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IRREGULAR OBLIQUE DERIVATIVE PROBLEMS FOR SECOND-ORDER NONLINEAR ELLIPTIC EQUATIONS ON INFINITE DOMAINS

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Abstract. In this article, we study irregular oblique derivative boundary-value problems for nonlinear elliptic equations of second order in an infinite domain. We first provide the formulation of the above boundary-value problem and obtain a representation theorem. Then we give a priori estimates of solutions by using the reduction to absurdity and the uniqueness of solutions. Finally by the above estimates and the Leray-Schauder theorem, the existence of solutions is proved.

1. Formulation of the problem

Let D be an $(N+1)$-connected domain including the infinite point with the boundary $\Gamma = \cup_{j=0}^{N}\Gamma_j$ in $\mathbb{C}$, where $\Gamma \in C^2_\mu\,(0 < \mu < 1)$. Without loss of generality, we assume that D is a circular domain in $|z| > 1$, where the boundary consists of $N+1$ circles $\Gamma_0 = \Gamma_{n+1} = \{|z| = 1\}$, $\Gamma_j = \{|z - z_j| = r_j\}$, $j = 1, \dots, N$ and $z = \infty \in D$. In this article, the notation is as the same in References [1, 2, 3, 4, 5, 6]. We consider the second-order nonlinear elliptic equation in the complex form

$$\begin{gathered} u_{z\bar z} = F(z, u, u_z, u_{zz}), \quad F = \operatorname{Re}[Qu_{zz} + A_1u_z] + \hat A_2u + A_3, \\ Q = Q(z, u, u_z, u_{zz}), A_j = A_j(z, u, u_z), \quad j = 1, 2, 3,\ \hat A_2 = A_2 + |u|^\sigma, \end{gathered} \tag{1.1}$$

satisfying the following conditions.

Condition (C). (1) $Q(z, u, w, U), A_j(z, u, w) (j = 1, 2, 3)$ are continuous in $u \in \mathbb{R}$, $w \in \mathbb{C}$ for almost every $z \in D$, $U \in \mathbb{C}$, and $Q = 0$, $A_j = 0$ $(j = 1, 2, 3)$ for $z \notin D$, σ is a positive number.

(2) The above functions are measurable in D for all continuous functions $u(z)$, $w(z)$ in $\overline{D}$, and satisfy

$$L_{p,2}[A_j(z, u, w), \overline{D}] \le k_0, \quad j = 1, 2,\ L_{p,2}[A_3(z, u, w), \bar D] \le k_1, \tag{1.2}$$

in which $p_0, p\,(2 < p_0 \le p), k_0, k_1$ are non-negative constants.

(3) Equation (1.1) satisfies the uniform ellipticity condition

$$|F(z, u, w, U_1) - F(z, u, w, U_2)| \le q_0|U_1 - U_2|, \quad A_2 \ge 0, \text{ in } D, \tag{1.3}$$

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for almost every point $z \in D$, any number $u \in \mathbb{R}$, $w, U_1, U_2 \in \mathbb{C}$, where $q_0(< 1)$ is a non-negative constant.

Problem (P). In the domain D, find a solution $u(z)$ of equation (1.1), which is continuous in $\overline{D}$, and satisfies the boundary conditions

$$\begin{gathered} \frac{1}{2}\frac{\partial u}{\partial \nu} + c_1(z)u(z) = c_2(z), \quad z \in \Gamma,\ u(a_j) = b_j, j = 0, 1, \dots, K',\ \text{i.e.} \\ \operatorname{Re}[\overline{\lambda(z)}u_z] + c_1(z)u = c_2(z), \quad z \in \Gamma,\ u(a_j) = b_j, j = 0, 1, \dots, K', \end{gathered} \tag{1.4}$$

where the vector ν $(\neq 0)$ can be arbitrary at every point on Γ, K' $(= 2K - 2N + J + 1 \geq 0)$, J are non-negative integers as stated below, $\lambda(z) = \cos(\nu, x) + i\sin(\nu, x) = e^{i(\nu,x)} \neq 0$, (ν, x) is the angle between ν and the x-axis, a_j $(\in \Gamma_j, j = 0, 1, \dots, K')$ are distinct points on Γ. Suppose that $\lambda(z), c_1(z), c_2(z)$, $b_j (j = 0, 1, \dots, K')$ satisfy the conditions

