

FACTORIZATION OF SECOND-ORDER STRICTLY HYPERBOLIC OPERATORS WITH NON-SMOOTH COEFFICIENTS AND MICROLOCAL DIAGONALIZATION

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ABSTRACT. We study strictly hyperbolic partial differential operators of second-order with non-smooth coefficients. After modeling them as semiclassical Colombeau equations of log-type we provide a factorization procedure on some time-space-frequency domain. As a result the operator is written as a product of two semiclassical first-order constituents of log-type which approximates the modelled operator microlocally at infinite points. We then present a diagonalization method so that microlocally at infinity the governing equation is equal to a coupled system of two semiclassical first-order strictly hyperbolic pseudodifferential equations. Furthermore we compute the coupling effect. We close with some remarks on the results and future directions.

1. INTRODUCTION

When studying strictly hyperbolic partial differential equations with generalized coefficients the results commonly depend on the appropriate choice of the asymptotic scale of the regularizing parameter. In this paper we consider certain types of second order partial differential equations with generalized coefficients of the form

$$LU = F \tag{1.1}$$

where U, F are Colombeau generalized functions (See section 2 for the precise asymptotic behavior in this case.) in $\mathcal{G}_{2,2}$ on \mathbb{R}^{n+1} and on the level of representatives the operator L acts as

$$(u_\varepsilon)_\varepsilon \mapsto (L_\varepsilon(x, z, D_t, D_x, D_z)u_\varepsilon)_\varepsilon \quad \forall (u_\varepsilon)_\varepsilon \in \mathcal{M}_{H^\infty}.$$

In detail the operator L_ε is considered to be of the form

$$L_\varepsilon := \partial_z^2 + \sum_{j=1}^{n-1} b_{j,\varepsilon}(x, z) \partial_{x_j}^2 - c_\varepsilon(x, z) \partial_t^2. \tag{1.2}$$

Here the coefficient $c_\varepsilon(x, z)$ is of log-type up to order $r \in \mathbb{N}$, i.e. for some constant $C > 0$ we have $\|\partial^\alpha c_\varepsilon\|_{L^\infty} = \mathcal{O}(\log^{|\alpha|/r}(C/\varepsilon))$ as $\varepsilon \rightarrow 0$, and strictly non-zero in the sense that there exists $\varepsilon_1 \in (0, 1]$ such that $\inf_{y \in \mathbb{R}^n} |c_\varepsilon(x, z)| \geq C$ for all $\varepsilon \in (0, \varepsilon_1]$

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and some constant $C > 0$ independent of ε . Also $b_{j,\varepsilon}$ is of log-type up to order r and strictly non-zero. Moreover we assume that $c_\varepsilon \rightarrow c$ and $b_{j,\varepsilon} \rightarrow b_j$ in the Hölder space $\mathcal{C}^{0,\mu}(\mathbb{R}^n)$ with exponent $\mu \in (0, 1)$ as $\varepsilon \rightarrow 0$.

The aim of this paper is to provide a symbolic calculus to explain a diagonalization with respect to the parameter z of the equation in (1.1). To do so we will reduce the diagonalization problem to a factorization theorem which in the case of smooth coefficients can be found in [29, Appendix II], [25, Chapter 23]. We note that in these references the results are only valid for operators with simple characteristics on the phase space with the zero section excluded. This is different to our approach as we are interested in factorizations with respect to the parameter z .

For recent contributions to a related topic we refer the reader to the work by Garetto and Oberguggenberger [20]. There the authors established existence and regularity results for solutions to strictly hyperbolic systems with Colombeau coefficients using symmetrisation up to regularizing errors. In detail, they proved existence of generalized solutions in the case of slow scale coefficients and, in addition they showed regularity in the case of logarithmic slow scale coefficients. Note that a net $r_\varepsilon \in \mathbb{R}^{(0,1]}$ is a logarithmic slow scale net if $|r_\varepsilon| = \mathcal{O}(\log^p(1/\varepsilon))$ for all $p > 0$ as $\varepsilon \rightarrow 0$ and a generalized coefficient $(a_\varepsilon)_\varepsilon$ is said to be logarithmic slow scale regular if for all $\alpha \in \mathbb{N}^n$ there exists a logarithmic slow scale net r_ε such that

$$\|\partial^\alpha a_\varepsilon\|_{L^\infty(\mathbb{R}^n)} = \mathcal{O}(r_\varepsilon) \quad \varepsilon \rightarrow 0.$$

Again, the square roots of the principal symbol of the operator are assumed to be simple on the phase space without the zero section in this case.

So when studying an equation of the form (1.1) with coefficients that satisfy strongly positive logarithmic slow scale estimates one can adapt the results in [17, 29] to obtain a diagonalization in some microlocal subregion of the phase space and microlocal regularity has to be understood in a $\mathcal{G}_{2,2}^\infty$ sense. Recall that $\mathcal{G}_{2,2}^\infty$ is the space of regular generalized functions in $\mathcal{G}_{2,2}$ and is characterized by uniform ε -growth in all derivatives. This is a generalization of the results in the smooth coefficient case which will be briefly explained in the beginning of Subsection 1.1. Since in the Colombeau framework the evolution behavior of propagating singularities is not yet sufficiently understood it is not clear how one can derive well-posed approximated Cauchy problems from the resulting microlocal first-order equations due to the diagonalization. For the theory for first-order hyperbolic pseudodifferential equations with generalized symbols we refer to [26, 33, 20].

However, to analyze operators as in (1.2) with less regular coefficients as in [20] we try a different approach. To overcome the necessity of the logarithmic slow scale assumption (e.g. construction of an approximative inverse) we associate to the operator L_ε in (1.2) a semiclassical modification $L_{\psi,\varepsilon}$ such that $L_{\psi,\varepsilon} = \varepsilon^2 L_\varepsilon$. Here the $\psi = \psi(\varepsilon)$ refers to the phase function in which the semiclassical scaling is kept to be retained and depend on the regularizing parameter $\varepsilon \in (0, 1]$. The detailed explanation of the correspondence between the operators L and L_ψ is given in Subsection 2.1. Then instead of working with the equation (1.1) we consider the corresponding semiclassical problem

$$L_\psi U = F \tag{1.3}$$

where U, F are generalized functions in $\mathcal{G}_{2,2}$ and the operator $L_\psi : \mathcal{G}_{2,2} \rightarrow \mathcal{G}_{2,2}$ acts on the level of representatives as

$$(u_\varepsilon)_\varepsilon \mapsto (L_{\psi,\varepsilon}(x, z, D_t, D_x, D_z)u_\varepsilon)_\varepsilon \quad \forall (u_\varepsilon)_\varepsilon \in \mathcal{M}_{H^\infty}.$$

Due to the additional ε -dependent semiclassical scaling parameter we then apply the theory of generalized semiclassical pseudodifferential operators which we call the ψ -pseudodifferential operators in the following for short. This enables us to state a factorization theorem for L_ψ in terms of two first-order ψ -pseudodifferential operators with respect to the parameter z modulo two different types of error operators. As already mentioned above the factorization is only valid when imposing certain restrictions on the underlying time-space-frequency domain which we left ε -independent. In detail the error operators are characterized by a semiclassical negligibility condition outside a compact set. Since global semiclassical negligible operators are easy to handle as they map moderate nets to negligible ones the interesting case is to understand the compactly supported error operators. But, due to the semiclassical scaling, worse ε -asymptotics are introduced when studying boundedness of such compactly supported operators. The basic notions and a general calculus of ψ -pseudodifferential operators can be found in Sections 2 and 3, the factorization procedure is explained in Section 4.

To eliminate the remainder terms produced in the factorization procedure we then proceed in Section 5 by presenting an adapted notion of microlocal regularity which essentially corresponds to the semiclassical wave front set at infinite points within the Colombeau framework. This allows us to overcome the difficulties that arise due to the compactly supported error operators occurring in the factorization. Concepts of the wave front set in Colombeau's theory have been explored in [31, 17, 18, 14]. For an introduction to semiclassical analysis and microlocalization we refer to [30, 11, 23, 1]. Again we remark that unlike to the case of smooth coefficients we lose the property that singularities are propagating on geometrically determined phase space trajectories.

In Section 6 we then combine the factorization with our notion of microlocal regularity in order to describe the desired microlocal diagonalization method. As a result we obtain a coupled system of two first-order ψ -pseudodifferential equations that approximate equation (1.3) microlocally at infinite points on an adequate subdomain of the phase space.

1.1. The smooth background case. A particular example of a strictly hyperbolic partial differential equation of second order is the wave equation describing phenomena such as propagation of elastic waves and vibrations. However, in this subsection of the introduction we give a motivation for the present survey and is devoted to one-way wave propagation in inhomogeneous acoustic media in the case of smooth background coefficients. For more information on the relevance in seismic imaging and migration models we refer the reader to [6, 4, 39, 40].

To start with a real life problem we may ask whether and how one can establish a tap-proof communication link between a source and a receiver in an underwater environment. In the following we analyze how the correlation between the source and the receiver location can help to detect a desired information in downward continuation problems. The mathematical reformulation of such problems corresponds to initial value problems of second-order partial differential equations with a space-like direction as evolution parameter and is in general not well-defined.

To overcome the ill-posedness the concept of wave-field decomposition enables us to rewrite the full wave equation into a coupled system of approximative one-way wave equations. Using microlocal techniques the system of one-way equations can then be decoupled assuming that the wave field propagates in one direction within

certain propagation angles and prohibits propagation in the opposite direction. Therefore the approximation allows to determine the high-frequency components of the solution as they propagate along curved trajectories in the phase space.

