Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 140, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

FAST PROPAGATION FOR NONLOCAL DELAY EQUATIONS WITH SLOWLY DECAYING INITIAL VALUES

FUGUO ZHU

Abstract. This article concerns the long time behavior of solutions to nonlocal delay equations when the initial values decay slowly at infinity towards the unstable steady state. By constructing proper auxiliary functions, it is proved that the lower bounds of asymptotic speed for spreading is larger any given positive constant, which implies the fast propagation.

1. Introduction

In this article, we shall investigate the initial-value problem of the following nonlocal delay equation arising in population dynamics,

$$\begin{gathered}\frac{\partial}{\partial t}u(x,t)=\Delta u(x,t)-du(x,t)+b((g*u)(x,t)),\quad x\in\mathbb{R},\ t>0,\\ u(x,s)=\phi(x,s),\quad x\in\mathbb{R},\ s\in[-\tau,0],\end{gathered}\tag{1.1}$$

in which $u(x,t)$ is the population density at time t at location $x\in\mathbb{R}$, $d>0$ reflects the death rate, $\tau>0$ formulates the maximal maturation period of the population, $b:\mathbb{R}^+\to\mathbb{R}^+$ denotes the birth function, $(g*u)(x,t)$ describes the random walk as well as the historical effect of the individuals and is defined by

$$(g*u)(x,t)=\int_{-\tau}^{0}\int_{\mathbb{R}}g(y,s)u(x-y,t+s)\,dy\,ds,$$

where $g:\mathbb{R}\times[-\tau,0]\to\mathbb{R}^+$ is a probability function satisfying $\int_{-\tau}^0\int_{\mathbb{R}}g(y,s)\,dy\,ds=1$.

In the past decades, much attention has been paid to the dynamics of (1.1); see for example So and Yang [18], Yi and Zou [21, 22]. To model the spatial-temporal patterns about transition process in population invasion and epidemic spreading, the traveling wave solutions and asymptotic spreading of (1.1) have been widely studied in the past ten years, see Fang and Zhao [2], Li et al. [5], Liang and Zhao [6], Ma [7], Mei et al. [9, 10], Schaaf [14], So et al. [17], Thieme and Zhao [19], Wang et al. [20].

When $b(u)$ is monotone, Wang et al. [20] obtained the existence, uniqueness and asymptotic stability of traveling wave solutions of (1.1). When $b(u)$ is not monotone, Fang and Zhao [2] studied the existence and uniqueness of traveling

---

2000 *Mathematics Subject Classification.* 35C07, 35K57, 37C65.
Key words and phrases. Fast propagation; traveling wave solutions; asymptotic spreading.

Submitted June 28, 2012. Published August 19, 2012.

wave solutions. In particular, the traveling wave solution grows like an exponential function when the traveling wave coordinate goes to negative infinity, which was proved by a Laplace transformation [2, 20]. Furthermore, the asymptotic stability of traveling wave solutions allows us to understand the long time dynamics of (1.1) when $\phi(x,s)$ is a spatial perturbation of the traveling wave solution in a weighted functional space [20]. Due to the asymptotic behavior of traveling wave solutions, these results are useful in reflecting the long time behavior of (1.1) when $\phi(x,s)$ likes an exponential function when $x \to -\infty$. Besides the traveling wave solutions, its asymptotic spreading has also been investigated in the past ten years, and some results of asymptotic speed of spreading were obtained [6, 19]. These results describe the long time behavior of (1.1) when the initial value admits nonempty compact support.

From the viewpoint of initial value, the results mentioned above formulate the propagation of (1.1) when the initial value decays very fast as $|x| \to \infty$. The purpose of this paper is the dynamics of (1.1) if the initial value decays slowly at the infinity towards the unstable steady state 0, which is formulated as follows:

(C): $\lim_{|x|\to\infty} \phi(x,s) = 0$ holds for $s \in [-\tau, 0]$, and for each $\epsilon > 0$, there exists $x_\epsilon > 0$ such that $\phi(x,s) \geq e^{-\epsilon|x|}$ for $|x| > x_\epsilon$.

For the reaction-diffusion equations with slowly decaying initial conditions, the property has been investigated by Hamel and Roques [3], in which the comparison principle plays a very important role. However, for the delayed model (1.1), the technique may fail due to the possible loss of comparison principle and a famous example is

$$b(u) = pue^{-au}, p > de, \tag{1.2}$$

in which all the parameters are positive. When $b(u)$ is the above form, the corresponding reaction model of (1.1) is the famous Nicholson's blowflies equation [4, 11, 12], and we refer to Fang and Zhao [2], Li et al. [5], Ma [7] for the existence of traveling wave solutions.