$$\begin{gathered} C_\alpha[\lambda(z), \Gamma] \leq k_0, \quad C_\alpha[c_1(z), \Gamma] \leq k_0, \quad C_\alpha[c_2(z), \Gamma] \leq k_2, \\ |b_j| \leq k_2, \quad j = 0, 1, \dots, K', \quad c_1(z)\cos(\nu, n) \geq 0 \quad \text{on } \Gamma, \end{gathered} \tag{1.5}$$

in which α $(1/2 < \alpha < 1)$, k_2 are non-negative constants. The boundary $\partial D = \Gamma$ can be divided into two parts, namely $E^+ \subset \{z \in \partial D, \cos(\nu, n) \geq 0, c_1 \geq 0\}$ and $E^- \subset \{z \in \partial D, \cos(\nu, n) \leq 0, c_1 \leq 0\}$, and $E^+ \cap E^- = \emptyset$, $E^+ \cup E^- = \Gamma$, $\overline{E^+} \cap \overline{E^-} = E^0$. For every component $L = \Gamma_j$ $(0 \leq j \leq N)$ of Γ, there are three cases:

1. $L \subset E^+$.
2. $L \subset E^-$. In these cases, if $\cos(\nu, n) \equiv 0$, $c_1(z) \equiv 0$ on Γ_j $(1 \leq j \leq J, J \leq N + 1)$, and the above identical formulas on Γ_j $(J < j \leq N + 1)$ do not hold, then we need the conditions $\int_{\Gamma_j} c_2(z)\mathrm{d}s = 0$ $(1 \leq j \leq J)$, and $u(a_j) = b_j$, $j = 0, 1, \dots, K'$ $(\geq J)$, in which a_j, b_j $(j = 0, 1, \dots, K')$ are as stated before, and denote $\Gamma' = \cup_{j=1}^{J}\Gamma_j$, $\Gamma'' = \cup_{j=J+1}^{N+1}\Gamma_j$.
3. There exists at least a point on each component of $L^+ = E^+ \cap L$ and $L^- = E^- \cap L$, such that $\cos(\nu, n) \neq 0$ at the point, and $E^0 \cap L \in \{a_0, a_1, \dots, a_{K'}\}$, such that every component of L^+, L^- includes its initial point and does not include its terminal point; and $a_j \in \overline{L^+} \cap L^-$ $(0 \leq j \leq K')$, when the direction of ν at a_j is equal to the direction of L; and $a_j \in L^+ \cap \overline{L^-}$ $(0 \leq j \leq K')$, when the direction of ν at a_j is opposite to the direction of L; and $\cos(\nu, n)$ changes the sign once on the two components with the end point a_j $(0 \leq j \leq K')$; we may assume that $u(a_j) = b_j$, $j = 0, 1, \dots, K'$. The number

$$K = \frac{1}{2}(K_1 + \dots + K_{N+1}), \quad K_j = \Delta_{\Gamma_j} \arg\lambda(z), \quad j = 1, \dots, N + 1, \tag{1.6}$$

is called the index of Problem (P). We can choose $K' = 2K - 2N + J + 1$. In the following, we shall prove the next theorem. Now we prove the uniqueness of solutions for Problem (P) of (1.1).

Theorem 1.1. *Suppose that* (1.1) *satisfy Condition* (C). *Then Problem* (P) *for equation* (1.1) *with the condition that $A_3 = 0$ in D, $c_2 = 0$ on Γ and $b_j = 0 (j = 0, 1, \dots, K')$ has only the trivial solution.*

Proof. Let $u(z)$ be any solution of Problem (P) for equation (1.1) with $A_3 = 0$, $c_2 = 0$ on Γ and $b_j = 0 (j = 0, 1, \dots, K')$. From Condition (C), it is easily seen that

$u(z)$ is a solution of the following uniformly elliptic equation

$$u_{z\bar{z}} = \mathrm{Re}[Qu_{zz} + A_1u_z] + \hat{A}_2u, \quad |Q| \le q_0 < 1, \hat{A}_2 = A_2 + |u|^\sigma \ge 0 \quad \text{in } D, \tag{1.7}$$

and satisfies the boundary condition

$$\frac{\partial u}{\partial \nu} + 2c_1(z)u(z) = 0 \quad \text{on } \Gamma^*,\ u(a_j) = 0,\ j = 0, 1, \dots, K'. \tag{1.8}$$