In the following we recall the crucial statements given by Stolk for the directional wave field decomposition in the present of smooth background data [37, 38]. For a discussion on the inverse scattering problem with simultaneous consideration of possible reflection data see for example [36, 40, 9] and the references therein. Also we refer to [34, 35] for investigations of the Cauchy problem of first-order pseudodifferential equations and to [43] for numerical implementations.

As in [37] our basic type of model will be the wave equation for inhomogeneous acoustic media in n -dimensions, $n \geq 2$. Since we are interested in approximative one-way equations we allocate the vertical direction z , which we call the depth, the lateral directions are denoted by x . The medium itself is described by the wave speed $c = c(x, z)$ and the fluid density $\rho = \rho(x, z)$. Also we let $U = U(t, x, z)$ denote the acoustic wave field and $F = F(t, x, z)$ a source which usually describes the initiation of an acoustic wave. Then the acoustic wave equation is given by

$$PU := \left(-\frac{1}{\rho} \frac{1}{c^2} \partial_t^2 + \sum_{j=1}^{n-1} \partial_{x_j} \frac{1}{\rho} \partial_{x_j} + \partial_z \frac{1}{\rho} \partial_z \right) U = F \quad (1.4)$$

where U and F are in the distribution space $\mathcal{D}'(\mathbb{R}^{n+1})$. Further we assume smooth background data, i.e. the wave speed and the density are functions in $C^\infty(\mathbb{R}^n)$, which shall also satisfy the boundedness conditions $0 < c(x, z), \rho(x, z) < \infty$ for all $(x, z) \in \mathbb{R}^{n-1} \times \mathbb{R}$.

In the case of slow varying media and the presence of a source function we follow Stolk's approach and restrict the analysis to a microlocal region I_{Θ_2} that is given by

$$I_{\Theta_2} := \{(t, x, z, \tau, \xi, \zeta) \mid (x, z, \tau, \xi) \in I'_{\Theta_2}, |\zeta| \leq C|\tau|\}$$

$$I'_{\Theta_2} := \{(x, z, \tau, \xi) \mid \tau \neq 0, |\xi| \leq \sin \Theta_2 |c(x, z)^{-1} \tau|\}$$

for some angle $\Theta_2 \in (0, \pi/2)$. An illustration of the first of these domains is presented in Figure 1; see [34].

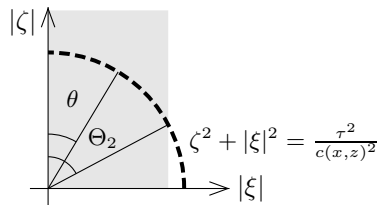


FIGURE 1. The shaded area corresponds to I_{Θ_2} at a given (t, x, z) and frequency τ . The bold dotted line designates the characteristic set $\text{Char}(P)$ and θ is the propagation angle of the singularities

Then under the assumption that the wave front set of U is contained in I_{Θ_2} , one can rewrite a microlocal equivalent model to (1.4) in terms of a coupled system of one-way wave equations with the depth as evolution parameter.

The main result is then the following. The equation

$$PU = F \quad \text{microlocally on } I_{\Theta_2}$$

is equivalent to the system of two first-order strictly hyperbolic partial differential equations of the form

$$P_{0,\pm}u_{\pm} := (\partial_z - iB_{\pm}(x, z, D_t, D_x))u_{\pm} = f_{\pm} \quad \text{microlocally on } I_{\Theta_2}, \quad (1.5)$$

where the plus-minus sign refers to downward and upward migration. Here u_{\pm}, f_{\pm} are distributions and B_{\pm} are pseudodifferential operators of order 1 that can be chosen selfadjoint. Furthermore the coupling effect of the counterpropagating constituents u_{\pm}, f_{\pm} and the original data U, F can be computed by a Douglis Nirenberg elliptic pseudodifferential operator transfer matrix.

To model wave propagation in the downward direction in the source-free case one decouples (1.5) microlocally and we are interested in the solutions to the problem

$$\begin{aligned} PU &= 0 \quad z > z_0 \\ \text{WF}(U) \cap \{z = z_0, \zeta/\tau > 0\} &= \emptyset \end{aligned}$$

for some initial depth $z_0 \in \mathbb{R}$ such that $U|_{z_0}$ is well-defined.

Using the geometry of the microlocal region we let the set $J_{\Theta,+}(z_0) \subset T^*\mathbb{R}^{n+1} \setminus 0$ consisting of points $(t, x, z, \tau, \xi, \zeta)$ so that the bicharacteristics $(t(z), x(z), \tau(z), \xi(z))$ corresponding to B_+ , parametrized by z , and with propagation angle $\theta(z)$, pass through $(t_0, x_0, \tau_0, \xi_0)$ at initial depth $z = z_0$ and the points $(x(z), z, \tau(z), \xi(z))$ remain in I'_{Θ} for all $z \in [0, Z]$. So $J_{\Theta,+}(z_0)$ is the set that can be reached from the initial depth $z = z_0$ while staying in $I_{\Theta,+}$ and the propagation angle $\theta(z)$ along the bicharacteristic is always smaller than Θ (cf. Figure 1).

Since the equation for downward migration in (1.5) holds only microlocally, one obtains approximative solutions when studying a perturbation P_+ of the operator $P_{0,+}$ including an additional damping term $C = C(x, z, D_t, D_x)$ which vanishes in I_{Θ_1} for some fixed positive angle $\Theta_1 < \Theta_2$ and suppresses singularities outside I_{Θ_2} . Also note that u_- is vanishing on $I_{\Theta_2} \cap \{z = z_0\}$ so that the perturbed initial value problem for downward propagation now reads

$$\begin{aligned} P_+u_+ &= (\partial_z - iB_+ + C)u_+ = 0 \quad \mathbb{R}^n \times (z_0, Z) \\ u_+|_{z_0} &= Q_+^{-1}(z_0)U|_{z_0} \end{aligned}$$

where Q_+^{-1} is the essential component of the transfer matrix mentioned above. As a result the solution to the initial value problem for the perturbed first-order pseudodifferential equation can be related to that of $P_{0,+}u_+ = 0$ using the geometry in the region $J_{\Theta,+}$. In detail one can recover the high frequency part of the original wave field on some subset $J_{\Theta_1,+}$ of the phase space that can be reached from an initial given depth z_0 while staying in I_{Θ_1} for propagation angles $\theta(z) < \Theta_1 < \Theta_2$.

2. BASIC NOTIONS

In this section we specify the basic notions that are needed for our constructions. As the problem is treated within the framework of Colombeau algebras we refer to the literature [7, 32, 31, 22, 16, 15] for a systematic treatment in this field.

One of the main objects in our setting are Colombeau generalized functions based on L^2 -norm estimates which were first introduced in [3]. The elements in

this algebra are given by equivalence classes $u := [(u_\varepsilon)_{\varepsilon \in (0,1]}]$ of nets of regularizing functions u_ε in the Sobolev space $H^\infty = \bigcap_{k \in \mathbb{Z}} H^k$ corresponding to certain asymptotic seminorm estimates. More precisely, we denote by \mathcal{M}_{H^∞} the nets of moderate growth whose elements are characterized by the property

$$\forall \alpha \in \mathbb{N}^n \exists N \in \mathbb{N} : \|\partial^\alpha u_\varepsilon\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(\varepsilon^{-N}) \quad \text{as } \varepsilon \rightarrow 0.$$

Negligible nets are denoted by \mathcal{N}_{H^∞} and are nets in \mathcal{M}_{H^∞} whose elements satisfy the additional condition:

$$\forall q \in \mathbb{N} : \|u_\varepsilon\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(\varepsilon^q) \quad \text{as } \varepsilon \rightarrow 0.$$

For properties of negligible nets see [12, Proposition 3.4]. Then the algebra of generalized functions based on L^2 -norm estimates is defined as the factor space $\mathcal{G}_{H^\infty} = \mathcal{M}_{H^\infty}/\mathcal{N}_{H^\infty}$ for which we continue to write $\mathcal{G}_{2,2}(\mathbb{R}^n)$ as in [3]. For simplicity, we shall also use the notation $(u_\varepsilon)_\varepsilon$ instead of $(u_\varepsilon)_{\varepsilon \in (0,1]}$ throughout the paper.

Using [3, Theorem 2.7] we first note that the distributions $H^{-\infty} = \bigcup_{k \in \mathbb{Z}} H^k$ are linearly embedded in $\mathcal{G}_{2,2}(\mathbb{R}^n)$ by convolution with a mollifier $\varphi_\varepsilon(x) = \varepsilon^{-n} \varphi(\varepsilon^{-1}x)$ where $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is a Schwartz function such that

$$\int \varphi(x) dx = 1, \quad \int x^\alpha \varphi(x) dx = 0 \quad \text{for all } |\alpha| \geq 1. \quad (2.1)$$

Further, by the same result, $H^\infty(\mathbb{R}^n)$ is embedded as a subalgebra of $\mathcal{G}_{2,2}(\mathbb{R}^n)$.