In this article, we first study the problem if $b(u)$ is monotone increasing which ensures the comparison principle on our desired interval. By constructing auxiliary monotone equations, we further consider the problem if the birth function $b(u) : \mathbb{R}^+ \to \mathbb{R}^+$ is bounded, which is motivated by [2, 7] and Smith [15, Section 7.3]. For both cases, we obtain the estimates of lower bounds of asymptotic speed of spreading for $u(x,t)$. From the viewpoint of population dynamics, our results imply that even if the initial population density decays towards 0 at the infinity, the fast propagation still occurs in the following sense: if an observer were to move to the right or left at any fixed speed, the local population density would be larger than a positive constant. In other words, this also indicates that the lower bounds of asymptotic speed of spreading of $u(x,t)$ defined by (1.1) is larger than any given positive constants (see Berestycki et al. [1] for an example in general domains).

The rest of this paper is organized as follows. In Section 2, we list some necessary preliminaries. In Section 3, with the help of comparison principle, we prove the fast propagation of $u(x,t)$ if the birth function is monotone. By constructing two auxiliary equations with monotone birth functions, the dynamics of (1.1) with bounded birth function is studied in the last section.

2. Preliminaries

In what follows, let

$$X = \{u(x) : u : \mathbb{R} \to \mathbb{R} \text{ is bounded and uniformly continuous}\}.$$

Then X is a Banach space with respect to the supremum norm $|\cdot|$. Denote

$$X^+ = \{u : u \in X \text{ and } u(x) \geq 0 \text{ for all } x \in \mathbb{R}\}.$$

If $a < b$, then

$$X_{[a,b]} = \{u : a \leq u(x) \leq b \text{ for all } x \in \mathbb{R}\}.$$

At the same time, we define $\mathcal{C} : [-\tau, 0] \to X$ as a continuous map with the supremum norm. Similarly, the mappings

$$\mathcal{C}^+ : [-\tau, 0] \to X^+, \quad \mathcal{C}_{[a,b]} : [-\tau, 0] \to X_{[a,b]}$$

are continuous. Moreover, $u(t) \in X$ will be interpreted as

$$u(t) =: (u(t))(x) = u(x,t).$$

For $t > 0$, we define $T(t) : X \to X$ as follows

$$T(t)u(x) = \frac{e^{-dt}}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} u(y)dy, \quad u(x) \in X.$$

Then $T(t) : X \to X$ is an analytic positive semigroup. For $u(s) \in X$, we also denote

$$T(t)u(s) =: T(t)u(x,s) = \frac{e^{-dt}}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} u(y,s)dy.$$

We now give the assumptions of $b(u)$ and $g(y,s)$ as follows:

- (b1) for any $\lambda > 0, c > 0$, $\int_{-\tau}^{0} \int_{\mathbb{R}} g(y,s) e^{\lambda(y+cs)}\, dy\, ds < \infty$;
- (b2) if $u > 0$, then $b(u) > 0$ is bounded;
- (b3) there exists $k > 0$ such that $b(k) = dk$, $b(0) = 0$ and $b(u) \neq du$, $u \in (0,k)$;
- (b4) there exists $\overline{k} \geq k$ such that $\overline{b}(\overline{k}) = d\overline{k}$, $\overline{b}(u) \neq du$, $u \in (0, \overline{k})$, where $\overline{b}(u) = \sup_{v \in [0,u]} b(v)$;
- (b5) there exists $\underline{k} \geq k$ such that $\underline{b}(\underline{k}) = d\underline{k}$, $\underline{b}(u) \neq du, u \in (0, \underline{k})$, where $\underline{b}(u) = \inf_{v \in [u, \overline{k}]} b(v)$;
- (b6) $b'(u)$ exists for $u \in [0, \overline{k}]$ and $b'(0) > d$;
- (b7) there exists $L > 0$ such that $0 < b'(0)u - b(u) \leq Lu^2, u \in (0, \overline{k}]$.

Remark 2.1. If $b(u)$ is monotone for $u \in [0,k]$, then $\overline{k} = \underline{k} = k$ and $\overline{b}(u) = b(u) = \underline{b}(u), u \in [0,k]$. Moreover, (b2) and (b6) imply that (b3)-(b5) are well defined and $\overline{b}(u), \underline{b}(u)$ are continuous for $u \in [0, \overline{k}]$, we list (b3)-(b5) for the sake of convenience.

Using the theory of abstract functional differential equations established by Martin and Smith [8], we have the following result (see Smith and Zhao [16], Wang et al. [20]).