Substitute the solution $u(z)$ into the coefficients of equation (1.7), we can find a solution $\Psi(z)$ of (1.7) satisfying the condition

$$\Psi(z) = 1 \quad \text{on } \Gamma,$$

thus the function $U(z) = u(z)/\Psi(z)$ is a solution of the equation

$$U_{z\bar{z}} = \mathrm{Re}[QU_{zz} + A_0U_z], \quad A_0 = -2(\log\Psi)_{\bar{z}} + 2Q(\log\Psi)_z + A_1, \tag{1.9}$$

satisfying the boundary conditions

$$\frac{\partial U}{\partial \nu} + c_1^*(z)U(z) = 0 \quad \text{on } \Gamma^*,\ U(a_j) = 0,\ j = 0, 1, \dots, K', \tag{1.10}$$

where $a_1^*(z) = c_1(z) + (\partial\Psi/\partial\nu)/\Psi(z)$, $c_1^*(z)\cos(\nu, n) \ge 0$ on Γ^*.

If $M = \max_{\overline{D}} U(z) > 0$ in D, then there exists a point $z^* \in \Gamma$ such that $M = U(z^*) = \max_{\overline{D}} U(z) > 0$. When $z^* \in \Gamma'$, noting that $\cos(\nu, n) \equiv 0$, $c_1(z) \equiv 0$, $\partial\Psi(z)/\partial\nu \equiv 0$ on Γ', we have $\partial U/\partial\nu \equiv 0$, $U(z) \equiv M$ on $\Gamma_j (1 \le j \le J')$, this contradicts the point conditions in (1.10). When $z^* \in \Gamma''$, if $\cos(\nu, n) > 0$ at z^*, from [3, Corollary 2.11, Chapter III], we have $\partial U/\partial\nu > 0$ at z^*, this contradicts (1.10) on Γ''. If $\cos(\nu, n) = 0$ and $c_1^*(z^*) \ne 0$ at z^*, then $\partial U/\partial\nu + c_1^*(z)U \ne 0$ at z^*, it is also impossible. Denote by L the longest curve of Γ including the point z^*, such that $\cos(\nu, n) = 0$ and $c^*(z) = 0$, thus $u(z) = M$ on L, from the point conditions in (1.10), any point of $\tilde{T} = \{z_0, z_1, \dots, z_{K'}\}$ cannot be an end point of L, then there exists a point $z' \in \Gamma''$, such that at z', $\cos(\nu, n) > 0$ (< 0), $\partial U/\partial n > 0$, $\cos(\nu, s) > 0$ (< 0), $\partial U/\partial s \ge 0$, or $\cos(\nu, n) < 0$ (> 0), $\partial U/\partial n > 0$, $\cos(\nu, s) > 0\,(< 0)$, $\partial U/\partial s \le 0$, hence

$$\frac{\partial U}{\partial \nu} = \cos(\nu, n)\frac{\partial U}{\partial n} + \cos(\nu, s)\frac{\partial U}{\partial s} > 0, \quad \text{or } < 0 \text{ at } z'$$

holds, where s is the tangent vector at $z' \in \Gamma''$, and then

$$\frac{\partial U}{\partial \nu} + c_1^*U > 0, \quad \text{or } \frac{\partial U}{\partial \nu} + c_1^*U < 0 \text{ at } z',$$

it is also impossible. This shows that $u(z)$ cannot attain its maximum M at a point $z^* \in \Gamma$. Similarly we can prove that $u(z)$ cannot attain its minimum at a point $z_* \in \Gamma$, hence $u(z) = 0$ on Γ, thus $u(z) = 0$ in $\overline{D}$. □

By a similar way as stated before, we can prove the uniqueness theorem of solutions of Problem (P) for equation (1.1) with $\sigma = 0$ as follows.

