Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 128, pp. 1–15.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

ENERGY DECAY IN THERMOELASTICITY TYPE III WITH VISCOELASTIC DAMPING AND DELAY TERM

TIJANI A. APALARA, SALIM A. MESSAOUDI, MUHAMMAD I. MUSTAFA

Abstract. In this article, we consider a thermoelastic system of type III with a viscoelastic damping and internal delay. We use the multiplier method to prove, under suitable assumptions, general energy decay results from which the exponential and polynomial types of decay are only special cases.

1. Introduction

In this article, we consider the problem

$$
\begin{gathered}
u_{tt}(x,t) - \mu\Delta u(x,t) - (\mu+\lambda)\nabla(\operatorname{div} u(x,t)) + \beta\nabla\theta(x,t) \\
+ \int_0^t g(s)\Delta u(x,t-s)ds + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0, \quad x\in\Omega,\ t>0 \\
\theta_{tt}(x,t) - \kappa\Delta\theta(x,t) - \delta\Delta\theta_t(x,t) + \beta \operatorname{div} u_{tt}(x,t), = 0, \quad x\in\Omega,\ t>0 \\
u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), \quad x\in\Omega, \\
\theta(x,0)=\theta_0(x), \quad \theta_t(x,0)=\theta_1(x), \quad x\in\Omega, \\
u_t(x,-t)=f_0(x,t), \quad x\in\Omega,\ t\in(0,\tau) \\
u(x,t)=\theta(x,t)=0, \quad x\in\partial\Omega,\ t\geq 0
\end{gathered}
\tag{1.1}
$$

where Ω is a bounded domain of $\mathbb{R}^n (n \geq 2)$ with a boundary $\partial\Omega$ of class C^2, $u = u(x,t) \in \mathbb{R}^n$ is the displacement vector, $\theta(x,t)$ is the difference temperature, the relaxation function g is positive and decreasing, the coefficients $\mu, \lambda, \beta, \mu_1, \kappa, \delta$ are positive constants, μ_2 is a real number, and $\tau > 0$ represents the time delay. This is a (type III) thermoelastic system with the presence of a viscoelastic damping and constant internal delay supplemented by initial data $u_0, u_1, \theta_0, \theta_1$ and a history function f_0.

Time delays so often arise in many physical, chemical, biological, thermal and economical phenomena. In recent years, the control of PDEs with time delay effects has become an active area of research, see for example [1, 24] and the references therein. The presence of delay may be a source of instability. See, for example [3, 16, 25], where it was proved that an arbitrarily small delay may destabilize a system which is uniformly asymptotically stable in the absence of delay unless additional conditions or control terms have been used.

2000 *Mathematics Subject Classification.* 35B37, 35L55, 74D05, 93D15, 93D20.
Key words and phrases. Damping; delay; relaxation function; thermoelasticity; viscoelasticity.

Submitted June 11, 2012. Published August 15, 2012.

Consider the system

$$\begin{gathered} u_{tt}(x,t) - \Delta u(x,t) = 0, \quad x \in \Omega,\ t > 0 \\ u(x,t) = 0, \quad x \in \Gamma_0,\ t > 0 \\ \frac{\partial u}{\partial \nu}(x,t) = -\mu_1 u_t(x,t) - \mu_2 u_t(x,t-\tau), \quad x \in \Gamma_1,\ t > 0. \end{gathered} \tag{1.2}$$

It is well known that in the absence of delay ($\mu_2 = 0, \mu_1 > 0$), system (1.2) is exponentially stable, see [6]–[8], [27]. Whereas, in the presence of delay ($\mu_2 > 0$), Nicaise and Pignotti [16] proved, under the assumption $\mu_2 < \mu_1$, that the energy is exponentially stable. However, for the opposite case ($\mu_2 \geq \mu_1$), they were able to construct a sequence of delays for which the corresponding solution is unstable. The same results were obtained for the case when both the damping and the delay act internally in the domain, see also [2] for the treatment of this problem in more general abstract form. Nicaise and Pignotti [17] treated the situation when the constant delay in system (1.2) is replaced with a distributed delay of the form

$$\int_{\tau_1}^{\tau_2} \mu_2(s) u_t(x,t-s) ds$$

and established an exponential stability result similar to the one in [16] under the condition that

$$\int_{\tau_1}^{\tau_2} \mu_2(s) ds < \mu_1.$$

Kirane and Said-Houari [5] considered a viscoelastic wave equation of the form

$$u_{tt}(x,t) - \Delta u(x,t) + \int_0^t g(t-s)\Delta u(x,s) ds + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0,$$

for $x \in \Omega,\ t > 0$, together with initial and Dirichlet boundary conditions. They established general energy decay results under the condition that $\mu_2 \leq \mu_1$. In fact, the presence of a viscoelastic damping together with a frictional damping allowed $\mu_2 = \mu_1$.

Recently, Pignotti [21] considered the equation

$$u_{tt}(x,t) - \Delta u(x,t) + a\chi_\omega u_t(x,t) + k u_t(x,t-\tau) = 0, \quad \text{in } \Omega \times (0,\infty)$$

for $a, \tau > 0$ and k a real number. She established, under some geometry condition on the domain, a well posedness of the problem and an exponential decay result for $|k| < a$.

In [15], Mustafa studied a thermoelastic system with boundary time-varying delay in one dimensional space and showed that the damping effect through heat conduction is still strong enough to uniformly stabilize the system even in the presence of boundary time-varying delay. For more results concerning time delay in one dimensional as well as multi-dimensional space, we refer the reader to [4], [18]–[20].

We also recall some results regarding thermoelastic systems of type III. In one space dimension, Quintanilla and Racke [23] considered the equation

$$\begin{gathered} u_{tt} - \alpha u_{xx} + \beta \theta_x = 0, \quad \text{in } [0,\infty) \times (0,L) \\ \theta_{tt} - \delta \theta_{xx} + \gamma u_{ttx} - \kappa \theta_{txx} = 0, \quad \text{in } [0,\infty) \times (0,L) \end{gathered}$$

and used the spectral analysis method and the energy method to obtain the exponential stability for various boundary conditions; (Dirichlet-Dirichlet or Dirichlet-Neuman). Furthermore, they proved an energy decay result for the radially symmetric situations in the multi-dimensional case ($n = 2, 3$). Zhang and Zuazua [26] analyzed the long time behavior of the solution of the n-dimensional system (1.1), when $g = \mu_1 = \mu_2 = 0$, and showed that (i) for most domains the energy of the system does not decay uniformly, (ii) under suitable conditions on the domain that may be described in terms of Geometric Optics, the energy of the system decays exponentially, and (iii) for most domains in two space dimensions, the energy of smooth solutions decays polynomially. Messaoudi and Soufyane [12] considered the system

$$\begin{gathered} u_{tt} - \mu\Delta u - (\mu + \lambda)\nabla(\operatorname{div} u) + \beta\nabla\theta = 0, \quad \text{in } \Omega \times \mathbb{R}^+ \\ \theta_{tt} - \kappa\Delta\theta - \delta\Delta\theta_t + \beta \operatorname{div} u_{tt} = 0, \quad \text{in } \Omega \times \mathbb{R}^+ \end{gathered}$$

subject to a boundary feedback of viscoelastic type that acts on a part of the boundary and established exponential and polynomial stability results. This result was later generalized by Messaoudi and Al-Shehri [10] by taking a wider class of relaxation functions. They proved a more general decay result, from which the exponential and polynomial decay estimates are only special cases.