Lemma 2.2. *Assume that $\phi \in \mathcal{C}_{[0,\overline{k}]}$. Then* (1.1) *has a mild solution $u(t) \in X_{[0,\overline{k}]}$ for all $t > 0$, and $u(t)$ is formulated by*

$$u(t) = T(t)\phi(0) + \int_0^t T(t-s)[(g * u)(s)]ds,$$

which is also a classical solution satisfying (1.1) *if $t > \tau$. Moreover, if $\phi(0) \in X_{[0,k]}$ admits nonempty support, then $u(t) \gg 0$ (namely, $u(x,t) > 0$ for all $x \in \mathbb{R}, t > 0$).*

When $b(u)$ is monotone for $u \in [0,k]$, the following comparison principle is true.

Lemma 2.3. *Assume that $\phi \in \mathcal{C}_{[0,k]}$ holds and $b(u)$ is monotone for $u \in [0,k]$. If $w(t) \in X_{[0,k]}$ satisfies*

$$w(t) \geq (\leq) T(t-s)w(s) + \int_s^t T(t-\theta)[(g * w)(\theta)]d\theta$$

for all $0 \leq s \leq t < t'(\leq \infty)$, then $w(t) \geq (\leq) u(t)$ for all $t \in (0,t')$.

In particular, suppose that

$$\begin{gathered} \frac{\partial}{\partial t}u_1(x,t) = \Delta u_1(x,t) - du_1(x,t) + b((g * u_1)(x,t)), \quad x \in \mathbb{R},\ t > 0, \\ u_1(x,s) = \phi_1(x,s), \quad x \in \mathbb{R},\ s \in [-\tau, 0], \end{gathered} \tag{2.1}$$

and

$$\begin{gathered} \frac{\partial}{\partial t}u_2(x,t) = \Delta u_2(x,t) - du_2(x,t) + b((g * u_2)(x,t)), x \in \mathbb{R}, t > 0, \\ u_2(x,s) = \phi_2(x,s), \quad x \in \mathbb{R},\ s \in [-\tau, 0]. \end{gathered} \tag{2.2}$$

If $\phi_1, \phi_2 \in \mathcal{C}$ satisfy

$$0 \leq \phi_1(x,s) \leq \phi_2(x,s) \leq k, \quad x \in \mathbb{R},\ s \in [-\tau, 0],$$

then $u_1(x,t) \leq u_2(x,t), x \in \mathbb{R}, t > 0$.

Moreover, we also have the following result on the asymptotic spreading (see Liang and Zhao [6], Thieme and Zhao [19]).

Lemma 2.4. *Assume that $\phi \in \mathcal{C}_{[0,k]}$ holds and $b(u)$ is monotone for $u \in [0,k]$. If $\phi(x,0)$ admits nonempty support, then $\lim_{t\to\infty} u(x,t) = k$ locally uniform in $x \in \mathbb{R}$.*

For $\lambda \geq 0$, $c > 0$, define

$$\Delta(\lambda, c) = \lambda^2 - c\lambda - d + b'(0) \int_{-\tau}^0 \int_{\mathbb{R}} g(y,s) e^{\lambda(y+cs)}\, dy\, ds.$$

Lemma 2.5. *There exists $c^* > 0$ such that for each $c > c^*$, $\Delta(\lambda, c) = 0$ has two positive real roots $\lambda_1(c) < \lambda_2(c)$. If $\lambda \in (\lambda_1(c), \lambda_2(c))$ holds, then $\Delta(\lambda, c) < 0$.*

3. Monotone birth function

In this part, we shall investigate the initial value problem (1.1) if $\phi \in \mathcal{C}_{[0,k]}$ holds and the birth function $b(u)$ is monotone for $u \in [0,k]$. In the remainder of this paper, (C) will be imposed without further illustration.

For any $c > c^*$, Lemma 2.5 implies that there exists $\eta \in (1,2)$ such that

$$\Delta(\eta\lambda(c), c) < 0.$$

In what follows, c, η will be fixed.

Lemma 3.1. *There exists $q > 1$ such that*

$$\phi(x,s) \geq \max\{e^{\lambda_1(c)(x+cs)} - qe^{\eta\lambda_1(c)(x+cs)}, e^{\lambda_1(c)(-x+cs)} - qe^{\eta\lambda_1(c)(-x+cs)}, 0\}$$

for $x \in \mathbb{R}, s \in [-\tau, 0]$.

Proof. If $q \to \infty$ is large, then

$$\max\{e^{\lambda_1(c)(x+ct)} - qe^{\eta\lambda_1(c)(x+ct)}, e^{\lambda_1(c)(-x+ct)} - qe^{\eta\lambda_1(c)(-x+ct)}, 0\} > 0$$

implies that $-|x| + ct \to -\infty$. By the condition (C), the lemma is clear. $\square$

By these constants, define continuous function

$$\underline{u}(x,t)=\max\{e^{\lambda_1(c)(x+ct)}-qe^{\eta\lambda_1(c)(x+ct)},0\}.$$

Lemma 3.2. *There exists $q>1$ large enough such that*

$$\underline{u}(t)\le T(t-s)\underline{u}(s)+\int_s^t T(t-\theta)[(g*\underline{u})(\theta)]d\theta$$

for all $0\le s<t\le 1+\tau$.