Corollary 1.2. *Suppose that*(1.1) *with* $\sigma = 0$ *satisfies Condition* (C) *and the following condition, for any real functions* $u_j(z) \in C^1(\overline{D}), V_j(z) \in L_{p_0,2}(\overline{D})(j = 1, 2)$*, the following equality holds:*

$$\begin{aligned} &F(z, u_1, u_{1z}, V_1) - F(z, u_2, u_{2z}, V_2) \\ &= \mathrm{Re}[\tilde{Q}(V_1 - V_2) + \tilde{A}_1(u_1 - u_2)_z] + \tilde{A}_2(u_1 - u_2) \quad \textit{in } D, \end{aligned}$$

where $|\tilde{Q}| \le q_0$ *in* D, $A_1, \tilde{A}_2 \in L_{p_0,2}(\overline{D})$. *Then Problem* (P) *for equation* (1.1) *has at most one solution.*

2. A priori estimates

We consider the nonlinear elliptic equations of second order

$$u_{z\bar{z}} - \operatorname{Re}[Qu_{zz} + A_1u_z] - \hat{A}_2u = A_3, \tag{2.1}$$

where $\hat{A}_2 = A_2 + |u|^\sigma$, σ is a positive number, and assume that the above equation satisfies Condition (C).

Theorem 2.1. *Let* (2.1) *satisfy Condition* (C). *Then any solution of Problem* (P) *for* (2.1) *satisfies the estimates*

$$\begin{gathered}\hat{C}_\beta[u, \overline{D}] = C^1_\beta[|u|^{\sigma+1}, \overline{D}] \le M_1, \quad \|u\|_{W^2_{p_0,2}(D)} \le M_1,\\ \hat{C}_\beta[u, \overline{D}] \le M_2(k_1 + k_2),\end{gathered} \tag{2.2}$$

in which $k = (k_0, k_1, k_2)$, β $(0 < \beta \le \alpha)$, $M_1 = M_1(q_0, p_0, \beta, k, D)$, $M_2 = M_2(q_0, p_0, \beta, k_0, p, D)$ *are non-negative constants.*

Proof. Using the reduction to absurdity, we shall prove that any solution $u(z)$ of Problem (P) satisfies the estimate

$$\hat{C}[u, \overline{D}] = C[|u|^{\sigma+1}, \overline{D}] + C[u_z, \overline{D}] \le M_3, \tag{2.3}$$

where $M_3 = M_3(q_0, p_0, \alpha, k, p, D)$ is a non-negative constant. Suppose that (2.3) is not true, then there exist sequences of coefficients $\{A^{(m)}_j\}$ $(j = 1, 2, 3)$, $\{Q^{(m)}\}$, $\{\lambda^{(m)}(z)\}$, $\{c^{(m)}_j\}$ $(j = 1, 2)$, $\{b^{(m)}_j\}(j = 0, 1, \dots, N_0)$, which satisfy the same conditions of Condition (C) and (1.6)–(1.8), such that $\{A^{(m)}_j\}$ $(j = 1, 2, 3)$, $\{Q^{(m)}\}$, $\{\lambda^{(m)}(z)\}$, $\{c^{(m)}_j\}$ $(j = 1, 2)$ and $\{b^{(m)}_j\}$ $(j = 0, 1, \dots, N_0)$ in $\overline{D}, \Gamma$ weakly converge or uniformly converge to $A^{(0)}_j$ $(j = 1, 2, 3)$, $Q^{(0)}$, $\lambda^{(0)}(z)$, $c^{(0)}_j(j = 1, 2)$, $b^{(0)}_j$ $(j = 0, 1, \dots, N_0)$, and the corresponding boundary-value problem

$$u_{z\bar{z}} - \operatorname{Re}[Q^{(m)}u_{zz} + A^{(m)}_1u_z] - \hat{A}^{(m)}_2u = A^{(m)}_3, \hat{A}^{(m)}_2 = A^{(m)}_2 + |u|^\sigma, \tag{2.4}$$

and

$$\frac{1}{2}\frac{\partial u}{\partial \nu} + a^{(m)}_1(z)u = c^{(m)}_2(z) \quad \text{on } \Gamma,\ u(a_j) = b_j,\ j = 0, 1, \dots, N_0, \tag{2.5}$$

