^2 dx + \varepsilon\int_0^1 |D^4 u_n|^2 dx\Big) \\
&\quad + |a_0|\Big(C(\varepsilon)\int_0^1 |D^2 u_n|^2 dx + \varepsilon\int_0^1 |D^4 u_n|^2 dx\Big) \\
&\quad + |a_2|\|Du_n\|_{L^\infty}^2\Big(\varepsilon\int_0^1 |D^4 u_n|^2 dx + C(\varepsilon)\int_0^1 u_n^2 dx\Big) \\
&\quad + C(\varepsilon)\|B^*\overline{\omega}\|_{V^*}^2 + \varepsilon\int_0^1 |D^4 u_n|^2 dx \\
&\leq \frac{\gamma}{2}\int_0^1 |D^4 u_n|^2 dx + C,
\end{aligned} \tag{2.21}$$

where ε is sufficiently small.

By the Gronwall's inequality, (2.21) implies

$$\iint_{Q_T} |D^4 u_n|^2 \, dx \, dt \leq C. \tag{2.22}$$

As we know

$$
\begin{aligned}
&\int_0^T\int_0^1 |D^3u_n|^2\,dx\,dt \\
&\le \frac{1}{2}\int_0^T\int_0^1 |D^2u_n|^2\,dx\,dt + \frac{1}{2}\int_0^T\int_0^1 |D^4u_n|^2\,dx\,dt \le C.
\end{aligned}
\tag{2.23}
$$

From a simple calculation, we have

$$
\|y_n\|_V = \|u_n - D^2u_n\|_V^2 \le C(\|u_n\| + \|Du_n\| + \|D^2u_n\| + \|D^3u_n\| + \|D^4u_n\|).
$$

From (2.14), (2.15), (2.22) and (2.23), we obtain

$$
\|y_n\|_{L^2(0,T;V)} \le C. \tag{2.24}
$$

Third step, from (2.2), (2.14), (2.15) and Sobolev embedding theorem, we have

$$
\begin{aligned}
\|y_{n,t}\|_{V^*} &\le \|B^*\overline{\omega}\|_{V^*} + \|D^4u_n\| + \|a(u_n)D^2u_n + \frac{a'(u_n)}{2}|Du_n|^2\| + \|f(u_n)\| \\
&\le \|B^*\overline{\omega}\|_{V^*} + \|D^4u_n\| + C\|u_n\|_{L^\infty}^2\|D^2u_n\| + C\|u_n\|_{L^\infty}\|Du_n\|^2 \\
&\quad + C\|u_n\|_{L^\infty}^6 + C \\
&\le \|D^4u_n\| + C.
\end{aligned}
$$

Then $\|y_{n,t}\|_{L^2(0,T;V^*)} \le C$. Thus, we have:

- (i) For every $t \in [0,T]$, the sequence $\{y_n\}_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;H)$ as well as in $L^2(0,T;V)$, which is independent of the dimension of amsatz space n.
- (ii) For every $t \in [0,T]$, the sequence $\{y_{n,t}\}_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;V^*)$, which is independent of the dimension of amsatz space n.

By theorem we get $\{y_{n,t}\}_{n\in\mathbb{N}} \subset W(0,T;V)$ and $W(0,T;V)$ is continuously embedded into $C(0,T;H)$. $\{y_{n,t}\}_{n\in\mathbb{N}}$ weak in $W(0,T;V)$, weak star in $L^\infty(0,T;H)$ and strong in $L^2(0,T;H)$ to a function $y(x,t) \in W(0,T;V)$. Obviously, the uniqueness of solution is easy to obtained [5]. We omit it here. $\square$

To ensure that the norm of weak solution in the space $W(0,T;V)$ can be controlled by initial value and control item, we need the following theorem.

Theorem 2.3. *If $B^*\overline{\omega} \in L^2(0,T;V^*)$, $u_0 \in H^2(\Omega)\cap H_0^1(\Omega)$ and $y_0 \in H$, then there exists a constant $C_3 > 0$ and $C_4 > 0$, such that*

$$
\|y\|_{W(0,T;V)}^2 \le C_3\Big(\|y_0\|_H^2 + \|\overline{\omega}\|_{L^2(0,T;Q_0)}^2\Big) + C_4.
$$

Proof. As in the proof of Theorem 2.2, we obtain

$$
\|u\| \le C, \quad \|Du\| \le C, \quad \|u\|_V \le C, \quad \|D^3u\| \le C. \tag{2.25}
$$