Proof. We now verify the inequality

$$\underline{u}(t)\le T(t-s)\underline{u}(s)+\int_s^t T(t-\theta)[b\left((g*\underline{u})(\theta)\right)]d\theta.$$

If $\underline{u}(x,t)=0$, then $\underline{u}(y,s)\ge 0, y\in\mathbb{R}, s\in[t-\tau,t]$ and

$$T(t-s)\underline{u}(s)+\int_s^t T(t-\theta)[b\left((g*\underline{u})(\theta)\right)]d\theta\ge 0$$

by the positivity of $T(t)$, which implies what we wanted.

By (b1), there exist $L_1>0, L_2>0$ such that

$$\int_{-\tau}^0\int_{\mathbb{R}} g(y,s)e^{\lambda(y+cs)}\,dy\,ds=L_1,\int_{-\tau}^0\int_{\mathbb{R}} g(y,s)e^{\eta\lambda(y+cs)}\,dy\,ds=L_2$$

and the definition of $\Delta(\lambda,c)$ indicates that

$$\begin{gathered}b'(0)L_1=d+c\lambda_1(c)-\lambda_1^2(c),\\ b'(0)L_2<d+c\eta\lambda_1(c)-\eta^2\lambda_1^2(c)=:L_3.\end{gathered}$$

If $\underline{u}(x,t)=e^{\lambda_1(c)(x+ct)}-qe^{\eta\lambda_1(c)(x+ct)}\ge 0$, then the positivity of $T(t)$ leads to

$$\begin{aligned}&T(t-s)\underline{u}(s)\\ &=:\frac{e^{-d(t-s)}}{\sqrt{4\pi(t-s)}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4(t-s)}}\underline{u}(y,s)dy\\ &\ge\frac{e^{-d(t-s)}}{\sqrt{4\pi(t-s)}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4(t-s)}}\left[e^{\lambda_1(c)(y+cs)}-qe^{\eta\lambda_1(c)(y+cs)}\right]dy\\ &=e^{-d(t-s)+(t-s)\lambda_1^2(c)+\lambda_1(c)x+\lambda_1(c)cs}-qe^{-d(t-s)+(t-s)\eta^2\lambda_1^2(c)+\eta\lambda_1(c)x+\eta\lambda_1(c)cs}\\ &=e^{\lambda_1(c)(x+ct)}e^{b'(0)L_1(s-t)}-qe^{\eta\lambda_1(c)(x+ct)}e^{L_3(s-t)}\\ &=I_1+I_2,\end{aligned}$$

where the definitions of I_1, I_2 are clear. From (b2) and (b7), we have

$$\begin{aligned}&\int_s^t T(t-\theta)[b\left((g*\underline{u})(\theta)\right)]d\theta\\ &=:\int_s^t\frac{e^{-d(t-\theta)}}{\sqrt{4\pi(t-\theta)}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4(t-\theta)}}\left[b\left((g*\underline{u})(y,\theta)\right)\right]dy\,d\theta\\ &\ge\int_s^t\frac{e^{-d(t-\theta)}}{\sqrt{4\pi(t-\theta)}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4(t-\theta)}}\left[b'(0)\left((g*\underline{u})(y,\theta)\right)\right]dy\,d\theta\\ &\quad-L\int_s^t\frac{e^{-d(t-\theta)}}{\sqrt{4\pi(t-\theta)}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4(t-\theta)}}\left[\left((g*\underline{u})(y,\theta)\right)\right]^2dy\,d\theta.\end{aligned}$$

By the above constants, we obtain

$$\underline{u}(y,s) \le e^{\lambda_1(c)(y+cs)}, \quad y \in \mathbb{R}, s \in [t-\tau, t]$$

such that

$$(g * \underline{u})(x,t) \le L_1 e^{\lambda_1(c)(x+ct)}, \quad x \in \mathbb{R},$$