have the solutions $\{u^{(m)}(z)\}$, where $\hat{C}[u^{(m)}(z), \overline{D}]\,(m = 1, 2, \dots)$ are unbounded. Hence we can choose a subsequence of $\{u^{(m)}(z)\}$ denoted by $\{u^{(m)}(z)\}$ again, such that $h_m = \hat{C}[u^{(m)}(z), \overline{G}] \to \infty$ as $m \to \infty$. We can assume $h_m \ge \max[k_1, k_2, 1]$. It is obvious that $\tilde{u}^{(m)}(z) = u^{(m)}(z)/h_m$ $(m = 1, 2, \dots)$ are solutions of the boundary-value problems

$$\tilde{u}_{z\bar{z}} - \operatorname{Re}[Q^{(m)}\tilde{u}_{zz} + A^{(m)}_1\tilde{u}_z] - \hat{A}^{(m)}_2\tilde{u} = A^{(m)}_3/h_m, \tag{2.6}$$

and

$$\frac{1}{2}\frac{\partial \tilde{u}}{\partial \nu} + c^{(m)}_1(z)\tilde{u} = c^{(m)}_2(z)/h_m \quad \text{on } \Gamma,\ \tilde{u}(a_j) = b^{(m)}_j,\ j = 0, 1, \dots, N_0. \tag{2.7}$$

We can see that the functions in the above equation and boundary conditions satisfy condition (C), (1.6)–(1.8), and

$$\begin{gathered} |u|^{\sigma+1}/h_m \le 1, \quad L_{p,2}[A_3^{(m)}/h_m, \overline{D}] \le 1, \\ |c_2^{(m)}/h_m| \le 1, \quad |b_j^{(m)}/h_m| \le 1, \quad j = 0, 1, \dots, N_0, \end{gathered} \tag{2.8}$$

hence from [3, Theorem 4.10, Chapter III], we obtain the estimate

$$\hat{C}_\beta[\tilde{u}^{(m)}(z), \overline{D}] \le M_4, \|\tilde{u}^{(m)}(z)\|_{W^2_{p_0,2}(D)} \le M_4,$$

in which $M_4 = M_4(q_0, p_0, \beta, k, D)$ is a non-negative constant. Thus from the sequence of functions $\{\tilde{u}^{(m)}(z)\}$, we can choose the subsequence denoted by $\{\tilde{u}^{(m)}(z)\}$, which converges uniformly to $\tilde{u}^{(0)}(z)$ in $\overline{D}$, and their partial derivatives $\tilde{u}_x^{(m)}, \tilde{u}_y^{(m)}$ in $\overline{D}$ are uniformly convergent and $\tilde{u}_{xx}^{(m)}, \tilde{u}_{yy}^{(m)}, \tilde{u}_{xy}^{(m)}$ in $\overline{D}$ weakly convergent. This shows $\tilde{u}_0(z)$ is a solution of the boundary-value problem

$$\tilde{u}_{0z\bar{z}} - \operatorname{Re}[Q^{(0)}\tilde{u}_{0zz} + A_1^{(0)}\tilde{u}_{0z}] - \hat{A}_2^{(0)}\tilde{u}_0 = 0, \tag{2.9}$$

and

$$\frac{1}{2}\frac{\partial \tilde{u}_0}{\partial \nu} + c_1^{(0)}(z)\tilde{u}_0 = 0 \quad \text{on } \Gamma, \ u_0(a_j) = 0, \ j = 0, 1, \dots, N_0. \tag{2.10}$$

We see that (2.9) possesses the condition $A_3^{(0)} = 0$ and (2.10) is the homogeneous boundary condition. On the basis of Theorem 1.1, the solution satisfies $\tilde{u}_0(z) = 0$. However, from $\hat{C}[\tilde{u}^{(m)}(z), \overline{D}] = 1$, we can derive that there exists a point $z^* \in \overline{D}$, such that $[|\tilde{u}_0(z)|^{\sigma+1} + |\tilde{u}_{0z}|]_{z=z^*} \ne 0$, which is impossible. This shows the first of two estimates in (2.2) is true. It is not difficult to verify the third estimate in (2.2). □