More generally, we introduce Colombeau algebras based on a locally convex vector space E topologized through a family of seminorms $\{p_i\}_{i \in I}$ as in [13, Section 3], [19, Section 1]. Again we call the elements of

$$\mathcal{M}_E = \{(u_\varepsilon)_\varepsilon \in E^{(0,1]} \mid \forall i \in I \exists N \in \mathbb{N} : p_i(u_\varepsilon) = \mathcal{O}(\varepsilon^{-N}) \text{ as } \varepsilon \rightarrow 0\}$$

E -moderate and the set

$$\mathcal{N}_E = \{(u_\varepsilon)_\varepsilon \in E^{(0,1]} \mid \forall i \in I \forall q \in \mathbb{N} : p_i(u_\varepsilon) = \mathcal{O}(\varepsilon^q) \text{ as } \varepsilon \rightarrow 0\}$$

is said to be E -negligible. Then \mathcal{N}_E is an ideal in \mathcal{M}_E and the space of Colombeau algebra based on E is defined by the factor space $\mathcal{G}_E = \mathcal{M}_E/\mathcal{N}_E$ and possesses the structure of a $\tilde{\mathbb{C}}$ -module. Here $\tilde{\mathbb{C}}$ denotes the ring of complex generalized numbers which is obtained by setting $E = \mathbb{C}$, this means $\tilde{\mathbb{C}} := \mathcal{M}_{\mathbb{C}}/\mathcal{N}_{\mathbb{C}}$. For more information about the topological structure of $\tilde{\mathbb{C}}$ we refer to [13].

To realize the log-type up to order $r \in \mathbb{N}$ conditions on the coefficients of the model operator (1.2) a rescaling in the mollification is required, see [27, 32]. In detail, we are going to consider regularized functions where the asymptotic growth is estimated in powers of $\omega_\varepsilon := (\log(C/\varepsilon))^{1/r}$ for some $r \in \mathbb{N}$ and a constant $C > 0$ such that $(\omega_\varepsilon)_\varepsilon$ is strongly positive. Here the exponent $1/r$ is not essential in the further considerations but we give some remarks in this regard in Section 6. In this case the regularization is obtained by convolution with the logarithmically scaled mollifier $\varphi_{\omega_\varepsilon^{-1}}(\cdot) := \omega_\varepsilon^n \varphi(\omega_\varepsilon \cdot)$ with $\varphi \in \mathcal{S}(\mathbb{R}^n)$ as in (2.1).

Further terminology. To establish a factorization theorem for generalized semiclassical partial differential operator as in (1.3) we will have to give a meaning to the square root of such operators. For this reason we introduce generalized semiclassical pseudodifferential operators which are characterized by symbols with respect to a certain phase function. Because the phase function is of simple type we shall initially discuss the main notions of generalized symbols which satisfy asymptotic

growth conditions with respect to $\omega_\varepsilon = (\log(C/\varepsilon))^{1/r}$ as already mentioned above. As usual, we use the notation $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$.

First, we let $m \in \mathbb{R}$ and denote by $S^m = S^m(\mathbb{R}^n \times \mathbb{R}^n)$ the set of symbols of order m as introduced by Hörmander in [25, Definition 18.1.1.]. Furthermore the space $S_{\text{hg}}^m = S_{\text{hg}}^m(\mathbb{R}^n \times (\mathbb{R}^n \setminus 0))$ consists of homogeneous functions $a \in C^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus 0))$ of order m ; i.e., $a(x, \lambda\xi) = \lambda^m a(x, \xi)$ for all $\lambda > 0$ and $\xi \neq 0$, such that

$$\forall \alpha, \beta \in \mathbb{N}^n \exists C > 0 : |\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \leq C |\xi|^{m-|\alpha|} \quad \text{for } (x, \xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus 0).$$

Since the symbol class S^m satisfies global estimates we remark that \mathcal{M}_{S^m} is different to the symbol classes in [19, Section 1.4]. We note that \mathcal{M}_{S^m} is the space \mathcal{M}_E by setting $E = S^m$. In the following we will typically encounter subspaces of \mathcal{M}_{S^m} subjected to two different asymptotic scales. This extends the definition of \mathcal{M}_{S^m} and \mathcal{N}_{S^m} in the following way:

Definition 2.1. Let ν be a non-negative real number and $l, k \in \mathbb{R}$. For $m \in \mathbb{R}$ we let $(a_\varepsilon)_\varepsilon$ be a family of symbols $a_\varepsilon \in S^m$. We then say that $(a_\varepsilon)_\varepsilon$ is in the generalized symbol class $\mathcal{M}_{S^{m,k}}^{(\nu,l)}$ if and only if

$$\begin{aligned} & \exists \eta \in (0, 1] \forall \alpha, \beta \in \mathbb{N}^n \exists C > 0 \forall \varepsilon \in (0, \eta] : \\ q_{\alpha,\beta}^{(m)}(a_\varepsilon) & := \sup_{(x,\xi) \in \mathbb{R}^{2n}} |\partial_\xi^\alpha \partial_x^\beta a_\varepsilon(x, \xi)| \langle \xi \rangle^{-m+|\alpha|} \leq C \varepsilon^k \omega_\varepsilon^{\nu|\beta|+l}. \end{aligned} \quad (2.2)$$

We call m, k and (ν, l) the order, the growth type and the log-type respectively of $\mathcal{M}_{S^{m,k}}^{(\nu,l)}$.

This definition is similar to that for generalized symbols $S_{\rho,\delta,\omega}^{m,\mu}$ in [16, Definition 4.1]. Also, since $(\omega_\varepsilon)_\varepsilon$ is a strongly positive slow scale net we note that a generalized symbol $(a_\varepsilon)_\varepsilon$ in $\mathcal{M}_{S^{m,k}}^{(\nu,l)}$ can always be estimated in the following way:

$$\begin{aligned} & \exists \eta \in (0, 1] \forall \alpha, \beta \in \mathbb{N}^n \exists C > 0 \text{ such that} \\ q_{\alpha,\beta}^{(m)}(a_\varepsilon) & \leq C \varepsilon^{k-1} \quad \forall \varepsilon \in (0, \eta]. \end{aligned} \quad (2.3)$$

Later in Lemma 6.1 we give the construction scheme for approximative inverse operators and the advantage of using (2.2) rather than (2.3) will be clear then. Hereafter we typically work in spaces $\mathcal{M}_{S^{m,k}}^{(\nu,l)}$ with $\nu \in \{0, 1\}$. Analogously to Definition 2.1 one introduces the space $\mathcal{M}_{S_{\text{hg}}^{m,k}}^{(\nu,l)}(\mathbb{R}^n \times (\mathbb{R}^n \setminus 0))$.

Furthermore the notion of negligibility in $\mathcal{M}_{S^{m,k}}^{(\nu,l)}$ is defined as follows:

Definition 2.2. An element of $\mathcal{M}_{S^{m,k}}^{(\nu,l)}$ is said to be negligible, denoted by \mathcal{N}_{S^m} , if the following condition is fulfilled:

$$\begin{aligned} & \exists \eta \in (0, 1] \forall q \in \mathbb{N} \forall \alpha, \beta \in \mathbb{N}^n \exists C > 0 \text{ such that} \\ q_{\alpha,\beta}^{(m)}(a_\varepsilon) & \leq C \varepsilon^q \quad \forall \varepsilon \in (0, \eta]. \end{aligned}$$

To give an example, let $P(x, D_x) = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha$ be a partial differential operator with bounded and measurable coefficients. Then the logarithmically scaled regularized coefficients $a_{\alpha,\varepsilon} := a_\alpha * \varphi_{\omega_\varepsilon^{-1}}$ satisfy the estimate

$$\|\partial^\alpha a_{\alpha,\varepsilon}\|_{L^\infty(\mathbb{R}^n)} = \mathcal{O}(\omega_\varepsilon^{|\alpha|}) = \mathcal{O}(\log^{|\alpha|/r}(C/\varepsilon)).$$

Therefore $(a_{\alpha,\varepsilon})_\varepsilon$ is of log-type up to order r , that is $\|\partial^\alpha a_{\alpha,\varepsilon}\|_{L^\infty(\mathbb{R}^n)} = \mathcal{O}(\log(C/\varepsilon))$ for all $|\alpha| \leq r$, but it is not logarithmic slow scale regular. Recall that the asymptotic norm estimate (of any order) of a logarithmic slow scale coefficient has a bound $\mathcal{O}(r_\varepsilon)$ where $|r_\varepsilon| = \mathcal{O}(\log^p(1/\varepsilon))$ for all $p > 0$ as ε tends to 0. In the same manner we obtain for the logarithmically scaled regularization of the operator a symbol $(p_\varepsilon)_\varepsilon$ where $p_\varepsilon(x, \xi) := (p(\cdot, \xi) * \varphi_{\omega_\varepsilon^{-1}})(x)$ which is of class $\mathcal{M}_{S^{m,0}}^{(1,0)}$ and of log-type up to order (∞, r) , cf. [26, Remark 3.2].

Also we will use the following microlocal symbol classes:

Definition 2.3. Let $U \subset \mathbb{R}^n \times \mathbb{R}^n$ be open and conic with respect to the second variable. We say that a generalized symbol $(a_\varepsilon)_\varepsilon$ is in $\mathcal{M}_{S^{m,k}}^{(\nu,l)}(U)$ if $a_\varepsilon \in \mathcal{C}^\infty(U)$ for fixed $\varepsilon \in (0, 1]$ and there is a constant $K > 0$ independent of ε such that:

$$\exists \eta \in (0, 1] \forall \alpha, \beta \in \mathbb{N}^n \exists C > 0 \text{ for which}$$

$$|\partial_\xi^\alpha \partial_x^\beta a_\varepsilon(x, \xi)| \leq C \varepsilon^k \omega_\varepsilon^{|\beta|+l} \langle \xi \rangle^{m-|\alpha|} \quad \text{for } (x, \xi) \in V_K, \varepsilon \in (0, \eta],$$

where $V_K := \{(x, \xi) \in U \mid d(\xi, \zeta) \geq K, \zeta \in \partial \text{pr}_2(U) \neq \emptyset\}$. We observe that, if $(x, \xi) \in V_K$ then $(x, \lambda \xi) \in V_K$ for all $\lambda \geq 1$.