Multiplying this equation by y and integrating the equation with respect to x, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|y\|_H^2 + \frac{\gamma}{k}\|D^2y\|_H^2 \\
&= \int_0^1 \frac{\gamma}{k}D^2yD^2udx - \int_0^1 (a(u)D^2u + \frac{a'(u)}{2}|Du|^2)D^2ydx \\
&\quad - \int_0^1 DyDf(u)dx + (B^*\overline{\omega}, y)_{V^*,V}.
\end{aligned}
\tag{2.26}
$$

From Hölder and Young inequalities, we have

$$\int_0^1 \frac{\gamma}{k} D^2 y D^2 u dx \le C(\varepsilon)\|D^2 u\|^2 + \varepsilon \|D^2 y\|^2. \tag{2.27}$$

From (2.25), we have

$$\begin{aligned}
&\int_0^1 (a(u)D^2 u + \frac{a'(u)}{2}|Du|^2) D^2 y\, dx \\
&\le \|u\|_{L^\infty}^3 \|D^2 y\| \|D^2 u\| + |a_2| \|u\|_{L^\infty} \|Du\|_{L^\infty}^2 \|D^2 y\| \\
&\le C(\epsilon) C \|D^2 u\|^2 + \varepsilon \|D^2 y\|^2 + CC(\epsilon) + \varepsilon \|D^2 y\|^2 \\
&\le \varepsilon \|D^2 y\|^2 + C,
\end{aligned} \tag{2.28}$$

and

$$-\int_0^1 DyDf(u)dx \le C_2 \int_0^1 |Dy| dx \le C\|Dy\| + C \le C. \tag{2.29}$$

Note that

$$(B^*\overline{\omega}, y)_{V^*,V} \le \|B^*\overline{\omega}\|_{V^*} \|y\|_V. \tag{2.30}$$

From (2.26)-(2.30), we have

$$\frac{1}{2}\frac{d}{dt}\|y\|_H^2 + \frac{\gamma}{k}\|D^2 y\|_H^2 \le \varepsilon \|D^2 y\|_H^2 + C\|B^*\overline{\omega}\|_{V^*}^2 + C. \tag{2.31}$$

Integrating the above inequality with respect to t yields

$$\|y\|_H^2 \le \|y_0\|_H^2 + C\|B^*\overline{\omega}\|_{L^2(0,T;V^*)}^2 + C. \tag{2.32}$$

By (2.32), (2.2) and (2.25), we deduce that

$$\begin{aligned}
\|y_t\|_{V^*}^2 &\le \|B^*\overline{\omega}\|_{V^*}^2 + \frac{\gamma}{k}\|y\|_V^2 + \frac{\gamma}{k}\|D^2 u\| + \|(a(u)D^2 u + \frac{a'(u)}{2}|Du|^2)\| + \|f(u)\| \\
&\le \|B^*\overline{\omega}\|_{V^*}^2 + C\|y\|_V^2 + C \\
&\le \|y_0\|_H^2 + C\|B^*\overline{\omega}\|_{L^2(0,T;V^*)}^2 + C.
\end{aligned} \tag{2.33}$$

From (2.32) and (2.33), we have

$$\begin{aligned}
\|y\|_{W(0,T;V)} &= \|y\|_{L^2(0,T;V)} + \|y_t\|_{L^2(0,T;V^*)} \\
&\le C_3 \Big(\|y_0\|_H^2 + \|\overline{\omega}\|_{L^2(0,T;Q_0)}^2 \Big) + C_4.
\end{aligned}$$

The proof is completed. □

3. Optimal problem

In this section, we will study the distributed optimal control of the viscous generalized Cahn-Hilliard equation and the existence of optimal solution is obtained based on Lions' theory.

We study the following Problem when $\overline{\omega} \in L^2(0,T;Q_0)$

$$\begin{gathered}
\min \mathcal{J}(y, \overline{\omega}) = \frac{1}{2}\|Cy - z\|_S^2 + \frac{\delta}{2}\|\overline{\omega}\|_{L^2(0,T;Q_0)}^2 \\
\text{s. t. } y_t + \frac{\gamma}{k} D^4 y - \frac{\gamma}{k} D^4 u + D^2(a(u)D^2 u + \frac{a'(u)}{2}|Du|^2) - D^2 f(u) = B^*\overline{\omega}, \\
y(x,0) = y_0 = u_0 - D^2 u(x,0),
\end{gathered}$$

$$u(x,t) = D^2u(x,t) = D^4u(x,t) = 0,$$

where $y = u - D^2u$.