Recently, Qin and Ma [22] considered the system

$$\begin{gathered} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + \nabla\theta = 0, \quad x \in \Omega,\ t > 0 \\ \theta_{tt} - \Delta\theta_t - \Delta\theta + \operatorname{div} u_{tt} = 0, \quad x \in \Omega,\ t > 0 \\ \theta = 0, \quad x \in \partial\Omega,\ t > 0 \\ u = 0, \quad x \in \Gamma_0,\ t > 0 \\ \frac{\partial u}{\partial \nu} - \int_0^t g(t-s)\Delta u(s)ds + H(u_t) = 0, \quad x \in \Gamma_1,\ t > 0 \end{gathered}$$

and established a general decay result depending on both g and H. This result extends the decay result obtained by Messaoudi and Mustafa [11] obtained earlier for wave equations. For more results on Thermoelasticity type III, we refer the reader to [9, 13, 14, 23] and references therein.

In this article, we investigate system (1.1) under suitable assumptions on the weight of the delay term and prove general decay result from which the exponential and polynomial types of decay are only special cases. This work extends the result obtained by Kirane and Said-Houari [5] for a viscoelastic wave equation to the thermoviscoelastic system with a delay. We should mention here that, to the best of our knowledge, there is no result concerning systems of thermoelasticty of type III with the presence of delays. The rest of our paper is organized as follows. In section 2, we introduce some transformations and assumptions needed in our work. Some technical lemmas and the statement with proof of our main results will be given in section 3 and section 4 respectively. Finally, we give some examples to illustrate our results.

2. Assumptions and Transformations

In this section, we present some materials needed in the proof of our results. We use the standard Lebesgue space $L^2(\Omega)$ and the Sobolev space $H_0^1(\Omega)$ with their usual scalar products and norms. Throughout this paper, c is used to denote a generic positive constant.

For the relaxation function g, we assume the following:

(A1) $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1 function satisfying
$$g(0) > 0, \quad \mu - \int_0^\infty g(s)ds = l > 0.$$

(A2) There exists a positive non-increasing differentiable function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying
$$g'(t) \le -\eta(t)g(t), \quad t \ge 0.$$

Remark 2.1. There are many functions that satisfy (A1) and (A2). Below are three examples of such functions with the assumptions that $a, b > 0$ and $a < \mu b$.

(1) If $g(t) = ae^{-bt}$, then $g'(t) = -\eta(t)g(t)$, where $\eta(t) = b$.
(2) If $g(t) = \frac{a}{(1+t)^{b+1}}$, then $g'(t) = -\eta(t)g(t)$, where $\eta(t) = \frac{b+1}{1+t}$.
(3) If $g(t) = \frac{a}{(e+t)[ln(e+t)]^{b+1}}$, then $g'(t) = -\eta(t)g(t)$, where
$$\eta(t) = \frac{1}{e+t} + \frac{b+1}{(e+t)\ln(e+t)}.$$

Now, as in [26], we introduce the new variable
$$v(x,t) = \int_0^t \theta(x,s)ds + \chi(x), \tag{2.1}$$
where $\chi(x)$ is the solution of
$$\begin{gathered} -\kappa\Delta\chi = \delta\Delta\theta_0 - \theta_1 - \beta \operatorname{div} u_1, \quad \text{in } \Omega, \\ \chi = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{2.2}$$
Then, integrating the second equation in (1.1) with respect to t and using (2.1) and (2.2), we have
$$v_{tt} - \kappa\Delta v - \delta\Delta v_t + \beta \operatorname{div} u_t = 0.$$
By introducing as in [16], another new dependent variable
$$z(x,\rho,t) = u_t(x, t - \tau\rho), \quad x \in \Omega,\ \rho \in (0,1),\ t > 0.$$
problem (1.1) takes the form
$$\begin{gathered} u_{tt}(x,t) - \mu\Delta u(x,t) - (\mu+\lambda)\nabla(\operatorname{div} u(x,t)) + \beta\nabla v_t(x,t) \\ + \int_0^t g(t-s)\Delta u(x,s)ds + \mu_1 u_t(x,t) + \mu_2 z(x,1,t) = 0, \quad x \in \Omega,\ t > 0 \\ v_{tt}(x,t) - \kappa\Delta v(x,t) - \delta\Delta v_t(x,t) + \beta \operatorname{div} u_t(x,t) = 0, \quad x \in \Omega,\ t > 0 \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \quad x \in \Omega, \rho \in (0,1),\ t > 0 \\ z(x,0,t) = u_t(x,t), \quad x \in \Omega,\ t > 0 \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \quad x \in \Omega, \\ z(x,\rho,0) = f_0(x,\tau\rho), \quad x \in \Omega,\ \rho \in (0,1) \\ u(x,t) = v(x,t) = 0, \quad x \in \partial\Omega,\ t \ge 0 \end{gathered} \tag{2.3}$$

Thus, we will consider problem (2.3) instead of (1.1). In what follows, we consider (u, v, z) to be a solution of system (2.3) with the regularity needed to justify the calculations in this paper. By repeating the arguments of [5], one can easily prove the existence and uniqueness of strong and weak solutions.