and

$$\begin{aligned}
&\int_s^t \frac{e^{-d(t-\theta)}}{\sqrt{4\pi(t-\theta)}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4(t-\theta)}} [b'(0)\left((g*\underline{u})(y,\theta)\right)]\, dy\, d\theta \\
&\ge L_1 b'(0) \int_s^t \frac{e^{-d(t-\theta)}}{\sqrt{4\pi(t-\theta)}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4(t-\theta)}} e^{\lambda_1(c)(y+c\theta)}\, dy\, d\theta \\
&\quad - qL_2 b'(0) \int_s^t \frac{e^{-d(t-\theta)}}{\sqrt{4\pi(t-\theta)}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4(t-\theta)}} e^{\eta\lambda_1(c)(y+c\theta)}\, dy\, d\theta \\
&= L_1 b'(0) \int_s^t e^{-d(t-\theta)+(t-\theta)\lambda_1^2(c)+\lambda_1(c)x+\lambda_1(c)c\theta} d\theta \\
&\quad - qL_2 b'(0) \int_s^t e^{-d(t-\theta)+(t-\theta)\eta^2\lambda_1^2(c)+\lambda_1(c)x+\eta\lambda_1(c)c\theta} d\theta \\
&= L_1 b'(0) e^{-dt+\lambda_1^2(c)t+\lambda_1(c)x} \int_s^t e^{d\theta-\lambda_1^2(c)\theta+\lambda_1(c)c\theta} d\theta \\
&\quad - qL_2 b'(0) e^{-dt+\eta^2\lambda_1^2(c)t+\eta\lambda_1(c)x} \int_s^t e^{d\theta-\eta^2\lambda_1^2(c)\theta+\eta\lambda_1(c)c\theta} d\theta \\
&= L_1 b'(0) e^{-dt+\lambda_1^2(c)t+\lambda_1(c)x} \int_s^t e^{b'(0)L_1\theta} d\theta \\
&\quad - qL_2 b'(0) e^{-dt+\eta^2\lambda_1^2(c)t+\eta\lambda_1(c)x} \int_s^t e^{L_3\theta} d\theta \\
&= e^{-dt+\lambda_1^2(c)t+\lambda_1(c)x} \left(e^{b'(0)L_1 t} - e^{b'(0)L_1 s} \right) \\
&\quad - \frac{L_2 q b'(0) e^{-dt+\eta^2\lambda_1^2(c)t+\eta\lambda_1(c)x}}{L_3} e^{L_3 t} \\
&\quad + \frac{L_2 q b'(0) e^{-dt+\eta^2\lambda_1^2(c)t+\eta\lambda_1(c)x}}{L_3} e^{L_3 s} \\
&= e^{\lambda_1(c)(x+ct)} \left(1 - e^{b'(0)L_1(s-t)} \right) - \frac{qL_2 b'(0) e^{\eta\lambda_1(c)(x+ct)}}{L_3} \\
&\quad + \frac{L_2 q b'(0) e^{\eta\lambda_1(c)(x+ct)}}{L_3} e^{L_3(s-t)} \\
&= I_3 + I_4 + I_5,
\end{aligned}$$

in which the definitions of I_3, I_4, I_5 are clear. At the same time, we also have

$$\begin{aligned}
&- L \int_s^t \frac{e^{-d(t-\theta)}}{\sqrt{4\pi(t-\theta)}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4(t-\theta)}} [((g*\underline{u})(y,\theta))]^2\, dy\, d\theta \\
&\ge -LL_1^2 \int_s^t \frac{e^{-d(t-\theta)}}{\sqrt{4\pi(t-\theta)}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4(t-\theta)}} e^{2\lambda_1(c)(y+c\theta)}\, dy\, d\theta
\end{aligned}$$

$$
\begin{aligned}
&= -LL_1^2 \int_s^t e^{-d(t-\theta)+(t-\theta)4\lambda_1^2(c)+2\lambda_1(c)x+2\lambda_1(c)c\theta} d\theta \\
&\geq -L_4 L L_1^2 e^{2\lambda_1(c)(x+ct)} (1 - e^{2\lambda_1(c)c(s-t)}) \\
&\geq -L_4 L L_1^2 e^{\eta\lambda_1(c)(x+ct)} (1 - e^{2\lambda_1(c)c(s-t)}) =: I_6,
\end{aligned}
$$

where the boundedness of $L_4 > 0$ is obtained by $0 \leq s \leq t \leq 1 + \tau$. Furthermore, direct calculations tell us