3. Solvability

By the above estimates and the Leray-Schauder theorem, we can prove the existence of solutions of Problem (P) for equation (1.1). We first introduce the nonlinear elliptic equation of second order

$$\begin{aligned} u_{z\bar{z}} = f_m(z, u, u_z, u_{zz}), \ & f_m(z, u, u_z, u_{zz}) \\ &= \operatorname{Re}[Q_m u_{zz} + A_{1m} u_z] + \hat{A}_{2m} u + A_3 \quad \text{in } D, \end{aligned} \tag{3.1}$$

with the coefficients

$$Q_m = \begin{cases} Q & \text{in } D_m \\ 0 & \text{in } \mathbb{C} \setminus D_m \end{cases} \qquad A_{jm} = \begin{cases} A_j & \text{in } D_m \\ 0 & \text{in } \mathbb{C} \setminus D_m \end{cases} \qquad j = 1, 3,$$

$$\hat{A}_{2m} = \begin{cases} \hat{A}_2 & \text{in } D_m \\ 0 & \text{in } \mathbb{C} \setminus D_m \end{cases}$$

where $D_m = \{z \in D : \operatorname{dist}(z, \Gamma \cup \{\infty\}) \ge 1/m\}$, m is a positive integer.

Theorem 3.1. *If* (3.1) *satisfies Condition* (C)*, and* $u(z)$ *is any solution of Problem* (P) *for equation* (3.1)*, then* $u(z)$ *can be expressed in the form*

$$u(z) = U(z) + \tilde{v}(z) = U(z) + \hat{v}(z) + v(z),$$

where $\tilde{v}(z) = \hat{v}(z) + v(z)$ *is a solution of* (3.1) *with the homogeneous Dirichlet boundary condition*

$$\tilde{v}(z) = 0 \quad \text{on } \partial D_0 = \{|z| = 1\}. \tag{3.2}$$

Here
$$v(z) = Hf_m = \frac{2}{\pi}\int\int_{D_0} \frac{f_m(1/\zeta)}{|\zeta|^4} \ln\Big|\frac{1-\zeta z}{\zeta}\Big| d\sigma_\zeta,$$
in which D_0 *is the image under the mapping* $z = 1/\zeta$, $U(z)$ *is a solution of the Dirichlet boundary-value problem for* $U_{z\bar{z}} = 0$ *in* D, *and* $U(z)$ *and* $\tilde{v}(z)$ *satisfy the estimates*
$$\hat{C}^1_\beta[U, \overline{D}] + \|U|_{W^2_{p_0,2}(D)} \le M_5, \quad \hat{C}^1_\beta[\tilde{v}, \overline{D_0}] + \|\tilde{v}\|_{W^2_{p_0,2}(D_0)} \le M_6, \tag{3.3}$$
where $\beta(>0)$, $M_j = M_j(q_0, p_0, \beta, k, D_m)$ $(j = 5, 6)$ *are non-negative constants.*

Proof. It is clear that the solution $u(z)$ can be expressed as before. On the basis of Theorem 2.1, it is easy to see that $\tilde{v}$ satisfies the second estimate in (3.3), and then we know that $U(z)$ satisfies the first estimate of (3.3). □

Theorem 3.2. *If* (1.1) *satisfies Condition* (C)*, then Problem* (P) *for equation* (1.1) *has a solution.*