Next, we assume that $|\mu_2| \le \mu_1$ and that ξ is a positive constant satisfying

$$\begin{gathered} \tau|\mu_2| < \xi < \tau(2\mu_1 - |\mu_2|), \quad \text{if } |\mu_2| < \mu_1, \\ \xi = \tau\mu_1, \quad \text{if } \mu_1 = |\mu_2|, \end{gathered} \tag{2.4}$$

The energy associated with problem (2.3) is

$$\begin{aligned} E(t) = \frac{1}{2}\int_\Omega |u_t|^2 dx + \frac{1}{2}\int_\Omega v_t^2 dx + \frac{1}{2}\Big(\mu - \int_0^t g(s)ds\Big)\int_\Omega |\nabla u|^2 dx + \frac{\kappa}{2}\int_\Omega |\nabla v|^2 dx \\ + \frac{(\mu+\lambda)}{2}\int_\Omega |\operatorname{div} u|^2 dx + \frac{1}{2}(g\circ \nabla u)(t) + \frac{\xi}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\, ds\, dx, \end{aligned} \tag{2.5}$$

where

$$(g\circ\nabla u)(t) = \int_\Omega\int_0^t g(t-s)\big|\ \nabla u(x,t) - \nabla u(x,s)\ \big|^2 ds\, dx.$$

3. Technical Lemmas

In this section we establish several lemmas needed for the proof of our main result.

Lemma 3.1. *Let* (u,v,z) *be the solution of* (2.3). *Then the energy functional, defined by* (2.5), *satisfies*

$$\begin{aligned} E'(t) \le -m_0\Big(\int_\Omega |u_t|^2 dx + \int_\Omega z^2(x,1,t)dx\Big) + \frac{1}{2}(g'\circ\nabla u)(t) \\ - \frac{1}{2}g(t)\int_\Omega |\nabla u|^2 dx - \delta\int_\Omega |\nabla v_t|^2 dx \le 0, \quad \forall t \ge 0, \end{aligned} \tag{3.1}$$

for some constant m_0, where $m_0 > 0$ if $|\mu_2| < \mu_1$ and $m_0 = 0$ if $\mu_1 = |\mu_2|$.

Proof. A multiplication of the first and the second equation in (2.3) by u_t and v_t respectively, and integration over Ω, using integration by parts and the boundary conditions, yield

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big\{\int_\Omega |u_t|^2 dx + \int_\Omega v_t^2 dx + \Big(\mu - \int_0^t g(s)ds\Big)\int_\Omega |\nabla u|^2 dx \\ &+ \kappa\int_\Omega |\nabla v|^2 dx + (\mu+\lambda)\int_\Omega |\operatorname{div} u|^2 dx + (g\circ\nabla u)(t)\Big\} \\ &= \frac{1}{2}(g'\circ\nabla u)(t) - \delta\int_\Omega |\nabla v_t|^2 dx - \frac{1}{2}g(t)\int_\Omega |\nabla u|^2 dx \\ &- \mu_1\int_\Omega |u_t|^2 dx - \mu_2\int_\Omega u_t\cdot z(x,1,t)dx. \end{aligned} \tag{3.2}$$

Now, multiplying the third equation in (2.3) by ξz and integrating over $\Omega\times(0,1)$, we obtain

$$\frac{\xi}{2}\frac{d}{dt}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx = -\frac{\xi}{2\tau}\int_\Omega z^2(x,1,t)dx + \frac{\xi}{2\tau}\int_\Omega |u_t|^2 dx. \tag{3.3}$$

A combination of (3.2) and (3.3), leads to

$$E'(t) = \frac{1}{2}(g'\circ\nabla u)(t) - \frac{1}{2}g(t)\int_\Omega |\nabla u|^2 dx - \delta\int_\Omega |\nabla v_t|^2 dx - \big(\mu_1 - \frac{\xi}{2\tau}\big)\int_\Omega |u_t|^2 dx$$

$$- \mu_2 \int_\Omega u_t \cdot z(x,1,t)dx - \frac{\xi}{2\tau}\int_\Omega z^2(x,1,t)dx.$$

Then by Young's inequality, we have

$$\begin{aligned} E'(t) &\le \frac{1}{2}(g'\circ\nabla u)(t) - \frac{1}{2}g(t)\int_\Omega |\nabla u|^2dx - \delta\int_\Omega |\nabla v_t|^2dx \\ &\quad - \big(\mu_1 - \frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\big)\int_\Omega |u_t|^2dx - \big(\frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\big)\int_\Omega z^2(x,1,t)dx. \end{aligned}$$

Consequently, using (2.4), estimate (3.1) follows. □

Lemma 3.2. *Suppose that* (A1) *and* (A2) *hold, and let* (u,v,z) *be the solution of* (2.3). *Then the functional*

$$F_1(t) := \int_\Omega u_t \cdot u dx$$

satisfies the following estimate, for some positive constant m_1,

$$\begin{aligned} F_1'(t) &\le c\Big(\int_\Omega |u_t|^2dx + \int_\Omega v_t^2dx + \int_\Omega z^2(x,1,t)dx + (g\circ\nabla u)(t)\Big) \\ &\quad - m_1\Big(\int_\Omega |\nabla u|^2dx \int_\Omega |\operatorname{div} u|^2dx\Big). \end{aligned} \tag{3.4}$$

Proof. Direct computations using the first equation in (2.3), yield

$$\begin{aligned} F_1'(t) &= \int_\Omega |u_t|^2dx - \mu\int_\Omega |\nabla u|^2dx - (\mu+\lambda)\int_\Omega |\operatorname{div} u|^2dx + \beta\int_\Omega v_t\cdot \operatorname{div} u dx \\ &\quad + \int_\Omega \nabla u\cdot\int_0^t g(t-s)\nabla u(s)\,ds\,dx - \mu_1\int_\Omega u\cdot u_t dx - \mu_2\int_\Omega z(x,1,t)\cdot u\,dx. \end{aligned}$$

Using Young's and Poincaré's inequalities,for $\delta_1 > 0$, we have

$$\begin{aligned} &F_1'(t) \le -\Big(\frac{\mu}{2} - \delta_1(\mu_1 + |\mu_2|)\Big)\int_\Omega |\nabla u|^2dx + \frac{1}{2\mu}\int_\Omega\Big(\int_0^t g(t-s)\nabla u(s)ds\Big)^2dx \\ &+ \big(1 + \frac{c\mu_1}{4\delta_1}\big)\int_\Omega |u_t|^2dx - (\mu+\lambda-\delta_1)\int_\Omega |\operatorname{div} u|^2dx + \frac{1}{4\delta_1}\int_\Omega v_t^2dx \\ &+ \frac{c|\mu_2|}{4\delta_1}\int_\Omega z^2(x,1,t)dx. \end{aligned} \tag{3.5}$$