Note that Definition 2.1 is equivalent to Definition 2.3 in the case that $U = \mathbb{R}^n \times \mathbb{R}^n$. Also, Definition 2.3 is different to a straightforward Colombeau generalization of global symbols $S^m(U)$. Also note the difference to the definition of the classical symbols which in the case of local symbols is given in [5, Definition 2.3, page 141].

In the following subsection we give some motivating remarks for the microlocal constructions used in Section 5.

2.1. Remarks on the governing equation in a semiclassical Colombeau setting. As this is a first investigation we will concentrate on operators as given in (1.2) which are characterized by homogeneity. Also note that our model operators have less regular log-type coefficients than logarithmic slow scale.

As pointed out in the motivation the main aim is to provide a diagonalization procedure using factorization theorems and microlocal analysis. Because it turns out that the log-type condition is not strong enough for our constructions we will introduce certain pseudodifferential operators defined on the Heisenberg group, so-called semiclassical generalized pseudodifferential operators.

In the usual theory of semiclassical pseudodifferential operators in the Kohn-Nirenberg or standard quantization one associates to a symbol $a \in S^m$ an operator $A : \mathcal{S} \rightarrow \mathcal{S}$ in the following way

$$\begin{aligned} A_\psi(x, D_x)u(x) &:= (2\pi)^{-n} \int (\mathcal{F}a)(q, p) e^{-ipq\hbar/2} \pi_\hbar(q, p) u(x) dq dp \\ &= (2\pi\hbar)^{-n} \int e^{i(x-y)\xi/\hbar} a(x, \xi) u(y) dy d\xi \end{aligned}$$

where in the first line $\mathcal{F}(a)$ denotes the Fourier transform of a on the projected Heisenberg group and $\pi_\hbar(q, p)$ is the projected Schrödinger representation of the Heisenberg group on \mathbb{R}^{2n} with parameter \hbar and the second line corresponds to the more familiar presentation of the first one. In detail we have

$$(\pi_\hbar(q, p)u)(x) := e^{i(qx + \hbar pq/2)} u(x + \hbar p)$$

and \hbar denotes the normalized Planck constant. Note that for $\hbar = 1$ this representation leads to the usual Kohn-Nirenberg calculus. Also, using representation theory

we observe for $\hbar \neq \hbar'$ that π_{\hbar} and $\pi_{\hbar'}$ are inequivalent representations on $L^2(\mathbb{R}^n)$ and moreover for $\hbar \neq 0$ the representation π_{\hbar} is irreducible and unitary on $L^2(\mathbb{R}^n)$.

Once introduced such an additional artificial parameter \hbar gives us now the opportunity to apply the theory of \hbar -pseudodifferential operators used in semiclassical analysis which is a formalism in an asymptotic regime $\hbar \ll 1$. As we want to combine this with the theory of generalized pseudodifferential operators we start writing $\hbar := \hbar(\varepsilon)$ with the restriction $\hbar(\varepsilon) \rightarrow 0$ in the case that $\varepsilon \rightarrow 0$.

For technical reasons (e.g. construction of an approximative inverse) we also have to prevent other possible values of the function $\hbar(\varepsilon)$. As this is a first approach in this field we have chosen $\hbar(\varepsilon) := \varepsilon$ at this point.

So when given a family $(a_\varepsilon)_\varepsilon \in \mathcal{M}_{S^m}$ we define the corresponding semiclassical pseudodifferential operator of Colombeau type to be the linear operator $A_\psi : \mathcal{G}_{2,2} \rightarrow \mathcal{G}_{2,2}$ which acts on the level of representatives as

$$(u_\varepsilon)_\varepsilon \mapsto (A_{\psi,\varepsilon}(x, D_x)u_\varepsilon)_\varepsilon \quad \forall (u_\varepsilon)_\varepsilon \in \mathcal{M}_{H^\infty}. \quad (2.4)$$

Here for fixed $\varepsilon \in (0, 1]$ the right hand side corresponds to the standard quantization of the symbol $(a_\varepsilon)_\varepsilon$ with parameter $\hbar(\varepsilon) = \varepsilon$; that is,

$$\begin{aligned} A_{\psi,\varepsilon}(x, D_x)u_\varepsilon(x) &:= (2\pi)^{-n} \int_{\mathbb{R}^n} (\mathcal{F}a_\varepsilon)(q, p) e^{-ipq/2} \pi_{\hbar}(q, p) u_\varepsilon(x) dq dp \\ &= (2\pi\varepsilon)^{-n} \int e^{i(x-y)\xi/\varepsilon} a_\varepsilon(x, \xi) u_\varepsilon(y) dy d\xi. \end{aligned}$$

Then with L_ε as in (1.2) the corresponding ψ -pseudodifferential operator L_ψ is essentially given by the oscillatory integral

$$L_{\psi,\varepsilon}u_\varepsilon(t, y) = (2\pi\varepsilon)^{-n-1} \int e^{it\tau/\varepsilon + iy\eta/\varepsilon} l_\varepsilon(y, \tau, \eta) \hat{u}_\varepsilon(\tau/\varepsilon, \eta/\varepsilon) d\tau d\eta$$

for fixed $\varepsilon \in (0, 1]$ and $(l_\varepsilon)_\varepsilon$ is called the generalized symbol of L_ψ and is given by

$$l_\varepsilon(y, \tau, \xi, \zeta) := -\zeta^2 - \langle b_\varepsilon(y)\xi, \xi \rangle + c_\varepsilon(y)\tau^2$$

where we have set $b_\varepsilon(y) := \text{diag}(b_{1,\varepsilon}(y), \dots, b_{n-1,\varepsilon}(y))$, $y := (x, z)$ and $\eta := (\xi, \zeta)$. Hence $(l_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{G}_{2,0}^{(1,0)}}$ and the operator L_ε can easily be reconstructed from $L_{\psi,\varepsilon}$; i.e., $L_\varepsilon = \varepsilon^{-2}L_{\psi,\varepsilon}$, because of the homogeneity of the operators.

Given such a scaled generalized operator as above we carry out all transformations within algebras of generalized functions from now on. More explicitly we will study the action of the linear operator $L_\psi = \text{OP}_\psi(l_\varepsilon)$ from $\mathcal{G}_{2,2}$ into itself in the following sense: on the level of representatives L_ψ acts as in (2.4). This explains our governing equation

$$L_\psi U = F$$

for which we will now present a microlocal diagonalization method.

3. ψ -PSEUDODIFFERENTIAL CALCULUS

In this section we introduce a general calculus for ψ -pseudodifferential operators which are certain semiclassical standard quantizations of generalized symbols as demonstrated in the previous section. For an introduction in semiclassical analysis we refer the reader to [30, 11, 23]. Since most of the techniques are similar to the classical theory of pseudodifferential operators we also want to give [25, 41] as references. Moreover a detailed discussion on pseudodifferential operators with Colombeau generalized symbols can be found in [16, 15].

To continue, given a generalized symbol $(a_\varepsilon(x, \xi))_\varepsilon$ one can assign the semiclassical standard quantization which we denote by $\text{OP}_\psi(a_\varepsilon) := a_\varepsilon(x, \varepsilon D_x)$. In this sense $\psi = \psi(\varepsilon)$ can be thought of as a scaled phase function on phase space. Therefore, we let

$$\psi_\varepsilon(x, \xi) := \langle x, \xi \rangle / \varepsilon := x\xi / \varepsilon \quad \varepsilon \in (0, 1].$$

throughout this article.

Moreover we choose the following convention for defining the Fourier transform \mathcal{F} of a function $u \in L^2(\mathbb{R}^n)$:

$$\mathcal{F}u(\xi) := \hat{u}(\xi) := \int e^{-ix\xi} u(x) dx := \lim_{\sigma \rightarrow 0^+} \int e^{-ix\xi - \sigma \langle x \rangle} u(x) dx.$$

Then the Fourier transform is an isomorphism on L^2 and the inverse Fourier transform of $u \in L^2(\mathbb{R}^n)$ is given by the following formula

$$\mathcal{F}^{-1}u(x) := (2\pi)^{-n} \int e^{ix\xi} u(\xi) d\xi := \lim_{\sigma \rightarrow 0^+} (2\pi)^{-n} \int e^{ix\xi - \sigma \langle \xi \rangle} u(\xi) d\xi.$$

More detailed pieces of information of the Fourier transform on $\mathcal{G}_{2,2}$ can be found in [2]. As already mentioned above we will focus on generalized pseudodifferential operators having the following phase-amplitude representation.