As we know that there exists a weak solution y to the equation (2.2), due to $u = (1-\partial_x^2)^{-1}y$, we know that there exists a weak solution u to the equation (2.1). Given an observation operator $C \in L(W(0,T;V), S)$, in which S is a real Hilbert space and C is continuous.

We choose the performance index of tracking type

$$\mathcal{J}(y,\overline{\omega}) = \frac{1}{2}\|Cy - z\|_S^2 + \frac{\delta}{2}\|\overline{\omega}\|^2_{L^2(0,T;Q_0)}, \tag{3.1}$$

where $z \in S$ is a desired state and $\delta > 0$ is fixed.

The optimal control problem about the sixth-order Cahn-hilliard equation is

$$\min \mathcal{J}(y,\overline{\omega}), \tag{3.2}$$

where $(y,\overline{\omega})$ satisfies (2.2).

Let $X = W(0,T;V) \times L^2(0,T;Q_0)$ and $Y = L^2(0,T;V) \times H$ We define an operator $e = e(e_1,e_2) : X \to Y$ by

$$e(y,\overline{\omega}) = e(e_1(y,\overline{\omega}), e_2(y,\overline{\omega})),$$

where

$$\begin{aligned} e_1(y,\overline{\omega}) = (\Delta^2)^{-1}\Big(&y_t + \frac{\gamma}{k}D^4y - \frac{\gamma}{k}D^4u + D^2(a(u)D^2u + \frac{a'(u)}{2}|Du|^2) \\ &- D^2f(u) - B^*\overline{\omega}\Big), \\ e_2 = y(x,0) - y_0, \end{aligned}$$

and Δ^2 is an operator from $H^2(0,1)$ to $H^{-2}(0,1)$.

Then equation (3.2) is rewritten as

$$\min \mathcal{J}(y,\overline{\omega}) \quad \text{subject to } e = e(y,\overline{\omega}) = 0.$$

Now, we have the following theorem.

Theorem 3.1. *There exists an optimal control solution to the above problem.*

Proof. Let $(y,\overline{\omega}) \in X$ satisfies the equation $e = e(y,\overline{\omega}) = 0$. In view of (3.1), we have

$$\mathcal{J}(y,\overline{\omega}) \geq \frac{\delta}{2}\|\overline{\omega}\|^2_{L^2(0,T;Q_0)}.$$

From Theorem 2.3, we have

$$\|y\|_{W(0,T;V)} \to \infty \quad \text{yields} \quad \|\overline{\omega}\|_{L^2(0,T;Q_0)} \to \infty.$$

Hence

$$\mathcal{J}(y,\overline{\omega}) \to +\infty \quad \text{when } \|y,\overline{\omega}\|_X \to \infty. \tag{3.3}$$

As the norm is weakly lowered semi-continuous [14], we achieve that $\mathcal{J}$ is weakly lowered semi-continuous.

Since $\mathcal{J}(y,\overline{\omega}) \geq 0$ for all $(y,\overline{\omega}) \in X$ holds, there exist

$$\eta = \inf\{\mathcal{J}(y,\overline{\omega}) | (y,\overline{\omega}) \in X \text{ such that } e(y,\overline{\omega}) = 0\},$$

which means that there exists a minimizing sequence $\{(y^n,\overline{\omega^n})\}_{n\in\mathbb{N}}$ in X such that

$$\eta = \lim_{n\to\infty} \mathcal{J}(y^n,\overline{\omega}^n) \quad \text{and} \quad e = e(y^n,\overline{\omega}^n) = 0, \quad \forall n \in \mathbb{N}.$$

From (3.3), there exists an element $(y^*, \overline{\omega}^*) \in X$ such that

$$y^n \rightharpoonup y^*, \quad y \in W(0,T;V), \tag{3.4}$$

$$\overline{\omega^n} \rightharpoonup \overline{\omega}^*, \quad \overline{\omega} \in L^2(0,T;Q_0), \tag{3.5}$$

when $n \to \infty$. From (3.4), we have

$$\lim_{n\to\infty} \int_0^T (y^n(t) - y^*(t), \phi(t))_{V^*,V} dt = 0, \quad \forall \phi \in L^2(0,T;V).$$

Since $W(0,T;V)$ is compactly embedded into $L^2(0,T;L^\infty)$, we have $y^n \to y^*$ strongly in $L^2(0,T;L^\infty)$. Then we also derive that $D^2u^n \to D^2u^*$ strongly in $L^2(0,T;L^\infty)$. On the other hand, by (2.25) and (2.33), we know that $u_n \in L^\infty(0,T;V)$ and $y_{nt} \in L^2(0,T;V^*)$. Hence by [12, Lemma 4] we have $u^n \to u^*$ strongly in $C(0,T;L^\infty)$, $Du^n \to Du^*$ strongly in $C(0,T;H)$, as $n \to \infty$.