Proof. To prove the existence of solutions of Problem (P) for (3.1) by using the Leray-Schauder theorem, we introduce the equation with the parameter $t \in [0, 1]$:
$$V_{z\bar{z}} = tf_m(z, u, u_z, (U + V)_{zz}) \quad \text{in } D. \tag{3.4}$$
Denote by B_M a bounded open set in the Banach space $B = \hat{W}^2_{p_0,2}(D_0) = \hat{C}^1_\beta(\overline{D_0}) \cap W^2_{p_0,2}(D_0)(0 < \beta \le \alpha)$, the elements of which are real functions $V(z)$ satisfying the inequalities
$$\hat{C}^1_\beta[V(z), \overline{D_0}] + \|V\|_{W^2_{p_0,2}(D_0)} < M_7 = M_6 + 1, \tag{3.5}$$
in which M_6 is a non-negative constants as stated in (3.3). We choose any function $V(z) \in \overline{B_M}$ and make an integral $v(z) = H\rho$ as follows:
$$v(z) = H\rho = \frac{2}{\pi}\int\int_{D_0} \frac{\rho(1/\zeta)}{|\zeta|^4} \log\left|\frac{1-\zeta z}{\zeta}\right| d\sigma_\zeta, \tag{3.6}$$
where $\rho(z) = V_{z\bar{z}}$. Next we find a solution $\hat{v}(z)$ of the boundary-value problem in D_0:
$$\hat{v}_{z\bar{z}} = 0 \quad \text{in } D_0, \tag{3.7}$$
$$\hat{v}(z) = -v(z) \quad \text{on } \partial D_0. \tag{3.8}$$
Denote $\tilde{v}(z) = \hat{v}(z) + v(z)$. Moreover we find a solution $U(z)$ of the boundary-value problem in D:
$$U_{z\bar{z}} = 0 \quad \text{in } D, \tag{3.9}$$
$$\frac{1}{2}\frac{\partial U}{\partial \nu} + c_1(z)U = c_2(z) - \frac{\partial \tilde{v}}{\partial \nu} - c_1(z)\tilde{v} \quad \text{on } \Gamma. \tag{3.10}$$
Now we discuss the equation
$$\tilde{V}_{z\bar{z}} = tf_m(z, u, u_z, U_{zz} + \tilde{v}_{zz}), \quad 0 \le t \le 1, \tag{3.11}$$
where $u(z) = U(z) + \tilde{v}(z)$. By Condition (C), the principle of contracting mapping and the results in Subsection 3.2, Problem (D) for the equation (3.11) in D_0 has a unique solution $\tilde{V}(z)$ with the boundary condition
$$\tilde{V}(z) = 0 \quad \text{on } \partial D_0.$$

Denote by $\tilde V = S(V,t)$ $(0 \le t \le 1)$ the mapping from V onto $\tilde V$. Furthermore, if $u(z)$ is a solution of Problem (P) in D for the equation

$$u_{z\bar z} = tf_m(z,u,u_z,u_{zz}), \quad 0 \le t \le 1, \tag{3.12}$$

then from Theorem 2.1, the solution $u(z)$ of Problem (P) for (3.12) satisfies (2.2), consequently $\tilde V(z) = u(z) - U(z) \in B_M$. Set $B_0 = B_M \times [0,1]$. In the following, we shall verify that the mapping $\tilde V = S(V,t)$ satisfies the following three conditions of Leray-Schauder theorem:

1. For every $t \in [0,1]$, $\tilde V = S(V,t)$ continuously maps the Banach space B into itself, and is completely continuous in B_M. Besides, for every function $V(z) \in \overline{B_M}$, $S(V,t)$ is uniformly continuous with respect to $t \in [0,1]$.

In fact, we arbitrarily choose $V_n(z) \in \overline{B_M}$, $n = 1,2,\dots$. It is clear that from $\{V_n(z)\}$ there exists a subsequence $\{V_{n_k}(z)\}$, such that $\{V_{n_k}(z)\}$, $\{V_{n_kz}(z)\}$ and corresponding functions $\{U_{n_k}(z)\}$, $\{U_{n_kz}(z)\}$ uniformly converge to $V_0(z)$, $V_{0z}(z)$, $U_0(z)$, $U_{0z}(z)$ in $\overline D$ respectively. We can find a solution $\tilde V_0(z)$ of Problem (D) for the equation

$$\tilde V_{0z\bar z} = tf_m(z,u_0,u_{0z},U_{0zz} + \tilde v_{0zz}), \quad 0 \le t \le 1.$$

Noting that $u_{n_kz\bar z} = U_{n_kz\bar z} + \tilde v_{n_kz\bar z}$, from $\tilde V_{n_k} = S(V_{n_k},t)$ and $\tilde V_0 = S(V_0,t)$, we have

$$\begin{aligned}(\tilde V_{n_k} - \tilde V_0)_{z\bar z} = t[&f_m(z,u_{n_k},u_{n_kz},U_{n_kzz} + \tilde v_{n_kzz})\\ &- f_m(z,u_{n_k},u_{n_kz},U_{n_kzz} + \tilde v_{0zz}) + C_{n_k}(z)], \quad 0 \le t \le 1,\end{aligned}$$

where

$$C_{n_k} = f_m(z,u_{n_k},u_{n_kz},U_{n_kzz} + \tilde v_{0zz}) - f_m(z,u_0,u_0,U_{0zz} + \tilde v_{0zz}),\ z \in D_0.$$