The second term in the right-hand side of (3.5) is estimated as follows:

$$\begin{aligned} &\int_\Omega\Big(\int_0^t g(t-s)|\nabla u(s)|ds\Big)^2dx \\ &\le \int_\Omega\Big(\int_0^t g(t-s)(|\nabla u(s) - \nabla u(t)| + |\nabla u(t)|)ds\Big)^2dx \\ &= \int_\Omega\Big(\int_0^t g(t-s)|\nabla u(s) - \nabla u(t)|ds\Big)^2dx + \int_\Omega\Big(\int_0^t g(t-s)|\nabla u(t)|ds\Big)^2dx \\ &\quad + 2\int_\Omega\Big(\int_0^t g(t-s)|\nabla u(s) - \nabla u(t)|ds\Big)\Big(\int_0^t g(t-s)|\nabla u(t)|ds\Big)dx. \end{aligned}$$

A simple calculation, using Cauchy-Schwarz and Young's inequalities, for $\eta > 0$, gives

$$
\begin{aligned}
&\int_\Omega \Big(\int_0^t g(t-s)|\nabla u(s)|ds\Big)^2 dx \\
&\leq (\mu - l)^2(1+\eta)\int_\Omega |\nabla u|^2 dx + (\mu - l)\big(1+\frac{1}{\eta}\big)(g\circ \nabla u)(t).
\end{aligned}
\tag{3.6}
$$

By inserting (3.6) into (3.5) and choosing $\eta = \frac{l}{\mu - l}$, we arrive at

$$
\begin{aligned}
F_1'(t) \leq &\big(1+\frac{c\mu_1}{4\delta_1}\big)\int_\Omega |u_t|^2 dx - \big(\frac{l}{2} - \delta_1(\mu_1 + |\mu_2|)\big)\int_\Omega |\nabla u|^2 dx \\
&- (\mu+\lambda-\delta_1)\int_\Omega |\operatorname{div} u|^2 dx + \frac{1}{4\delta_1}\int_\Omega v_t^2 dx + \frac{(\mu - l)}{2l}(g\circ\nabla u)(t) \\
&+ \frac{c|\mu_2|}{4\delta_1}\int_\Omega z^2(x,1,t)dx.
\end{aligned}
$$

By taking δ_1 small enough, (3.4) follows. □

Lemma 3.3. *let (u, v, z) be the solution of (2.3). Then the functional*

$$
F_2(t) := \int_\Omega v_t v dx + \beta \int_\Omega v \operatorname{div} u dx + \frac{\delta}{2}\int_\Omega |\nabla u|^2 dx
$$

satisfies the following estimate, for any positive constant δ_2,

$$
F_2'(t) \leq \big(1+\frac{\beta}{4\delta_2}\big)\int_\Omega v_t^2 dx + \beta\delta_2 \int_\Omega |\operatorname{div} u|^2 dx - \kappa \int_\Omega |\nabla v|^2 dx. \tag{3.7}
$$

Proof. Taking the derivative of $F_2(t)$ and using the second equation in (2.3), it follows that

$$
F_2'(t) = \int_\Omega v_t^2 dx + \kappa\int_\Omega v\Delta v dx + \delta \int_\Omega v \Delta v_t dx + \beta\int_\Omega v_t \operatorname{div} u dx + \delta \int_\Omega \nabla v\cdot\nabla v_t dx.
$$

Use of Green's formula and the boundary conditions lead to

$$
F_2'(t) = \int_\Omega v_t^2 dx - \kappa \int_\Omega |\nabla v|^2 dx + \beta \int_\Omega v_t \operatorname{div} u dx.
$$

By exploiting Young's inequality for $\delta_2 > 0$, estimate (3.7) is established. □

Lemma 3.4. *let (u, v, z) be the solution of (2.3). Then the functional*

$$
F_3(t) := \tau \int_\Omega \int_0^1 e^{-\tau\rho} z^2(x,\rho,t) d\rho dx,
$$

satisfies the following estimate, for some positive constant m_2,

$$
F_3' \leq -m_2\Big(\int_\Omega z^2(x,1,t)dx + \tau\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx\Big) + \int_\Omega |u_t|^2 dx. \tag{3.8}
$$

Proof. By differentiating $F_3(t)$ and using the third equation in (2.3), we obtain

$$
\begin{aligned}
F_3'(t) &= -2\int_\Omega\int_0^1 e^{-\tau\rho} z(x,\rho,t) z_\rho(x,\rho,t) d\rho dx \\
&= -\frac{d}{d\rho}\int_\Omega\int_0^1 e^{-\tau\rho} z^2(x,\rho,t)d\rho dx - \tau\int_\Omega\int_0^1 e^{-\tau\rho} z^2(x,\rho,t) d\rho dx \\
&= -\int_\Omega [e^{-\tau} z^2(x,1,t) - z^2(x,0,t)]dx - \tau \int_\Omega\int_0^1 e^{-\tau\rho} z^2(x,\rho,t)d\rho dx
\end{aligned}
$$

$$\le -m_2\Big(\int_\Omega z^2(x,1,t)dx + \tau\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx\Big) + \int_\Omega |u_t|^2dx.$$

which gives (3.8). □

Lemma 3.5. *Suppose that* (A1) *and* (A2) *hold and let* (u, v, z) *be the solution of* (2.3). *Then for* $\mu_1 = |\mu_2|$ *and for any* $t_0 > 0$, *the functional*

$$F_4(t) := -\int_\Omega u_t\cdot\int_0^t g(t-s)(u(t)-u(s))\,ds\,dx,$$

satisfies the following estimate, for some positive constant m_3, *and for any positive* δ_3, δ_4, δ_5,

$$\begin{aligned} F_4'(t) \le& -m_3\int_\Omega |u_t|^2dx + \frac{\beta}{2}\int_\Omega |\nabla v_t|^2dx + \delta_3 c\int_\Omega |\nabla u|^2dx \\ &+ \delta_4(\mu+\lambda)\int_\Omega |\operatorname{div} u|^2dx + C_\delta(g\circ\nabla u)(t) + \delta_5\mu_1\int_\Omega z^2(x,1,t)dx \\ &- c(g'\circ\nabla u)(t), \quad \forall t\ge t_0>0. \end{aligned} \tag{3.9}$$