As the sequence $\{y^n\}_{n\in\mathbb{N}}$ converges weakly, then $\|y^n\|_{W(0,T;V)}$ is bounded. And $\|y^n\|_{L^2(0,T;L^\infty)}$ is also bounded based on the embedding theorem.

Since $y^n \to y^*$ strongly in $L^2(0,T;L^\infty)$, then we derive that $\|y^*\|_{L^2(0,T;L^\infty)}$, $\|u^*\|_{L^2(0,T;L^\infty)}$ and $\|D^2u^*\|_{L^2(0,T;L^\infty)}$ are bounded.

Notice that

$$\begin{aligned}
&\Big|\int_0^T \int_0^1 (D^2 f(u^n) - D^2 f(u^*))\psi \,dx\,dt\Big| \\
&= \Big|\int_0^T \int_0^1 (f(u^n) - f(u^*))D^2\psi \,dx\,dt\Big| \\
&\le \int_0^T \int_0^1 |(u^n - u^*)\Big(6\gamma_1((u^n)^4 + (u^n)^2(u^*)^2 + (u^n)^3u^* + u^n(u^*)^3 + (u^*)^4) \\
&\quad + 4(h_0-2)((u^2)^2 + (u^*)^2 + u^nu^*) + 2(1-2h_0)\Big)|D^2\psi| \,dx\,dt \\
&\le \big(\|u^n\|^4_{L^8(0,T;L^\infty)} + \|u^*\|^4_{L^8(0,T;L^\infty)}\big)\|u^n - u^*\|_{C(0,T;H)}\|D^2\psi\|_{L^2(0,T;H)} \\
&\to 0, \quad \forall \psi \in L^2(0,T;V).
\end{aligned}$$

As we know

$$\begin{aligned}
&\Big|\int_0^T \int_0^1 \Big(D^2(a(u^n)D^2u^n + \frac{a'(u^n)}{2}|Du^n|^2) \\
&\quad - D^2(a(u^*)D^2u^* + \frac{a'(u^*)}{2}|Du^*|^2)\Big)\psi \,dx\,dt\Big| \\
&= \Big|\int_0^T \int_0^1 D^2(a(u^n)D^2u^n - a(u^*)D^2u^*)\psi \,dx\,dt \\
&\quad + \int_0^T \int_0^1 D^2(\frac{a'(u^n)}{2}|Du^n|^2 - \frac{a'(u^*)}{2}|Du^*|^2)\psi \,dx\,dt\Big| \\
&= |I_1 + I_2|.
\end{aligned}$$

Note that

$$\begin{aligned}
|I_1| &= \Big|\int_0^T \int_0^1 D^2(a(u^n)D^2u^n - a(u^*)D^2u^*)\psi \,dx\,dt\Big| \\
&= \Big|\int_0^T \int_0^1 (a(u^n)D^2u^n - a(u^*)D^2u^*)D^2\psi \,dx\,dt\Big|
\end{aligned}$$

$$
\begin{aligned}
&= \Big| \int_0^T \int_0^1 ((a_2(u^n)^2 + a_0)D^2u^n - (a_2(u^*)^2 + a_0)D^2u^*)D^2\psi \,dx\,dt \Big| \\
&= \Big| \int_0^T \int_0^1 (a_2(u^n)^2 D^2u^n - a_2(u^*)^2 D^2u^*)D^2\psi dx dt \\
&\quad + \int_0^T \int_0^1 (a_0 D^2u^n - a_0 D^2u^*)D^2\psi \,dx\,dt \Big| \\
&= |I_1^1 + I_1^2|.
\end{aligned}
$$