$$
I_1 + I_3 = e^{\lambda_1(c)(x+ct)}.
$$

Then it suffices to prove that

$$
I_2 + I_4 + I_5 + I_6 \geq -qe^{\eta\lambda_1(c)(x+ct)}.
$$

Let

$$
I(t,s) = I_2 + I_4 + I_5 + I_6 + qe^{\eta\lambda_1(c)(x+ct)},
$$

then $I(t,t) = 0$ for all $t \in [0, 1+\tau]$. Moreover,

$$
\begin{aligned}
I(t,s) &= -\frac{L_2 q b'(0) e^{\eta\lambda_1(c)(x+ct)}}{L_3} - qe^{\eta\lambda_1(c)(x+ct)} e^{L_3(s-t)} \\
&\quad + \frac{L_2 b'(0) q e^{\eta\lambda_1(c)(x+ct)}}{L_3} e^{L_3(s-t)} \\
&\quad - L_4 L L_1^2 e^{\eta\lambda_1(c)(x+ct)} \left(1 - e^{2\lambda_1(c)c(s-t)}\right) + qe^{\eta\lambda_1(c)(x+ct)} \\
&= -\frac{L_2 q b'(0) e^{\eta\lambda_1(c)(x+ct)}}{L_3} + \frac{\left(L_2 b'(0) - L_3\right) q e^{\eta\lambda_1(c)(x+ct)}}{L_3} e^{L_3(s-t)} \\
&\quad - L_4 L L_1^2 e^{\eta\lambda_1(c)(x+ct)} \left(1 - e^{2\lambda_1(c)c(s-t)}\right) + qe^{\eta\lambda_1(c)(x+ct)}.
\end{aligned}
$$

Clearly, $I(t,s)$ is differentiable in $s \in [0,t]$. Denote

$$
I_7(t,s) = \frac{q\left(L_2 b'(0) - L_3\right)}{L_3} e^{L_3(s-t)} - L_4 L L_1^2 \left(1 - e^{2\lambda_1(c)c(s-t)}\right).
$$

Let $q > 1$ be large, then

$$
\frac{\partial I_7}{\partial s}(t,s) < 0, \quad s \in [0,t]
$$

by $L_3 > L_2 b'(0)$ and the boundedness of $t - s$. Moreover, it is clear that

$$
\frac{\partial I_7}{\partial s}(t,s) = e^{-\eta\lambda_1(c)(x+ct)} \frac{\partial}{\partial s} I(t,s) < 0,
$$

which also implies that $I(t,s) \geq 0$ and we complete the proof. □

Note that q is uniform for $t - s \in [0,\tau]$ in the proof of Lemma 3.2, so we can *fix* $q > 1$ satisfying Lemmas 3.1-3.2. Furthermore, for such a $q > 1$, we can obtain the following conclusion by a discussion similar to the proof of Lemma 3.2.

Lemma 3.3. *For each $n \in \mathbb{N}$, $\underline{u}(t) \in X_{[0,k]}$ satisfies*

$$
\underline{u}(t) \leq T(t-s)\underline{u}(s) + \int_s^t T(t-\theta)[(g * \underline{u})(\theta)] d\theta
$$

for all $n + n\tau \leq s < t \leq (n+1) + (n+1)\tau$.

Applying Lemma 2.3, the following result is true.

Lemma 3.4. $u(x,t) \geq \underline{u}(x,t)$ *for all* $t > 0, x \in \mathbb{R}$.

From Lemma 3.1, we further obtain a conclusion as follows.

Lemma 3.5. *For $x \in \mathbb{R}$, $t > 0$, $u(x,t)$ satisfies*

$$u(x,t) \geq \max\{e^{\lambda_1(c)(x+ct)} - qe^{\eta\lambda_1(c)(x+ct)}, e^{\lambda_1(c)(-x+ct)} - qe^{\eta\lambda_1(c)(-x+ct)}, 0\}.$$

Assume that $x + ct$ satisfies

$$x + ct = -\frac{\ln q}{(\eta - 1)\lambda_1(c)}. \tag{3.1}$$

Then Lemma 3.5 implies that there exist $\delta_1 > 0$, $\delta_2 > 0$ such that

$$u(y,s) > \delta_1, |x - y| \leq \delta_2, t - s \in [0,\tau].$$

By Lemmas 2.3-2.4, the following holds.

Lemma 3.6. *For any $\epsilon > 0$, there exists $T_1 = T_1(\epsilon) > 0$ such that*

$$u(x, t+T) > k - \epsilon \quad \text{for any } T > T_1$$

if x, t satisfy (3.1) *and $t \geq 3\tau + 1$.*

If $t = 3\tau + 1$ with

$$|x| \leq \Big| -\frac{\ln q}{(\eta - 1)\lambda_1(c)} - ct\Big|, \tag{3.2}$$

then the uniform continuity implies that there exist $\delta_3 > 0, \delta_4 > 0$ such that

$$u(y,s) > \delta_3, \quad t - s \in [0,\tau],\ |x - y| \leq \delta_4.$$

Applying Lemmas 2.3-2.4, $u(x,t)$ defined by (1.1) satisfies the following property.

Lemma 3.7. *For any $\epsilon > 0$, there exists $T_2 = T_2(\epsilon) > 0$ such that*

$$u(x, t+T) > k - \epsilon \quad \text{for any } T > T_2$$

if x, t satisfy (3.2) *and $t = 3\tau + 1$.*

By what we have done, we have the following result.