Proof. Differentiation of $F_4(t)$, using (2.3) and integrating by parts together with the boundary conditions, yield

$$\begin{aligned} F_4'(t) =& \mu\int_\Omega \nabla u\cdot\Big(\int_0^t g(t-s)(\nabla u(s)-\nabla u(t))ds\Big)dx \\ &+ (\mu+\lambda)\int_\Omega (\operatorname{div} u)\cdot\Big(\int_0^t g(t-s)(\operatorname{div} u(s)-\operatorname{div} u(t))ds\Big)dx \\ &- \beta\int_\Omega \nabla v_t\cdot\Big(\int_0^t g(t-s)(u(s)-u(t))ds\Big)dx \\ &- \int_\Omega\Big(\int_0^t g(t-s)\nabla u(s)ds\Big)\cdot\Big(\int_0^t g(t-s)(\nabla u(s)-\nabla u(t))ds\Big)dx \\ &+ \mu_1\int_\Omega u_t\cdot\int_0^t g(t-s)(u(s)-u(t))dsdx \\ &+ \mu_2\int_\Omega z(x,1,t)\cdot\int_0^t g(t-s)(u(s)-u(t))dsdx - \Big(\int_0^t g(s)ds\Big)\int_\Omega |u_t|^2dx \\ &- \int_\Omega u_t\cdot\int_0^t g'(t-s)(u(s)-u(t))\,ds\,dx. \end{aligned} \tag{3.10}$$

Now, we estimate the terms in the right hand side of (3.10) using Young's, Cauchy-Schwarz, and Poincaré's inequalities. So, for $\delta_3, \delta_4, \delta_5, \delta_6 > 0$, we obtain

$$\begin{aligned} I_1 =& \int_\Omega \nabla u\cdot\Big(\int_0^t g(t-s)(\nabla u(s)-\nabla u(t))ds\Big)dx \\ \le& \delta_3\int_\Omega |\nabla u|^2dx + \frac{1}{4\delta_3}\int_\Omega\Big(\int_0^t g(t-s)|\nabla u(s)-\nabla u(t)|ds\Big)^2dx \\ \le& \delta_3\int_\Omega |\nabla u|^2dx + \frac{1}{4\delta_3}\int_\Omega\Big(\int_0^t g(s)ds\Big)\Big(\int_0^t g(t-s)|\nabla u(s)-\nabla u(t)|^2ds\Big)dx \\ \le& \delta_3\int_\Omega |\nabla u|^2dx + \frac{\mu-l}{4\delta_3}(g\circ\nabla u)(t). \end{aligned} \tag{3.11}$$

$$
\begin{aligned}
I_2 &= \int_\Omega (\operatorname{div} u) \cdot \Big(\int_0^t g(t-s)(\operatorname{div} u(s) - \operatorname{div} u(t)) ds \Big) dx \\
&\le \delta_4 \int_\Omega |\operatorname{div} u|^2 dx + \frac{1}{4\delta_4} \int_\Omega \Big(\int_0^t g(t-s) \left(\operatorname{div} u(s) - \operatorname{div} u(t)\right) ds \Big)^2 dx \\
&\le \delta_4 \int_\Omega |\operatorname{div} u|^2 dx + \frac{\mu - l}{4\delta_4} \int_\Omega \int_0^t g(t-s) |\operatorname{div} u(s) - \operatorname{div} u(t)|^2 \, ds \, dx \\
&\le \delta_4 \int_\Omega |\operatorname{div} u|^2 dx + \frac{\mu - l}{2\delta_4} (g \circ \nabla u)(t).
\end{aligned} \tag{3.12}
$$

$$
\begin{aligned}
I_3 &= -\int_\Omega \nabla v_t \cdot \Big(\int_0^t g(t-s)(u(s) - u(t)) ds \Big) dx \\
&\le \frac{1}{2} \int_\Omega |\nabla v_t|^2 dx + \frac{c(\mu - l)}{2} (g \circ \nabla u)(t).
\end{aligned} \tag{3.13}
$$

$$
\begin{aligned}
I_4 &= -\int_\Omega \Big(\int_0^t g(t-s) \nabla u(s) ds \Big) \cdot \Big(\int_0^t g(t-s)(\nabla u(s) - \nabla u(t)) ds \Big) dx \\
&\le \delta_3 \int_\Omega \Big(\int_0^t g(t-s) |\nabla u(s)| ds \Big)^2 dx \\
&\quad + \frac{1}{4\delta_3} \int_\Omega \Big(\int_0^t g(t-s) |\nabla u(s) - \nabla u(t)| ds \Big)^2 dx \\
&\le 2(\mu - l)^2 \delta_3 \int_\Omega |\nabla u|^2 dx + (\mu - l) \left(2\delta_3 + \frac{1}{4\delta} \right) (g \circ \nabla u)(t).
\end{aligned} \tag{3.14}
$$

$$
\begin{aligned}
I_5 &= \int_\Omega u_t \cdot \int_0^t g(t-s)(u(s) - u(t)) ds dx \\
&\le \delta_6 \int_\Omega |u_t|^2 dx + \frac{c(\mu - l)}{4\delta_6} (g \circ \nabla u)(t).
\end{aligned} \tag{3.15}
$$

$$
\begin{aligned}
I_6 &= \int_\Omega z(x,1,t) \cdot \int_0^t g(t-s)(u(s) - u(t)) ds dx \\
&\le \delta_5 \int_\Omega z^2(x,1,t) dx + \frac{c(\mu - l)}{4\delta_5} (g \circ \nabla u)(t).
\end{aligned} \tag{3.16}
$$

$$
\begin{aligned}
I_7 &= -\int_\Omega u_t \cdot \int_0^t g'(t-s)(u(s) - u(t)) \, ds \, dx \\
&\le \delta_6 \int_\Omega |u_t|^2 dx + \frac{1}{4\delta_6} \int_\Omega \Big(\int_0^t g'(t-s)(u(s) - u(t)) ds \Big)^2 dx \\
&\le \delta_6 \int_\Omega |u_t|^2 dx + \frac{1}{4\delta_6} \int_\Omega \Big(\int_0^t -g'(s) ds \Big) \Big(\int_0^t -g'(t-s) |(u(s) - u(t)|^2 ds \Big) dx \\
&\le \delta_6 \int_\Omega |u_t|^2 dx - \frac{c g(0)}{4\delta_6} (g' \circ \nabla u)(t).
\end{aligned} \tag{3.17}
$$