Now, we deal with I_1^1 and I_1^2,

$$
\begin{aligned}
I_1^1 &= \int_0^T \int_0^1 (a_2(u^n)^2 D^2u^n - a_2(u^*)^2 D^2u^*)D^2\psi \,dx\,dt \\
&= \int_0^T \int_0^1 \Big(a_2(u^n)^2 D^2u^n - a_2(u^n)^2 D^2u^* \\
&\quad + a_2(u^n)^2 D^2u^* - a_2(u^*)^2 D^2u^* \Big) D^2\psi \,dx\,dt \\
&= \int_0^T \int_0^1 (a_2(u^n)^2 D^2u^n - a_2(u^n)^2 D^2u^*)D^2\psi \,dx\,dt \\
&\quad + \int_0^T \int_0^1 (a_2(u^n)^2 D^2u^* - a_2(u^*)^2 D^2u^*)D^2\psi \,dx\,dt \\
&\le \int_0^T \int_0^1 a_2(u^n)^2 (D^2u^n - D^2u^*)D^2\psi \,dx\,dt \\
&\quad + \int_0^T \int_0^1 (a_2(u^n)^2 - a_2(u^*)^2) D^2u^* D^2\psi \,dx\,dt \\
&\le \int_0^T a_2\|u^n\|_{L^4}^2 \|D^2u^n - D^2u^*\|_{L^\infty} \|D^2\psi\|_H dt \\
&\quad + \int_0^T a_2\|(u^n)^2 - (u^*)^2\|_H \|D^2u^*\|_{L^\infty} \|D^2\psi\|_H dt \\
&\le a_2\|u^n\|_{C(0,T;L^4)}^2 \|D^2u^n - D^2u^*\|_{L^2(0,T;L^\infty)} \|D^2\psi\|_{L^2(0,T;H)} \\
&\quad + a_2\|u^n - u^*\|_{C(0,T;H)} (\|u^n\|_{C(0,T;L^\infty)} + \|u^*\|_{C(0,T;L^\infty)}) \\
&\quad \cdot \|D^2u^*\|_{L^2(0,T;L^\infty)} \|D^2\psi\|_{L^2(0,T;H)} \to 0, \quad \forall \psi \in L^2(0,T;V),
\end{aligned}
$$

and

$$
\begin{aligned}
I_1^2 &= \int_0^T \int_0^1 (a_0 D^2u^n - a_0 D^2u^*)D^2\psi \,dx\,dt \\
&\le \int_0^T |a_0| \|D^2u^n - D^2u^*\|_H \|D^2\psi\|_H dt \\
&\le |a_0| \|D^2u^n - D^2u^*\|_{L^2(0,T;H)} \|D^2\psi\|_{L^2(0,T;H)} \\
&\to 0, \quad \forall \psi \in L^2(0,T;V).
\end{aligned}
$$

Further,

$$
I_2 = \int_0^T \int_0^1 D^2(a_2u^n|Du^n|^2 - a_2u^*|Du^*|^2)\psi \,dx\,dt
$$

$$
\begin{aligned}
&= \int_0^T \int_0^1 (a_2 u^n |Du^n|^2 - a_2 u^* |Du^*|^2) D^2\psi \, dx \, dt \\
&= \int_0^T \int_0^1 (a_2 u^n |Du^n|^2 - a_2 u^n |Du^*|^2) D^2\psi \, dx \, dt \\
&\quad + \int_0^T \int_0^1 (a_2 u^n |Du^*|^2 - a_2 u^* |Du^*|^2) D^2\psi \, dx \, dt \\
&\le \int_0^T a_2 \|u^n\|_{L^\infty} \|Du^n - Du^*\|_H (\|Du^n\|_{L^\infty} + \|Du^*\|_{L^\infty}) \|D^2\psi\|_H \\
&\quad + \int_0^T a_2 \|u^n - u^*\|_{L^\infty} \||Du^*|^2\|_H \|D^2\psi\|_H dt \\
&\le a_2 \|u^n\|_{C(0,T;L^\infty)} \|Du^n\|_{L^2(0,T;L^\infty)} \|Du^n - Du^*\|_{C(0,T;H)} \|D^2\psi\|_{L^2(0,T;H)} \\
&\quad + a_2 \|u^n\|_{C(0,T;L^\infty)} \|Du^*\|_{L^2(0,T;L^\infty)} \|Du^n - Du^*\|_{C(0,T;H)} \|D^2\psi\|_{L^2(0,T;H)} \\
&\quad + a_2 \|u^n - u^*\|_{L^2(0,T;L^\infty)} \||Du^*|^2\|_{C(0,T;H)} \|D^2\psi\|_{L^2(0,T;H)} \\
&\to 0, \quad \forall \psi \in L^2(0,T;V).
\end{aligned}
$$