Theorem 3.8. *For any $\epsilon > 0$, there exists $T_3 = T_3(\epsilon)$ such that*

$$u(x, t+T) > k - \epsilon \quad \text{for any } |x| \leq ct, T > T_3. \tag{3.3}$$

Due to the arbitrary of c, we can present the main result of this section.

Theorem 3.9. *For any $c_1 > 0$, $u(x,t)$ satisfies*

$$\liminf_{t\to\infty} \inf_{|x|\leq c_1 t} u(x,t) = \limsup_{t\to\infty} \sup_{|x|\leq c_1 t} u(x,t) = k. \tag{3.4}$$

Proof. We now prove the result by the idea in Pan [13, Theorem 3.3]. For each fixed c_1 and $\epsilon > 0$, there exists $c = c_1 + 1$ such that

$$u(x, t+T) > k - \epsilon, \quad |x| \leq ct,\ T > T_3(\epsilon) \tag{3.5}$$

by (3.3) and comparison principle.

Let $c_1 s = ct$ with large $t > 0$, then $s - t > T_3(\epsilon)$ and (3.5) imply that

$$\liminf_{s\to\infty} \inf_{|x|\leq c_1 s} u(x,s) > k - \epsilon.$$

By the arbitrariness of ϵ and $u(x,t) \leq k$, the proof is complete. $\square$

4. Bounded birth function

In the previous section, we investigate the dynamics of (1.1) if $b(u)$ is monotone. In this part, we shall establish a conclusion of (1.1) with bounded $b(u)$.

Lemma 4.1. *Assume that $\phi \in \mathcal{C}_{[0,\overline{k}]}$ and $u(x,t)$ is defined by* (1.1)*. Let*

$$\begin{gathered} \frac{\partial}{\partial t}u_1(x,t) = \Delta u_1(x,t) - du_1(x,t) + \underline{b}((g*u_1)(x,t)), \quad x \in \mathbb{R},\ t>0, \\ u_1(x,s) = \phi(x,s), \quad x \in \mathbb{R},\ s \in [-\tau,0] \end{gathered} \tag{4.1}$$

and

$$\begin{gathered} \frac{\partial}{\partial t}u_2(x,t) = \Delta u_2(x,t) - du_2(x,t) + \overline{b}((g*u_2)(x,t)), \quad x \in \mathbb{R},\ t>0, \\ u_2(x,s) = \phi(x,s), \quad x \in \mathbb{R},\ s \in [-\tau,0], \end{gathered} \tag{4.2}$$

then $u_1(x,t) \le u(x,t) \le u_2(x,t)$ for all $x \in \mathbb{R}, t>0$.

Proof. Because of the monotonicity of $\overline{b}(u)$ and $\underline{b}(u)$ for $u \in [0,\overline{k}]$, then the conclusion is evident by

$$\underline{b}(u) \le b(u) \le \overline{b}(u), \quad u \in [0,\overline{k}]$$

and Lemma 2.3. The proof is complete. □

Since both $\overline{b}(u)$ and $\underline{b}(u)$ are monotone in $u \in [0,\overline{k}]$, then Theorem 3.9 and Lemma 4.1 imply the following result.

Theorem 4.2. *For any $c_1>0$, $u(x,t)$ satisfies*

$$0 < \underline{k} < \liminf_{t\to\infty} \inf_{|x|\le c_1 t} u(x,t) \le \limsup_{t\to\infty} \sup_{|x|\le c_1 t} u(x,t) \le \overline{k}. \tag{4.3}$$

By [7, 22], we also have the following result.

Corollary 4.3. *Assume that $b(u)$ is defined by* (1.2) *and $p/d \in (1,e^2]$ holds. Then* (3.4) *with $k = \frac{1}{a}\ln\frac{p}{d}$ is true.*

In fact, we can weaken the initial value condition as follows.

Theorem 4.4. *If $\phi(s) \in \mathcal{C}_{[0,\overline{k}]}$ and for any $\epsilon>0$, there exists $x_\epsilon>0$ such that*

$$\phi(x,0) \ge e^{-\epsilon|x|}, |x| > x_\epsilon,$$

then Theorems 3.9 and 4.2 still hold.

Proof. We can verify that

$$u(x,t) \ge \max\{e^{\lambda_1(c)(x+ct)} - qe^{\eta\lambda_1(c)(x+ct)}, e^{\lambda_1(c)(-x+ct)} - qe^{\eta\lambda_1(c)(-x+ct)}, 0\}$$

for $x \in \mathbb{R}, t \in [0,1+\tau]$ and large $q>1$, and the result is clear. □

Before ending the paper, we give the following remark.