Since the function g is positive, continuous and $g(0) > 0$, then for any $t \geq t_0 > 0$, we have

$$\int_0^t g(s)ds \geq \int_0^{t_0} g(s)ds = g_0. \tag{3.18}$$

A combination of (3.10)–(3.18), bearing in mind that $\mu_1 = |\mu_2|$ leads to

$$\begin{aligned} F_4'(t) \leq & -[g_0 - \delta_6(1+\mu_1)] \int_\Omega |u_t|^2 dx + \delta_5 \mu_1 \int_\Omega z^2(x,1,t)dx - \frac{cg(0)}{4\delta_6}(g' o \nabla u)(t) \\ & + \frac{\beta}{2} \int_\Omega |\nabla v_t|^2 dx + \delta_3[\mu + 2(\mu - l)^2] \int_\Omega |\nabla u|^2 dx + \delta_4(\mu+\lambda) \int_\Omega |\operatorname{div} u|^2 dx \\ & + (\mu - l)\Big[\frac{\mu+1}{4\delta_3} + \frac{\mu+\lambda}{2\delta_4} + 2\delta_3 + \frac{c\mu_1}{4}\Big(\frac{1}{\delta_5} + \frac{1}{\delta_6}\Big) + \frac{c\beta}{2}\Big](g \circ \nabla u)(t), \end{aligned}$$

for all $t \geq t_0$. Next, we choose δ_6 small enough to obtain (3.9). □

4. Asymptotic Stability

This section is divided into two parts. In the first subsection, we discuss the case where $|\mu_2| < \mu_1$ and in the second, we discuss the case where $\mu_1 = |\mu_2|$.

4.1. General stability for $|\mu_2| < \mu_1$.

For $\varepsilon > 0$, to be chosen appropriately later, we let

$$\mathcal{L}(t) := E(t) + \varepsilon F_1(t) + \varepsilon F_2(t) + \varepsilon F_3(t). \tag{4.1}$$

Lemma 4.1. *There exist two positive constants α_1 and α_2 such that*

$$\alpha_1 E(t) \leq L(t) \leq \alpha_2 E(t), \quad \forall t \geq 0, \tag{4.2}$$

for ε small enough

Proof. Let

$$\mathcal{G}(t) = \varepsilon F_1(t) + \varepsilon F_2(t) + \varepsilon F_3(t).$$

By using Young's and Poincaré's inequalities, we obtain

$$\begin{aligned} |\mathcal{G}(t)| \leq & \frac{\varepsilon}{2} \int_\Omega \Big(|u_t|^2 + v_t^2 + c|\nabla u|^2 + (c(1+\beta)+\delta)\, |\nabla v|^2 + |\operatorname{div} u|^2 \Big) dx \\ & + \varepsilon\tau \int_\Omega \int_0^1 z^2(x,\rho,t) d\rho dx \\ & \leq \varepsilon c E(t). \end{aligned}$$

Consequently, $|\mathcal{L}(t) - E(t)| \leq \varepsilon c E(t)$, which yields

$$(1 - \varepsilon c)E(t) \leq \mathcal{L}(t) \leq (1 + \varepsilon c)E(t).$$

By choosing ε small enough, (4.2) follows. □

Theorem 4.2. *let (u, v, z) be the solution of* (2.3)*. Assume $|\mu_2| < \mu_1$ and* (A1), (A2) *hold. Then, there exist two positive constants c_0 and c_1 such that the energy functional for the system* (2.3) *satisfies*

$$E(t) \leq c_0 e^{-c_1 \int_0^t \eta(s) ds}, \quad \forall t \geq 0. \tag{4.3}$$

Proof. By differentiating (4.1) and using (3.1), (3.4), (3.7) and (3.8), and Poincaré's inequality, we obtain

$$\begin{aligned}
\mathcal{L}'(t) \le & -[m_0 - \varepsilon c] \int_\Omega |u_t|^2 dx - \varepsilon m_1 \int_\Omega |\nabla u|^2 dx - \varepsilon \kappa \int_\Omega |\nabla v|^2 dx \\
& - \varepsilon \left[m_1 - \beta \delta_2\right] \int_\Omega |\operatorname{div} u|^2 dx - \varepsilon m_2 \tau \int_\Omega \int_0^1 z^2(x,\rho,t) d\rho dx \\
& + \varepsilon c (g \circ \nabla u)(t) - \big[\delta - \varepsilon c \big(c + \frac{\beta}{4\delta_2}\big)\big] \int_\Omega |\nabla v_t|^2 dx \\
& - [(m_0 - \varepsilon c) + \varepsilon m_2] \int_\Omega z^2(x,1,t) dx.
\end{aligned}$$

At this point, we choose δ_2 small enough such that $(m_1 - \beta\delta_2) > 0$. Next, by picking

$$\varepsilon < \ min\{\frac{m_0}{c}, \frac{\delta}{c(c + \frac{\beta}{4\delta_2})}\},$$

we obtain

$$\begin{aligned}
\mathcal{L}'(t) \le & \, k_1 (g \circ \nabla u)(t) - k_2 \Big\{ \int_\Omega |u_t|^2 dx + \int_\Omega |\nabla u|^2 dx + \int_\Omega |\nabla v|^2 dx \\
& + \int_\Omega |\operatorname{div} u|^2 dx + \int_\Omega \int_0^1 z^2(x,\rho,t) d\rho dx + \int_\Omega |\nabla v_t|^2 dx \Big\},
\end{aligned}$$

for positive constants k_1 and k_2. Then, using Poincaré's inequality and (2.5), we obtain

$$\mathcal{L}'(t) \le -k_0 E(t) + k_1 (g \circ \nabla u)(t), \quad \forall t \ge 0, \tag{4.4}$$

for a positive constant k_0. By multiplying (4.4) by $\eta(t)$ and using (A2) and (3.1), we arrive at

$$\eta(t)\mathcal{L}'(t) \le -k_0 \eta(t) E(t) - 2k_1 E'(t), \quad \forall t \ge 0,$$

which can be rewritten as

$$(\eta(t)\mathcal{L}(t) + 2k_1 E(t))' - \eta'(t)\mathcal{L}(t) \le -k_0 \eta(t) E(t), \ \forall t \ge 0.$$

Using the fact that $\eta'(t) \le 0, \forall t \ge 0$, we have

$$(\eta(t)\mathcal{L}(t) + 2k_1 E(t))' \le -k_0 \eta(t) E(t), \quad \forall t \ge 0.$$

By exploiting (4.2), it can easily be shown that

$$\mathcal{R}(t) = \eta(t)\mathcal{L}(t) + 2k_1 E(t) \sim E(t). \tag{4.5}$$

Consequently, for some positive constant c_1, we obtain

$$\mathcal{R}'(t) \le -c_1 \eta(t) \mathcal{R}(t), \quad \forall t \ge 0. \tag{4.6}$$

A simple integration of (4.6) over $(0,t)$ leads to

$$\mathcal{R}(t) \le \mathcal{R}(0) e^{-c_1 \int_0^t \eta(s) ds}, \quad \forall t \ge 0. \tag{4.7}$$

The conclusion of the theorem follows by combining (4.5) and (4.7). $\square$

4.2. **General stability for $|\mu_2| = \mu_1$.** By recalling (2.4), we have $\xi = \tau\mu_1$. Hence, (3.1) takes the form

$$E'(t) \le \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t)\int_\Omega |\nabla u|^2 dx - \delta \int_\Omega |\nabla v_t|^2 dx \le 0, \quad \forall t \ge 0. \tag{4.8}$$

We then use (3.4), (3.7), and (3.8) with $\mu_1 = |\mu_2|$ and define another Lyapunov functional

$$\tilde{\mathcal{L}}(t) := NE(t) + \varepsilon_1 F_1(t) + F_2(t) + \varepsilon_2 F_3(t) + F_4(t), \tag{4.9}$$

where N, ε_1 and ε_2 are positive real numbers which will be chosen properly later.