Definition 3.1. Let $(a_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{S}^{m,k}}^{(\nu,l)}(\mathbb{R}^n \times \mathbb{R}^n)$. We define the corresponding linear operator $A_\psi : \mathcal{G}_{2,2} \rightarrow \mathcal{G}_{2,2}$ such that on the level of representatives we have

$$(u_\varepsilon)_\varepsilon \mapsto (A_{\psi,\varepsilon}(x, D_x)u_\varepsilon)_\varepsilon \quad \forall (u_\varepsilon)_\varepsilon \in \mathcal{M}_{H^\infty}$$

and

$$\begin{aligned} A_{\psi,\varepsilon}(x, D_x)u_\varepsilon(x) &:= (2\pi\varepsilon)^{-n} \int e^{i(x-y)\xi/\varepsilon} a_\varepsilon(x, \xi) u_\varepsilon(y) dy d\xi \\ &= \varepsilon^{-n} \mathcal{F}_{\xi \rightarrow x/\varepsilon}^{-1} \left(a_\varepsilon(x, \xi) \hat{u}_\varepsilon(\xi/\varepsilon) \right) \end{aligned} \quad (3.1)$$

where the above integral is interpreted as an oscillatory integral. We call A_ψ the ψ -pseudodifferential operator with generalized symbol $(a_\varepsilon)_\varepsilon = (a_\varepsilon(x, \xi))_\varepsilon$. Also we will often write $A_\psi \in \text{OP}_\psi \mathcal{M}_{\mathcal{S}^{m,k}}^{(\nu,l)}$ when the generalized symbol of A_ψ belongs to the class $\mathcal{M}_{\mathcal{S}^{m,k}}^{(\nu,l)}$.

Remark 3.2. (a) Note that the map in the above definition preserves moderateness and negligibility, respectively, so that A_ψ is well-defined on equivalence classes and continuous (cf. [26, Section 1.2] and [29, Theorem 2.7]). Further, we remark that $(A_{\psi,\varepsilon}(x, D_x)u_\varepsilon)_\varepsilon = (A_\varepsilon(x, \varepsilon D_x)u_\varepsilon)_\varepsilon$ since formally we can always perform the rescaling. The problem of rescaling will be discussed in the next subsection.

(b) Note that Definition 3.1 agrees with the notion of the semiclassical standard quantization of a generalized symbol $(a_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{S}^{m,k}}^{(\nu,l)}(\mathbb{R}^n \times \mathbb{R}^n)$ and can simply written as in (3.1) using scaled Fourier transforms. Another commonly used quantization is the Weyl quantization $a^W(x, \varepsilon D)$ which has the nice property that real symbols correspond to self-adjoint Weyl quantizations. Since we will exclusively deal with the standard quantization of generalized symbols we decided to call them ψ -pseudodifferential operators.

To prepare the factorization theorem of Section 4 we will have to consider products of ψ -pseudodifferential operators. We therefore start with some general observations concerning the notion of asymptotic expansion of a generalized symbol in $\mathcal{M}_{\mathcal{S}^{m,k}}^{(\nu,l)}$.

3.1. Asymptotic expansion of the first kind. First we present the asymptotic expansion of the first kind which is inspired by the results given in [15, Section 2.5] but also uses a relevant technical aspect from the semiclassical approach. Since this first notion of an asymptotic expansion turns out to have no suitable invariant character under rescaling we will further introduce the asymptotic expansion of second kind which behaves slightly different when performing the rescaling. We should emphasize here that this invariance property of the asymptotic expansion of the second kind will be essential from Section 5 onwards, where we discuss microlocal regularity at infinite points.

The definition of the asymptotic expansion of the first kind is now the following:

Definition 3.3. For $j \in \mathbb{N}$ let $\{m_j\}_j$ be a strictly decreasing sequence of real numbers with $m_j \searrow -\infty$ as $j \rightarrow \infty$, $m_0 = m$ and $\{l_j\}_j$ be a sequence of the form $l_j = \sigma j + l$ for some fixed $\sigma, l \in \mathbb{R}$ and $\sigma \geq 0$. Further let $\{(a_{j,\varepsilon})_\varepsilon\}_j$ be a sequence with $(a_{j,\varepsilon})_\varepsilon \in \mathcal{M}_{\mathcal{S}^{m_j,k}}^{(\nu,l_j)}$ so that the following uniform growth type condition is satisfied:

$$\begin{aligned} \exists \eta \in (0, 1] \forall j \in \mathbb{N} \forall \alpha, \beta \in \mathbb{N}^n \exists C > 0 \text{ for which} \\ q_{\alpha,\beta}^{(m_j)}(a_{j,\varepsilon}) \leq C \varepsilon^k \omega_\varepsilon^{\nu|\beta|+l_j} \quad \forall \varepsilon \in (0, \eta]. \end{aligned} \tag{3.2}$$

We say that $\sum_{j=0}^\infty (\varepsilon^j a_{j,\varepsilon})_\varepsilon$ is the asymptotic expansion of the first kind for $(a_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{S}^{m,k}}^{(\nu,l)}$, denoted by $(a_\varepsilon)_\varepsilon \sim \sum_j (\varepsilon^j a_{j,\varepsilon})_\varepsilon$, if and only if

$$(a_\varepsilon - \sum_{j=0}^{N-1} \varepsilon^j a_{j,\varepsilon})_\varepsilon \in \mathcal{M}_{\mathcal{S}^{m_N, N+k-1}}^{(\nu,l)} \quad \forall N \geq 1.$$

Moreover $(a_{0,\varepsilon})_\varepsilon$ is said to be the principal symbol of $(a_\varepsilon)_\varepsilon$ if $a_{0,\varepsilon}$ is not identically vanishing for any fixed $\varepsilon \in (0, \eta]$.

We are now in a position to state our first result.

Lemma 3.4. *Let $\{l_j\}_j$, $\{m_j\}_j$ and $\{(a_{j,\varepsilon})_\varepsilon\}_j$ be as in Definition 3.3. Then there exists a generalized symbol $(a_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{S}^{m,k}}^{(\nu,l)}$ such that $(a_\varepsilon)_\varepsilon \sim \sum_j (\varepsilon^j a_{j,\varepsilon})_\varepsilon$ in $\mathcal{M}_{\mathcal{S}^{m,k}}^{(\nu,l)}$. Moreover the asymptotic expansion of the first kind determines $(a_\varepsilon)_\varepsilon$ uniquely modulo $\mathcal{N}_{\mathcal{S}^{-\infty}}$.*

Before presenting the proof of Lemma 3.4 we need a technical auxiliary result.

Lemma 3.5. *Let $\{(a_{j,\varepsilon})_\varepsilon\}_j$ be as in Definition 3.3 and χ a smooth function such that $\chi \equiv 0$ on $[0, 1]$, $\chi \equiv 1$ on $[2, \infty)$ and $0 \leq \chi \leq 1$. Then there exists a zero sequence $\{\mu_j\}_j$ such that for some fixed $\eta \in (0, 1]$ we have for every $j \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{N}^n$ with $|\alpha + \beta| \leq j$:*

$$|\chi(\mu_j(\varepsilon\omega_\varepsilon^\sigma)^{-1})q_{\alpha,\beta}^{(m_j)}(a_{j,\varepsilon})| \leq 2^{-j-1} \varepsilon^k \omega_\varepsilon^{\nu|\beta|+l_j} (\varepsilon\omega_\varepsilon^\sigma)^{-1} \quad \forall \varepsilon \in (0, \eta],$$

with the same σ as in Definition 3.3.

Proof. Using the uniformity condition (3.2) of the sequence $\{(a_{j,\varepsilon})_\varepsilon\}_j$ we obtain that for some $\eta \in (0, 1]$ and $\forall j \in \mathbb{N}, \forall \alpha, \beta \in \mathbb{N}^n, \exists C_{j,\alpha,\beta}^{(1)} > 0$ and for all $\varepsilon \in (0, \eta]$ we have

$$\begin{aligned} \varepsilon^{-k} \omega_\varepsilon^{-\nu|\beta|-l_j} |\chi(\mu_j(\varepsilon\omega_\varepsilon^\sigma)^{-1}) q_{\alpha,\beta}^{(m_j)}(a_{j,\varepsilon})| &\leq C_{j,\alpha,\beta}^{(1)} \chi(\mu_j(\varepsilon\omega_\varepsilon^\sigma)^{-1}) \\ &= C_{j,\alpha,\beta}^{(1)} \frac{\mu_j}{\varepsilon\omega_\varepsilon^\sigma} \chi(\mu_j(\varepsilon\omega_\varepsilon^\sigma)^{-1}) \frac{\varepsilon\omega_\varepsilon^\sigma}{\mu_j} \\ &\leq C_{j,\alpha,\beta}^{(1)} \frac{\mu_j}{\varepsilon\omega_\varepsilon^\sigma} \end{aligned}$$

since $\chi(\mu_j(\varepsilon\omega_\varepsilon^\sigma)^{-1}) = 0$ for $\mu_j \leq \varepsilon\omega_\varepsilon^\sigma$ and $0 \leq \chi \leq 1$. We now choose the sequence $\{\mu_j\}_j$ in the following way: for all $j \in \mathbb{N}$ and all $\alpha, \beta \in \mathbb{N}^n$ with $|\alpha + \beta| \leq j$ we have

$$C_{j,\alpha,\beta}^{(1)} \mu_j \leq 2^{-j-1} \tag{3.3}$$

and the proof is complete. □

Proof of Lemma 3.4. In the following proof we basically combine techniques from the theory of non-linear generalized functions, [15, Theorem 2.2], and Borel’s Theorem from the semiclassical approach as in [11, Theorem 4.11] or alternatively [30, Proposition 2.3.2]. To avoid overload calculations, we may assume without loss of generality that $k = 0$ in Lemma 3.4 and therefore also in the uniformity condition (3.2).