Remark 4.5. Our result remains true for the model in Hamel and Roques [3], and [3, Theorem 1.1] holds if $u_0(x)$ satisfies

$$u_0(x) \ge 0, \lim_{|x|\to\infty} u_0(x) = 0$$

and for any $\epsilon>0$, there exists $x_\epsilon>0$ such that $u_0(x) \ge e^{-\epsilon|x|}$, $|x|>x_\epsilon$.

References

[1] H. Berestycki, F. Hamel, N. Nadirashvili: *The speed of propagation for KPP type problems. II. General domains,* J. Amer. Math. Soc. **23** (2010), 1-34.

[2] J. Fang, X. Q. Zhao: *Existence and uniqueness of traveling waves for non-monotone integral equations with applications,* J. Differential Equations **248** (2010), 2199-2226.

[3] F. Hamel, L. Roques: *Fast propagation for KPP equations with slowly decaying initial conditions,* J. Differential Equations **249** (2010), 1726-1745.

[4] W.S.C. Gurney, S. P. Blythe, R. M. Nisbet: *Nicholson's blowflies revisited,* Nature **287** (1990), 17-21.

[5] W. T. Li, S. Ruan, Z. C. Wang: *On the diffusive nicholson's blowflies equation with nonlocal delay,* J. Nonlinear Sci. **17** (2007), 505-525.

[6] X. Liang, X. Q. Zhao: *Asymptotic speeds of spread and traveling waves for monotone semiflows with applications,* Comm. Pure Appl. Math. **60** (2007), 1-40.

[7] S. Ma: *Traveling waves for non-local delayed diffusion equations via auxiliary equations,* J. Differential Equations **237** (2007), 259-277.

[8] R. H. Martin, H. L. Smith: *Abstract functional differential equations and reaction-diffusion systems,* Trans. Amer. Math. Soc. **321** (1990), 1-44.

[9] M. Mei, C.-K. Lin, C.-T. Lin, J. W.-H. So: *Traveling wavefronts for time-delayed reaction-diffusion equation: (II) Nonlocal nonlinearity,* J. Differential Equations **247** (2009), 511-529.

[10] M. Mei, J. W.-H. So, M. Y. Li, S. S. P. Shen: *Asymptotic stability of travelling waves for Nicholson's blowflies equation with diffusion,* Proc. Royal Soc. Edinburgh A **134** (2004), 579-594.

[11] A. J. Nicholson: *Compensatory reactions of populations to stresses, and their evolutionary significance,* Aust. J. Zool., **2** (1954), 1-8.

[12] A. J. Nicholson: *An outline of the dynamics of animal populations,* Aust. J. Zool. **2** (1954), 9-65.

[13] S. Pan: *Persistence of spreading speed for the delayed Fisher equation,* Electron. J. Differential Equations, Vol. **2012**, No. 113, 7 pp.

[14] K.W. Schaaf: *Asymptotic behavior and traveling wave solutions for parabolic functional differential equations,* Trans. Amer. Math. Soc. **302** (1987), 587–615.

[15] H.L. Smith: *Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems.* AMS, Providence, 1995.

[16] H. L. Smith, X. Q. Zhao: *Global asymptotic stability of traveling waves in delayed reaction-diffusion equations,* SIAM J. Math. Anal. **31** (2000), 514-534.

[17] J.W.H. So, J. Wu, X. Zou: *A reaction-diffusion model for a single species with age structure I. Travelling wavefronts on unbounded domains,* Proc. Roy. Soc. London Ser. A **457** (2001), 1841-1853.

[18] J.W.H. So, Y. Yang: *Dirichlet problem for the diffusive Nicholson's blowflies equation,* J. Differential Equations **150** (1998), 317-348.

[19] H. R. Thieme, X. Q. Zhao: *Asymptotic speeds of spread and traveling waves for integral equations and delayed reaction diffusion models,* J. Differential Equations **195** (2003), 430-470.

[20] Z. C. Wang, W. T. Li, S. Ruan: *Traveling fronts in monostable equations with nonlocal delayed effects,* J. Dynam. Diff. Eqns. **20** (2008), 573–603.

[21] T. Yi, X. Zou: *Global attractivity of the diffusive Nicholson blowflies equation with Neumann boundary condition: a non-monotone case,* J. Differential Equations **245** (2008), 3376-3388.

[22] T. Yi, X. Zou: *Global dynamics of a delay differential equation with spatial non-locality in an unbounded domain,* J. Differential Equations **251** (2011), 2598-2611.

Fuguo Zhu

School of Mathematics and Statistics, Hexi University, Zhangye, Gansu 734000, China

E-mail address: `fugzhu@yeah.net`