Lemma 4.3. *For N large enough, $\tilde{\mathcal{L}}(t)$ and $E(t)$ satisfy*

$$\alpha_3 E(t) \le \tilde{\mathcal{L}}(t) \le \alpha_4 E(t), \quad \forall t \ge 0, \tag{4.10}$$

for two positive constants α_3 and α_4.

The inequality in the above lemma is established with similar steps as in the proof of Lemma 4.1.

Theorem 4.4. *let (u, v, z) be the solution of* (2.3). *Assume $|\mu_2| = \mu_1$ and* (A1), (A2) *hold. Then, for any $t_0 > 0$, there exist positive constants c_2 and c_3 independent of t such that the energy functional of the system* (2.3) *satisfies*

$$E(t) \le c_2 e^{-c_3 \int_{t_0}^t \eta(s)ds}, \quad \forall t \ge t_0. \tag{4.11}$$

Proof. Differentiating $\tilde{\mathcal{L}}(t)$ and using (3.4), (3.7), (3.8), (3.9), (4.8) and Poincaré's inequality, we obtain

$$\begin{aligned}
\tilde{\mathcal{L}}'(t) \le & -[m_3 - \varepsilon_1 c - \varepsilon_2]\int_\Omega |u_t|^2 dx - [\varepsilon_1 m_1 - \delta_3 c]\int_\Omega |\nabla u|^2 dx - \kappa \int_\Omega |\nabla v|^2 dx \\
& - \varepsilon_2 m_2 \tau \int_\Omega \int_0^1 z^2(x,\rho,t)d\rho dx - [\varepsilon_1 m_1 - \beta\delta_2 - \delta_4(\mu+\lambda)]\int_\Omega |\operatorname{div} u|^2 dx \\
& - \big[N\delta - \frac{\beta c}{2} - c\big(1 + \frac{\beta}{4\delta_2} + \varepsilon_1 c\big)\big]\int_\Omega |\nabla v_t|^2 dx + [\frac{N}{2} - c](g' \circ \nabla u)(t) \\
& - [\varepsilon_2 m_2 - \varepsilon_1 c - \delta_5 \mu_1]\int_\Omega z^2(x,1,t)dx + [\varepsilon_1 c + C_\delta](g \circ \nabla u)(t).
\end{aligned}$$

Now, we let

$$\varepsilon_2 = \frac{m3}{2}, \quad \delta_3 = \frac{\varepsilon_1 m_1}{2c}, \quad \delta_4 = \frac{\varepsilon_1 m_1}{2(\mu+\lambda)}.$$

Next, we choose ε_1 small enough so that

$$\tilde{k}_1 := [\frac{m_3}{2} - \varepsilon_1 c] > 0, \quad \tilde{k}_2 := [\frac{m_2 m_3}{2} - \varepsilon_1 c] > 0.$$

Once ε_1 is fixed, we then take $\delta_5 = \tilde{k}_2/(2\mu_1)$ and choose δ_2 small enough so that

$$\tilde{k}_3 := [\frac{\varepsilon_1 m_1}{2} - \beta\delta_2] > 0.$$

Finally, we choose N so large that (4.10) remains valid and, furthermore,

$$\tilde{k}_4 := \big[N\delta - \frac{\beta c}{2} - c\big(1 + \frac{\beta}{4\delta_2} + \varepsilon_1 c\big)\big] > 0, \quad [\frac{N}{2} - c] > 0.$$

Hence, we arrive at

$$\tilde{\mathcal{L}}'(t) \le -\tilde{k}_1 \int_\Omega |u_t|^2 dx - \frac{\varepsilon_1 m_1}{2}\int_\Omega |\nabla u|^2 dx - \kappa \int_\Omega |\nabla v|^2 dx$$

$$
\begin{aligned}
&- \tilde{k}_4 \int_\Omega |\nabla v_t|^2 dx - \tilde{k}_3 \int_\Omega |\operatorname{div} u|^2 dx + \tilde{k}_5 (g \circ \nabla u)(t) \\
&- \frac{m_2 m_3 \tau}{2} \int_\Omega \int_0^1 z^2(x,\rho,t) d\rho dx.
\end{aligned}
$$

Using Poincaré's inequality, we obtain

$$
\tilde{\mathcal{L}}'(t) \le -\tilde{k}_0 E(t) + \tilde{k}_5 (g \circ \nabla u)(t), \quad \forall t \ge t_0, \tag{4.12}
$$

where $\tilde{k}_0$ and $\tilde{k}_5$ are two positive constants.

By multiplying (4.12) by $\eta(t)$ and using (A_2) and (4.8), we obtain

$$
\begin{gathered}
\eta(t)\tilde{\mathcal{L}}'(t) \le -\tilde{k}_0 \eta(t) E(t) - 2\tilde{k}_5 E'(t), \quad \forall t \ge t_0, \\
\Big(\eta(t)\tilde{\mathcal{L}}(t) + 2\tilde{k}_5 E(t)\Big)' \le -\tilde{k}_0 \eta(t) E(t), \quad \forall t \ge t_0.
\end{gathered}
$$

If we set

$$
\tilde{\mathcal{R}}(t) = \eta(t)\tilde{\mathcal{L}}(t) + 2\tilde{k}_5 E(t) \sim E(t), \tag{4.13}
$$

and follow the same steps as in Theorem 4.2, we arrive at

$$
\tilde{\mathcal{R}}(t) \le \tilde{\mathcal{R}}(t_0) e^{-\tilde{c}_3 \int_0^t \eta(s) ds}, \quad \forall t \ge t_0. \tag{4.14}
$$

Consequently,(4.11) is established by virtue of (4.13) and (4.14).