From (3.5), we have

$$
\Big| \int_0^T \int_0^1 (B^* \overline{\omega}^n - B^* \overline{\omega}^*) \psi \Big| \to 0, \quad \forall \psi \in L^2(0,T;V).
$$

In view of the above discussion, we can conclude that

$$
e_1(y^*, \overline{\omega}^*) = 0, \quad \forall n \in \mathbb{N}.
$$

Since $y^* \in W(0,T;V)$, we have $y^*(0) \in H$. From $y^n \rightharpoonup y^*$ in $W(0,T;V)$, we can infer that $y^n(0) \rightharpoonup y^*(0)$. Thus we obtain

$$
(y^n(0) - y^*(0), \psi) \to 0, \quad \forall \psi \in H,
$$

which means that $e_2(y^*, \overline{\omega}^*) = 0$, for all $n \in \mathbb{N}$. Hence, we can derive that $e(y^*, \overline{\omega}^*) = 0$, for all $n \in \mathbb{N}$.

In conclusion, there exists an optimal solution $(y^*, \overline{\omega}^*)$ to the problem. And we can infer that there exists an optimal solution $(y^*, \overline{\omega}^*)$ to the viscous generalized Cahn-Hilliard equation due to $u = (1 - \partial_x^2)^{-1} y$. □

Acknowledgments. The authors would like to express their deep thanks to the anonymous referees for their valuable suggestions for the revision of this article.

References

[1] A. Armaou, P. D. christofides; *Feedback control of kuramto-sivashing equation*, Physica D., 137 (2000), 49-61.

[2] G. Gompper, M. Kraus; *Ginzburg-Landau theory of ternary amphiphilic systems. I. Gaussian interface uctuations*, Phys. Rev. E., 47 (1993), 4289-4300.

[3] G. Gompper, M. Kraus; *Ginzburg-Landau theory of ternary amphiphilic systems. II. Monte Carlo simulations*, Phys. Rev. E., 47 (1993), 4301-4312.

[4] G. Gompper, J. Goos; *Fluctuating interfaces in microemulsion and sponge phases*, Phys. Rev. E., 50 (1994), 1325-1335.

[5] M. Hinze, S. Volkwein; *Analysis of instantaneous control for the Burgers equation*, Nonlinear Anal. TMA, 50 (1) (2002), 1-26.

[6] K. Kunisch, S. Volkwein; *Control of the Burgers equation by a reduced-order approach using proper orthogonal decomposition*, J. Optim. Theory Appl., 102(2)(1999), 345-371.

[7] C. Liu; *Regularity of solutions for a sixth order nonlinear parabolic equation in two space dimensions*, Annales Polonici Mathematici, In press.
[8] I. Pawłow, W. Zajączkowski; *A sixth order Cahn-Hilliard type equation arising in oil-water-surfactant mixtures*, Commun. Pure Appl. Anal., 10(2011), 1823-1847.
[9] G. Schimperna, I. Pawłow; *On a class of Cahn-Hilliard models with nonlinear diffusion*, to appear, 2011.
[10] C. Shen, L. Tian, A. Gao; *Optimal control of the viscous Dullin-Gottwalld-Holm equation*, Nonlinear Anal. RWA, 11 (2010), 480-491.
[11] C. Shen, A. Gao, L. Tian; *Optimal control of the viscous generalized Camassa-Holm equation*, Nonlinear Anal. RWA, 11 (2010), 1835-1846.
[12] J. Simon; *Nonhomogeneous viscous incompressible fluids: existence of velocity, density, and pressure*, SIAM J. Math. Anal., 21(5) (1990), 1093-1117.
[13] L. Tian, C. shen, D. Ding; *Optimal control of the viscous Camassa-Holm equation*, Nonlinear Anal. RWA, 10 (1)(2009), 519-530.
[14] A. Wouk; *A Course of Applied Functional Analysis*, Wiley-Interscience, New York, 1979.
[15] X. Zhao, C. Liu; *Optimal control problem for viscous Cahn-Hilliard equation*, Nolinear Analysis, 74 (17) (2011), 6348-6357.

Changchun Liu
Department of Mathematics, Jilin University, Changchun 130012, China
E-mail address: liucc@jlu.edu.cn

Zhao Wang
Department of Mathematics, Jilin University, Changchun 130012, China
E-mail address: wangzhao2717@163.com