We let $\phi \in C^\infty(\mathbb{R}^n), 0 \leq \phi \leq 1$ such that $\phi(\xi) = 0$ for $|\xi| \leq 1$ and $\phi(\xi) = 1$ for $|\xi| \geq 2$. Further let χ and $\{\mu_j\}_j$ be as in Lemma 3.5. We introduce functions

$$\begin{aligned} b_{j,\varepsilon}^{(1)}(x, \xi) &:= \chi(\mu_j(\varepsilon\omega_\varepsilon^\sigma)^{-1}) a_{j,\varepsilon}(x, \xi) \\ b_{j,\varepsilon}^{(2)}(x, \xi) &:= (1 - \chi(\mu_j(\varepsilon\omega_\varepsilon^\sigma)^{-1})) \phi(\lambda_j \xi) a_{j,\varepsilon}(x, \xi) \end{aligned}$$

where $\{\lambda_j\}_j$ is a strictly decreasing zero sequence such that for all $j : \lambda_j \leq 1$ and will be specified later. For fixed $\varepsilon \in (0, 1]$ we also define

$$a_\varepsilon(x, \xi) := \sum_{j \geq 0} \varepsilon^j b_{j,\varepsilon}^{(1)}(x, \xi) + \sum_{j \geq 0} \varepsilon^j b_{j,\varepsilon}^{(2)}(x, \xi).$$

By construction the first sum consists of at most finitely many nonzero terms (depending on $\varepsilon \in (0, 1]$ fixed), since $\mu_j \rightarrow 0$ as $j \rightarrow \infty$. Moreover the second sum is locally finite and we conclude that $(a_\varepsilon)_\varepsilon \in \mathcal{E}(\mathbb{R}^n \times \mathbb{R}^n)^{(0,1]}$.

Step 1: In the first part of the proof we show that $(a_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{S}_{m,0}}^{(\nu,l)}$. Since $(a_{j,\varepsilon})_\varepsilon \in \mathcal{M}_{\mathcal{S}_{m_j,0}}^{(\nu,l_j)}$ satisfy the uniform growth type condition with $k = 0$ we obtain that there exists $\eta \in (0, 1]$ such that for all $j \in \mathbb{N}$, and all $\alpha, \beta \in \mathbb{N}^n$ we have

$$|\partial_\xi^\alpha \partial_x^\beta b_{j,\varepsilon}^{(1)}(x, \xi)| \leq C_{j,\alpha,\beta}^{(1)} \omega_\varepsilon^{\nu|\beta|+l_j} \langle \xi \rangle^{m_j-|\alpha|} \quad \forall \varepsilon \in (0, \eta] \tag{3.4}$$

and the constant $C_{j,\alpha,\beta}^{(1)} > 0$ is the same as in the proof of Lemma 3.5.

Concerning $b_{j,\varepsilon}^{(2)}$ we observe that $\text{supp}(\partial^\alpha \phi)(\lambda_j \xi) \subset \{\xi : \lambda_j^{-1} \leq |\xi| \leq 2\lambda_j^{-1}\}$ for every $|\alpha| \geq 1$. Accordingly, if $|\alpha| \geq 1$, we can assume $\lambda_j \leq 2/|\xi| \leq 4\langle \xi \rangle^{-1}$ on $\text{supp}(\partial^\alpha \phi)(\lambda_j \xi)$ and $b_{j,\varepsilon}^{(2)}$ can be estimated as follows: $\exists \eta \in (0, 1]$ independent of

α, β and j such that

$$\begin{aligned} |\partial_\xi^\alpha \partial_x^\beta b_{j,\varepsilon}^{(2)}(x, \xi)| &\leq \langle \xi \rangle^{m_j - |\alpha|} \sum_{\gamma \leq \alpha} c(\chi, \phi, \gamma) 4^{|\alpha - \gamma|} q_{\gamma, \beta}^{(m_j)}(a_{j,\varepsilon}) \\ &\leq C_{j,\alpha,\beta}^{(2)} \omega_\varepsilon^{\nu|\beta| + l_j} \langle \xi \rangle^{m_j - |\alpha|} \end{aligned} \tag{3.5}$$

for all $\varepsilon \in (0, \eta]$. At this point we choose the sequence $\{\lambda_j\}_j$ strictly decreasing and so that

$$\forall j \in \mathbb{N} \forall \alpha, \beta \in \mathbb{N}^n \text{ with } |\alpha + \beta| \leq j \text{ we have: } C_{j,\alpha,\beta}^{(2)} \lambda_j \leq 2^{-j-1}. \tag{3.6}$$

To show that $(a_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{S}^{m,0}}^{(\nu,l)}$ we note that

$$\forall \alpha, \beta \in \mathbb{N}^n \exists j_0 \in \mathbb{N} : |\alpha + \beta| \leq j_0, m_{j_0} + 1 \leq m \tag{3.7}$$

and decompose $(a_\varepsilon)_\varepsilon$ in an appropriate way, that now is

$$a_\varepsilon(x, \xi) = \sum_{j \leq j_0 - 1} \varepsilon^j b_{j,\varepsilon}(x, \xi) + \sum_{j \geq j_0} \varepsilon^j b_{j,\varepsilon}(x, \xi) =: f_\varepsilon(x, \xi) + s_\varepsilon(x, \xi) \tag{3.8}$$

where we have set $b_{j,\varepsilon}(x, \xi) := b_{j,\varepsilon}^{(1)}(x, \xi) + b_{j,\varepsilon}^{(2)}(x, \xi)$. Then $\exists \eta \in (0, 1]$ such that for all $\alpha, \beta \in \mathbb{N}^n$

$$\begin{aligned} |\partial_\xi^\alpha \partial_x^\beta f_\varepsilon(x, \xi)| &= \left| \sum_{j \leq j_0 - 1} \varepsilon^j \partial_\xi^\alpha \partial_x^\beta \{b_{j,\varepsilon}^{(1)}(x, \xi) + b_{j,\varepsilon}^{(2)}(x, \xi)\} \right| \\ &\leq \omega_\varepsilon^{\nu|\beta| + l} \langle \xi \rangle^{m - |\alpha|} \sum_{j \leq j_0 - 1} (\varepsilon \omega_\varepsilon^\sigma)^j (C_{j,\alpha,\beta}^{(1)} + C_{j,\alpha,\beta}^{(2)}) \\ &\leq C_{j_0} \omega_\varepsilon^{\nu|\beta| + l} \langle \xi \rangle^{m - |\alpha|} \end{aligned}$$

for all $\varepsilon \in (0, \eta]$ by (3.4) and (3.5). We next estimate the remainder s_ε . For this purpose we apply (3.3) from Lemma 3.5 and use (3.6) to obtain

$$\begin{aligned} \omega_\varepsilon^{-\nu|\beta| - l} |\partial_\xi^\alpha \partial_x^\beta s_\varepsilon(x, \xi)| &\leq \sum_{j \geq j_0} (\varepsilon \omega_\varepsilon^\sigma)^j \langle \xi \rangle^{m_j - |\alpha|} 2^{-j-1} \{\mu_j^{-1} + \lambda_j^{-1}\} \\ &\leq (\varepsilon \omega_\varepsilon^\sigma)^{j_0 - 1} \langle \xi \rangle^{m_{j_0} + 1 - |\alpha|} \sum_{j \geq j_0} 2^{-j} (\varepsilon \omega_\varepsilon^\sigma)^{j - j_0} \\ &\leq (\varepsilon \omega_\varepsilon^\sigma)^{j_0 - 1} \langle \xi \rangle^{m_{j_0} + 1 - |\alpha|} \leq \langle \xi \rangle^{m - |\alpha|} \quad \forall \varepsilon \in (0, \eta] \end{aligned}$$

for some $\eta \in (0, 1]$ independent of the order of differentiation. Note that in the second inequality we used that $\mu_j^{-1} \leq (\varepsilon \omega_\varepsilon^\sigma)^{-1}$ on $\text{supp}(b_{j,\varepsilon}^{(1)})$ and $\lambda_j^{-1} \leq \langle \xi \rangle$ on $\text{supp}(b_{j,\varepsilon}^{(2)})$. And so, $(a_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{S}^{m,0}}^{(\nu,l)}$ as required.