Similarly to [6, (2.4.18), Chapter 2], we obtain

$$L_{p_0,2}[C_{n_k},\overline{D_0}] \to 0 \quad \text{as } k \to \infty.$$

Similarly to (2.2)–(2.10), we obtain

$$\|\tilde V_{n_k} - \tilde V_0\|_{\hat W^2_{p_0,2}(D_0)} \le L_{p_0,2}[C_{n_k},\overline{D_0}]/[1-q_0], \tag{3.13}$$

where $q_0 < 1$. It is easy to show that $\|\tilde V_{n_k} - \tilde V_0\|_{\hat W^2_{p_0,2}(D)} \to 0$ as $k \to \infty$. Moreover, from Theorem 2.1, we can verify that from $\{\tilde V_{n_k}(z) - \tilde V_0(z)\}$, there exists a subsequence, denoted by $\{\tilde V_{n_k}(z) - \tilde V_0(z)\}$ again, such that $C^1_\beta[\tilde V_{n_k} - \tilde V_0, \overline{D_0}] \to 0$ as $k \to \infty$. This shows that the complete continuity of $\tilde V = S(V,t)(0 \le t \le 1)$ in $\overline{B_M}$. By using a similar method, we can prove that $\tilde V = S(V,t)(0 \le t \le 1)$ continuously maps $\overline{B_M}$ into B, and $\tilde V = S(V,t)$ is uniformly continuous with respect to $t \in [0,1]$ for $V \in \overline{B_M}$.

2. For $t = 0$, from Theorem 2.1 and (3.5). It is clear that $\tilde V(z) = S(V,0) \in B_M$.

3. From Theorem 2.1 and (3.5), we see that $\tilde V = S(V,t)(0 \le t \le 1)$ does not have a solution $\tilde V(z)$ on the boundary $\partial B_M = \overline{B_M} \setminus B_M$.

Hence by the Leray-Schauder theorem, we know that Problem (P) for the equation (3.4) with $t = 1$, namely (3.1) has a solution $u(z) = U(z) + \tilde v(z) = U(z) + \hat v(z) + v(z) \in B_M$. □

Theorem 3.3. *Under the conditions in Theorem 3.1, Problem* (P) *for equation* (1.1) *has a solution.*

Proof. By Theorems 2.1 and 3.2, Problem (P) for the equation (3.1) possesses a solution $u_m(z)$, and the solution $u_m(z)$ of Problem (P) for (3.1) satisfies the estimate (2.2), where $m = 1, 2, \dots$. Thus, we can choose a subsequence $\{u_{m_k}(z)\}$, such that $\{u_{m_k}(z)\}$, $\{u_{m_k z}(z)\}$ in $\overline{D}$ uniformly converge to $u_0(z)$, $u_{0z}(z)$ respectively. Obviously, $u_0(z)$ satisfies the boundary conditions of Problem (P) for equation (1.1). $\square$

We can choose $K' = 2K - 2N + J + 1$. By using the similar method as Section 1-3, we can prove the following theorem.

Theorem 3.4. *Under the above conditions, Problem* (P) *for the equation* (1.1) *has a solution. Moreover we have the solvability result of Problem* (P) *for* (1.1) *with the boundary condition*

$$\frac{1}{2}\frac{\partial u}{\partial \nu} + c_1(z)u(z) = c_2(z), \quad z \in \Gamma.$$

When $K \geq N - 1/2$, the general solution includes $K' + 1 = 2K - 2N + 2 + J$ arbitrary real constants.

References

[1] A.-V. Bitsadze; *Some classes of partial differential equations,* Gordon and Breach, New York, (1988).

[2] I.-N. Vekua; *Generalized analytic functions,* Pergamon, Oxford, (1962).

[3] G.-C, Wen, H. Begehr; *boundary-value problems for elliptic equations and systems,* Longman Scientific and Technical Company, Harlow, (1990).

[4] G.-C, Wen; *Conformal mappings and boundary-value problems,* Translations of Mathematics Monographs 106, Amer. Math. Soc., Providence, RI, (1992).

[5] G.-C, Wen, D.-C, Chen, Z.-L. Xu; *Nonlinear complex analysis and its applications,* Mathematics Monograph Series 1, Science Press, Beijing, (2008).

[6] G.-C, Wen; *Recent progress in theory and applications of modern complex analysis,* Science Press, Beijing, (2010).

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