Note that Estimate (4.11) also holds for $t \in [0, t_0]$ by the continuity and boundedness of $E(t)$ and $\eta(t)$. □

Now, we give some examples to illustrate the energy decay rates obtained by Theorem 4.2 which is also valid for Theorem 4.4. We consider the three examples under Remark 2.1 with the same assumptions on a and b as stated before.

(1) If $g(t) = ae^{-bt}$, then

$$
E(t) \le c_0 e^{-bc_1 t}, \quad \forall t \ge 0.
$$

(2) If $g(t) = \frac{a}{(1+t)^{b+1}}$, then

$$
E(t) \le \frac{c_0}{(1+t)^{(b+1)c_1}}, \quad \forall t \ge 0.
$$

(3) If $g(t) = \frac{a}{(e+t)[ln(e+t)]^{b+1}}$, then

$$
E(t) \le \frac{c_0 e^{c_1}}{\{(e+t)[ln(e+t)]^{b+1}\}^{c_1}}, \quad \forall t \ge 0.
$$

Remark 4.5. As in Pignotti [21], we do not require that μ_2 be positive. Our result extends, in a way, the result of Kirane and Said-Houari [5], where μ_2 is taken to be positive.

Acknowledgements. The authors thank KFUPM for its continuous support and the referee for pointing out a valuable reference which improved this work a lot. This work has been funded by KFUPM under Project # FT 111002.

References

[1] Abdallah, C.; Dorato, P.; Benitez-Read, J.; Byrne, R.; *Delayed positive feedback can stabilize oscillatory system*, ACC. San Francisco, 3106-3107 (1993).

[2] Ait Benhassi, E. M.; Ammari, K.; Boulite, S.; Maniar, L.; *Feedback stabilization of a class of evolution equations with delay*, J. Evol. Equ. (9), 103-121 (2009).

[3] Datko, R.; Lagnese, J.; Polis, M. P.; *An example on the effect of time delays in boundary feedback stabilization of wave equations*, SIAM J. Control Optim. (24), 152-156 (1986).

[4] Fridman, E.; Nicaise, S.; Valein, J.; *Stabilization of second order evolution equations with unbounded feedback with time-dependent delay*, SIAM J. Control Optim. 48 (8), 5028-5052 (2010).

[5] Kirane, M.; Said-Houari, B.; *Existence and asymptotic stability of a viscoelastic wave equation with a delay*, Z. Angew. Math. Phys. (62), 1065-1082 (2011).

[6] Komornik, V.; Zuazua, E.; *A direct method for the boundary stabilization of the wave equation*, J. Math. Pures Appl. (69), 35-54 (1990).

[7] Lasiecka, I.; *Stabilization of wave and plate-like equations with nonlinear dissipation on the boundary*, J. Differential Equations (79), 340-381 (1989).

[8] Lasiecka, I.; *Global uniform decay rates for the solution to the wave equation with nonlinear boundary conditions*, Appl. Anal. (47), 191-212 (1992).

[9] Liu, W. J.; *The exponential stabilization of higher-dimensional linear system of thermoviscoelasticity*, J. Math. Pures Appl. 77, 355386 (1998).

[10] Messaoudi, S. A.; Al-Shehri, A.; *General boundary stabilization of memory type in thermoelasticity of type III*, Z. Angew. Math. Phys. 62, 469-481 (2011).

[11] Messaoudi, S. A.; Mustafa, M. I.; *On the control of solutions of viscoelastic equations with boundary feedback*, Nonlinear Anal. TMA (10), 3132-3140. (2009).

[12] Messaoudi, S. A.; Soufyane, A.; *Boundary stabilization of memory type in thermoelasticity of type III* Applicable Analysis 87 (1), 13-28 (2008).

[13] Messaoudi, S. A.; Said-Houari, B.; *Energy decay in a Timoshenko-type system of thermoelasticity of type III* J. Math. Anal. Appl. 348 298307 (2008).

[14] Messaoudi, S. A.; Said-Houari, B.; *Energy decay in a Timoshenko-type system with history in thermoelasticity of type III*, Adv. Difference Equ. 4 (3-4), 375-400 (2009).

[15] Mustafa, M. I.; *Uniform stability for thermoelastic systems with boundary time-varying delay*, J. Math. Anal. Appl. 383, 490-498 (2011).

[16] Nicaise, S.; Pignotti, C.; *Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks*. SIAM J. Control Optim. 45, 1561-1585 (2006).

[17] Nicaise, S.; Pignotti, C.; *Stabilization of the wave equation with boundary or internal distributed delay*, Diff. Int. Equs. 21 (9-10), 935-958 (2008).

[18] Nicaise, S.,; Pignotti C.; *Interior feedback stabilization of wave equations with time dependent delay*, Elect. J. Differ. Eqns. 2011 (41), 1-20 (2011).

[19] Nicaise, S.; Pignotti, C.; Valein, J.; *Exponential stability of the wave equation with boundary time-varying delay*, Discrete Contin. Dyn. Syst. Ser. S 4 (3), 693-722 (2011).

[20] Nicaise, S.; Valein, J.,; Fridman, E.; *Stability of the heat and of the wave equations with boundary time-varying delays* Discrete Contin. Dyn. Syst. Ser. S 2 (3), 559-581 (2009).

[21] Pignotti, C.; *A note on stabilization of locally damped wave equations with time delay*, Systems and control letters 61, 92-97 (2012)

[22] Qin, Y.; Ma, Z.; *Energy decay and global attractors for thermoviscoelastic systems*, Acta Appl. Math. 117, 195-214 (2012).

[23] Quintanilla, R.; Racke, R.; *Stability in thermoelasticity of type III*, Discrete and Continuous Dynamical system, Series B. 3 (3), 383-400 (2003).

[24] Suh, I. H.; Bien, Z.; *Use of time delay action in the controller design*, IEEE Trans. Automat. Control 25, 600-603 (1980).

[25] Xu, G. Q.; Yung, S., P.; Li, L., K.; *Stabilization of wave systems with input delay in the boundary control* ESAIM: COCV 12, 770-785 (2006)

[26] Zhang, X.; Zuazua, E.; *Decay of solutions of the system of thermoelasticity of type III*, Commun. Contemp. Math. 5, 25-83 (2003).

[27] Zuazua, E.; *Uniform Stabilization of the wave equation by nonlinear boundary feedback* SIAM J. Control Optim. 28 (2), 466-477 (1990).

Tijani A. Apalara
Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia
E-mail address: tijani@kfupm.edu.sa

Salim A. Messaoudi
Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia
E-mail address: messaoud@kfupm.edu.sa

Muhammad I. Mustafa
Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia
E-mail address: mmustafa@kfupm.edu.sa