Step 2: We complete the proof by showing that for every $N \geq 1$ we have

$$(a_\varepsilon - \sum_{j \leq N-1} \varepsilon^j a_{j,\varepsilon})_\varepsilon \in \mathcal{M}_{\mathcal{S}^{m,N},N}^{(\nu,l_N)}(\mathbb{R}^n \times \mathbb{R}^n). \tag{3.9}$$

Therefore, we let $N \geq 1$ be a fixed natural number and write

$$\begin{aligned} a_\varepsilon(x, \xi) - \sum_{j \leq N-1} \varepsilon^j a_{j,\varepsilon}(x, \xi) &= \sum_{j \leq N-1} \varepsilon^j (1 - \chi(\mu_j (\varepsilon \omega_\varepsilon^\sigma)^{-1})) (\phi(\lambda_j \xi) - 1) a_{j,\varepsilon}(x, \xi) \\ &\quad + \sum_{j \geq N} \varepsilon^j b_{j,\varepsilon}(x, \xi) =: g_\varepsilon(x, \xi) + t_\varepsilon(x, \xi). \end{aligned}$$

We first calculate the contribution of g_ε . For this note that $|\xi| \leq 2\lambda_j^{-1}$ on the support of $(\phi(\lambda_j\xi) - 1)$ and we obtain that $\exists \eta \in (0, 1]$ such that $\forall j \leq N - 1$ we have

$$\begin{aligned} & |\partial_\xi^\alpha \partial_x^\beta (\phi(\lambda_j\xi) - 1) a_{j,\varepsilon}(x, \xi)| \\ & \leq \sum_{\gamma \leq \alpha} c(\phi, \gamma) \lambda_j^{|\gamma|} q_{\alpha-\gamma, \beta}^{(m_j)}(a_{j,\varepsilon})(\xi)^{m_j - |\alpha - \gamma|} \\ & \leq \langle \xi \rangle^{m_N - |\alpha|} \sum_{\gamma \leq \alpha} c(\phi, \gamma) \lambda_j^{|\gamma|} \langle 2\lambda_{N-1}^{-1} \rangle^{m_j - m_N + |\gamma|} q_{\alpha-\gamma, \beta}^{(m_j)}(a_{j,\varepsilon}) \\ & \leq C_{N, \alpha, \beta} \omega_\varepsilon^{\nu|\beta| + l_j} \langle \xi \rangle^{m_N - |\alpha|} \quad \forall \varepsilon \in (0, \eta]. \end{aligned}$$

Now taking into account that $1 - \chi(\mu_j(\varepsilon\omega_\varepsilon^\sigma)^{-1}) \neq 0$ only if $\mu_j \leq 2\varepsilon\omega_\varepsilon^\sigma$ we compute

$$\begin{aligned} & \omega_\varepsilon^{-\nu|\beta| - l} |\partial_\xi^\alpha \partial_x^\beta g_\varepsilon(x, \xi)| \\ & \leq C_{N, \alpha, \beta} \langle \xi \rangle^{m_N - |\alpha|} \sum_{j \leq N-1} (\varepsilon\omega_\varepsilon^\sigma)^j (1 - \chi(\mu_j(\varepsilon\omega_\varepsilon^\sigma)^{-1})) \\ & = C_{N, \alpha, \beta} (\varepsilon\omega_\varepsilon^\sigma)^N \langle \xi \rangle^{m_N - |\alpha|} \sum_{j \leq N-1} \mu_j^{j-N} \left(\frac{\mu_j}{\varepsilon\omega_\varepsilon^\sigma}\right)^{N-j} (1 - \chi(\mu_j(\varepsilon\omega_\varepsilon^\sigma)^{-1})) \\ & \leq \tilde{C}_{N, \alpha, \beta} (\varepsilon\omega_\varepsilon^\sigma)^N \langle \xi \rangle^{m_N - |\alpha|} \quad \forall \varepsilon \in (0, \eta]. \end{aligned}$$

We proceed with the estimation of t_ε in which we again use a decomposition as in (3.7) and (3.8). Therefore we let $N \in \mathbb{N}$, $N \geq 1$ be fixed. Then

$$\forall \alpha, \beta \in \mathbb{N}^n \exists j_0 \in \mathbb{N} : |\alpha + \beta| \leq j_0, m_{j_0} + 1 \leq m_N \text{ and } j_0 - 1 \geq N;$$

we write

$$t_\varepsilon(x, \xi) = \sum_{j=N}^{j_0-1} \varepsilon^j b_{j,\varepsilon}(x, \xi) + s_\varepsilon(x, \xi) \tag{3.10}$$

with s_ε as in (3.8). As already shown in the estimate for s_ε we also have

$$|\partial_\xi^\alpha \partial_x^\beta s_\varepsilon(x, \xi)| \leq \omega_\varepsilon^{\nu|\beta| + l} (\varepsilon\omega_\varepsilon^\sigma)^{j_0-1} \langle \xi \rangle^{m_{j_0} + 1 - |\alpha|}.$$

Furthermore we obtain for the first term on the right hand side of (3.10)

$$\begin{aligned} |\partial_\xi^\alpha \partial_x^\beta \sum_{j=N}^{j_0-1} \varepsilon^j b_{j,\varepsilon}(x, \xi)| & \leq \omega_\varepsilon^{\nu|\beta| + l} \langle \xi \rangle^{m_N - |\alpha|} \sum_{j=N}^{j_0-1} (\varepsilon\omega_\varepsilon^\sigma)^j (C_{j, \alpha, \beta}^{(1)} + C_{j, \alpha, \beta}^{(2)}) \\ & \leq C_{j_0} (\varepsilon\omega_\varepsilon^\sigma)^N \omega_\varepsilon^{\nu|\beta| + l} \langle \xi \rangle^{m_N - |\alpha|}. \end{aligned}$$

Putting this together we get (3.9) which completes the proof. □

Remark 3.6. Before we proceed we make some observations concerning the construction of the generalized symbol $(a_\varepsilon)_\varepsilon$ in Lemma 3.4 which was given by the expression

$$a_\varepsilon(x, \xi) = \sum_{j \geq 0} \varepsilon^j \chi\left(\frac{\mu_j}{\varepsilon\omega_\varepsilon^\sigma}\right) a_{j,\varepsilon}(x, \xi) + \sum_{j \geq 0} \varepsilon^j \left\{ 1 - \chi\left(\frac{\mu_j}{\varepsilon\omega_\varepsilon^\sigma}\right) \right\} \phi(\lambda_j\xi) a_{j,\varepsilon}(x, \xi).$$

Here the first sum is inspired from the semiclassical approach whereas the second sum is also used in the usual Colombeau framework. Then it is worth to mention the following facts:

(a) To construct the square root of the homogeneous operator L_ψ we also need to discuss polyhomogeneous symbols. Then the second sum in the definition of a_ε also makes sense for homogeneous symbols $a_{j,\varepsilon}$, but the first sum would induce a finite number of singularities (at $\xi = 0$) for fixed $\varepsilon \in (0, 1]$. To overcome this problem we will introduce generalized symbols that are smoothed off at the zero section so that we can require a symbol to have an asymptotic expansion in homogeneous terms if we allow some additional error terms in the asymptotic expansion. For the precise description of these so-called polyhomogeneous generalized symbols we refer to Definition 4.1.

(b) Another important property concerns the effect of rescaling. This will be used in Section 5 where we test microlocal regularity of generalized functions using L^2 -boundedness results of zeroth-order ψ -pseudodifferential operators. To exemplify this let $\{(a_{j,\varepsilon})_\varepsilon\}_j$ be a sequence with $(a_{j,\varepsilon})_\varepsilon \in \mathcal{M}_{\mathcal{S}^{-j,k}}^{(\nu,l_j)}$ and l_j as in Definition 3.3, $j \geq 0$. By Lemma 3.4 there exists $(a_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{S}^{0,k}}^{(\nu,l)}$ such that for all $N \geq 1$ we have

$$(r_\varepsilon)_\varepsilon := (a_\varepsilon - \sum_{j \leq N-1} \varepsilon^j a_{j,\varepsilon})_\varepsilon \in \mathcal{M}_{\mathcal{S}^{-N,N+k-1}}^{(\nu,l)}. \tag{3.11}$$

Now using the rescaling $(x, \xi) \mapsto (r_\varepsilon(x, \varepsilon\xi))_\varepsilon$ we obtain for every $N \geq 1$

$$(x, \xi) \mapsto (r_\varepsilon(x, \varepsilon\xi))_\varepsilon \in \mathcal{M}_{\mathcal{S}^{-N,k-1}}^{(\nu,l)}$$

concluding that the asymptotic expansion of the first kind keeps the order of the error operator invariant under rescaling but not its growth type. According to the L^2 -boundedness it seems suitable to have an asymptotic expansion which preserves the growth type of the error operator in (3.11) under rescaling rather than the order of the same.

Therefore, note that $(r_\varepsilon)_\varepsilon$ in (3.11) can also be considered as a symbol of class $\mathcal{N}_{\mathcal{S}^0}$. Then the rescaling does not influence the negligibility of the symbol $(r_\varepsilon)_\varepsilon$ since

$$(x, \xi) \mapsto (r_\varepsilon(x, \varepsilon\xi))_\varepsilon \in \mathcal{N}_{\mathcal{S}^0}.$$

Using this we now introduce the asymptotic expansion of the second kind to achieve an invariant growth type under rescaling at least when studying zeroth-order operators. In detail we will study asymptotic expansions of the form $(\sum_{j \geq 0} \varepsilon^j a_{j,\varepsilon})_\varepsilon$ which are similar those in Definition 3.3 but where the sequence $\{(a_{j,\varepsilon})_\varepsilon\}_j$ is uniformly bounded in the generalized symbol class of order $m = m_0$.

3.2. Asymptotic expansion of the second kind. To eliminate the rescaling problem arising in Remark 3.6(b) we now present the asymptotic expansion of the second kind.

Definition 3.7. Let $m, k \in \mathbb{R}$ and $\{l_j\}_j$ be a sequence of real numbers of the form $l_j = \sigma j + l$ for some fixed $\sigma, l \in \mathbb{R}$ and $\sigma \geq 0, j \in \mathbb{N}$. Further we let $\{(a_{j,\varepsilon})_\varepsilon\}_j$ be a sequence with $(a_{j,\varepsilon})_\varepsilon \in \mathcal{M}_{\mathcal{S}^{m,k}}^{(\nu,l_j)}$ satisfying the uniformity condition:

$$\begin{aligned} &\exists \eta \in (0, 1] \forall j \in \mathbb{N} \forall \alpha, \beta \in \mathbb{N}^n \exists C > 0 \text{ such that} \\ &q_{\alpha,\beta}^{(m)}(a_{j,\varepsilon}) \leq C \varepsilon^k \omega_\varepsilon^{|\beta|+l_j} \quad \forall \varepsilon \in (0, \eta]. \end{aligned}$$

We say that the $\sum_{j=0}^{\infty}(\varepsilon^j a_{j,\varepsilon})_{\varepsilon}$ is the asymptotic expansion of the second kind for the symbol $(a_{\varepsilon})_{\varepsilon} \in \mathcal{M}_{\mathcal{S}^{m,k}}^{(\nu,l)}$, denoted by $(a_{\varepsilon})_{\varepsilon} \approx \sum_j(\varepsilon^j a_{j,\varepsilon})_{\varepsilon}$, if

$$(a_{\varepsilon} - \sum_{j=0}^{N-1} \varepsilon^j a_{j,\varepsilon})_{\varepsilon} \in \mathcal{M}_{\mathcal{S}^{m,N}}^{(\nu,l_N)} \quad \forall N \geq 1.$$

Again we call $(a_{0,\varepsilon})_{\varepsilon}$ the principal symbol of $(a_{\varepsilon})_{\varepsilon}$ if $a_{0,\varepsilon}$ is not identically zero for every fixed $\varepsilon > 0$ sufficiently small.

Comparing Definition 3.3 and 3.7 we observe that any generalized symbol $(a_{\varepsilon})_{\varepsilon}$ from Lemma 3.4 also admits an asymptotic expansion of the second kind.

Furthermore we note that our notions of an asymptotic expansion are different from the one used in the semiclassical approach. A crucial difference is that in the semiclassical setting one uses the Hörmander symbol class $S_{0,0}^m$ as the underlying classical space and not $S^m := S_{1,0}^m$ as in our case, cf. [24]. As above, the semiclassical asymptotic expansion provides an invariant character of zeroth-order operators under rescaling. For more details on the semiclassical asymptotic expansion we refer to [10, 30, 23].

3.3. Composition of ψ -pseudodifferential operators. In this subsection we establish the composition law of two ψ -pseudodifferential operators utilizing the method of stationary phase to determine the asymptotic behavior in the occurring oscillatory integrals. Because of the specific construction of the ψ -pseudodifferential operators it suffices to study a corollary of the stationary phase formula which can be found in [30, Corollary 2.6.3] and [21, Example 2.2].

In detail, this simplified formula says that given a function $a \in C_c^{\infty}(\mathbb{R}^{2n})$ then for every $N \geq 1$ one has

$$\left(\frac{\lambda}{2\pi}\right)^n \int e^{-i\lambda xy} a(x, y) dx dy = \sum_{|\alpha| \leq N-1} \frac{1}{\alpha! \lambda^{|\alpha|}} (D_x^{\alpha} \partial_y^{\alpha} a)(0, 0) + S_N(a; \lambda)$$

where the parameter λ is considered in the limit $\lambda \rightarrow \infty$. Moreover the remainder term $S_N(a; \lambda)$ can be estimated as follows

$$|S_N(a; \lambda)| \leq \frac{C}{N! \lambda^N} \sum_{|\alpha+\beta| \leq 2n+1} \|\partial_x^{\alpha} \partial_y^{\beta} (\partial_x \cdot \partial_y)^N a\|_{L^1(\mathbb{R}^{2n})} \quad \lambda \geq 1$$

where $\partial_x \cdot \partial_y = \sum_{j=1}^n \partial_{x_j} \partial_{y_j}$ and the constant $C > 0$ is independent of λ . In the application we will work with smooth functions a which are not compactly supported but are well-behaved at infinity and also depend on the parameter λ . For a more general discussion on the method of stationary phase we refer the reader to [30, Section 2.6] or alternatively to [21, Section 2].

Using the above formula we can show the following result.

Proposition 3.8. *Let $A_{\psi}(x, D_x)$ and $B_{\psi}(x, D_x)$ be two ψ -pseudodifferential operators with generalized symbols $(a_{\varepsilon})_{\varepsilon} \in \mathcal{M}_{\mathcal{S}^{m_1,k_1}}^{(\nu,l_1)}$ and $(b_{\varepsilon})_{\varepsilon} \in \mathcal{M}_{\mathcal{S}^{m_2,k_2}}^{(\nu,l_2)}$ respectively. Then the product $A_{\psi} B_{\psi}$ is well-defined and maps $\mathcal{G}_{2,2}$ into itself. Moreover $A_{\psi} B_{\psi}$ is a ψ -pseudodifferential operator with generalized symbol $(a_{\varepsilon} \# b_{\varepsilon})_{\varepsilon}$ in $\mathcal{M}_{\mathcal{S}^{m_1+m_2,k_1+k_2}}^{(\nu,l_1+l_2)}$*

and we have the following representation

$$(a_\varepsilon \# b_\varepsilon)_\varepsilon \sim \sum_{|\alpha| \geq 0} \left(\frac{\varepsilon^{|\alpha|}}{\alpha!} D_\xi^\alpha a_\varepsilon(x, \xi) \partial_x^\alpha b_\varepsilon(x, \xi) \right)_\varepsilon. \tag{3.12}$$

Note that in Proposition 3.8 the generalized symbol $(a_\varepsilon \# b_\varepsilon)_\varepsilon$ is given by its asymptotic expansion of the first kind. Hence it can also be reinterpreted as a symbol having an asymptotic expansion of the second kind.

The following proof is an adaption of [30, Theorem 2.6.5] and [11, Theorem 9.13].

Proof of Proposition 3.8. Without loss of generality we assume that $(a_\varepsilon)_\varepsilon$ and $(b_\varepsilon)_\varepsilon$ are of growth type 0; i.e., $k_1 = k_2 = 0$, as the proof for more general growth type assumptions requires only slight changes in the argumentation. Let $u \in \mathcal{G}_{2,2}(\mathbb{R}^n)$ having $(u_\varepsilon)_\varepsilon$ as representative and let $\varepsilon \in (0, 1]$ be fixed and arbitrary. Then $A_\psi B_\psi u$ makes sense as an oscillatory integral and we write

$$\begin{aligned} A_{\psi,\varepsilon} B_{\psi,\varepsilon} u_\varepsilon(x) &= \frac{1}{(2\pi\varepsilon)^n} \lim_{\sigma \rightarrow 0_+} \int e^{i(x-y)\eta/\varepsilon - \sigma\langle y \rangle - \sigma\langle \eta \rangle} a_\varepsilon(x, \eta) (B_{\psi,\varepsilon} u_\varepsilon)(y) dy d\eta \\ &= \frac{1}{(2\pi\varepsilon)^n} \lim_{\sigma \rightarrow 0_+, \tau \rightarrow 0_+} \int e^{ix\xi/\varepsilon - \tau\langle \xi \rangle} c_{\sigma,\varepsilon}(x, \xi) \hat{u}_\varepsilon(\xi/\varepsilon) d\xi \end{aligned} \tag{3.13}$$

where we have set

$$c_{\sigma,\varepsilon}(x, \xi) = (2\pi\varepsilon)^{-n} \int e^{i(x-y)(\eta-\xi)/\varepsilon - \sigma\langle y \rangle - \sigma\langle \eta \rangle} a_\varepsilon(x, \eta) b_\varepsilon(y, \xi) dy d\eta.$$

In what follows we show that $c_{\sigma,\varepsilon}(x, \xi)$ corresponds to a generalized symbol in $\mathcal{M}_{\mathcal{S}^{m_1+m_2,0}}^{(\nu, l_1+l_2)}$ uniformly for all $\sigma > 0$. Here the relevant asymptotic behavior of $c_{\sigma,\varepsilon}$ as $\varepsilon \rightarrow 0$ is described by using the method of stationary phase. Finally, by passing to the limit $\sigma \rightarrow 0_+$, we obtain by Lebesgue’s dominated convergence theorem that the limit of $(c_{\sigma,\varepsilon})_\varepsilon$ exists in $\mathcal{M}_{\mathcal{S}^{m_1+m_2,0}}^{(\nu, l_1+l_2)}$ and admits a representation as stated in (3.12).

For that purpose we first split the integral representation of $c_{\sigma,\varepsilon}(x, \xi)$ into three parts. Therefore, let $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$ such that $\chi(s) = 1$ for $|s| \leq 1/4$ and $\chi(s) = 0$ for $|s| \geq 1/2$ and set $\chi_1(x, y) = \chi(|x - y|)$ for $x, y \in \mathbb{R}^n$. For $|\xi| \geq 1$ we write

$$c_{\sigma,\varepsilon}(x, \xi) = d_{\sigma,\varepsilon}(x, \xi) + e_{\sigma,\varepsilon}(x, \xi) + f_{\sigma,\varepsilon}(x, \xi), \tag{3.14}$$

where $d_{\sigma,\varepsilon}, e_{\sigma,\varepsilon}$ and $f_{\sigma,\varepsilon}$ are defined by the additional factors $1 - \chi(|\xi - \eta|/|\xi|)$, $\chi(|\xi - \eta|/|\xi|)(1 - \chi_1(x, y))$ and $\chi(|\xi - \eta|/|\xi|)\chi_1(x, y)$ in the integrand of $c_{\sigma,\varepsilon}$. Also we note that $f_{\sigma,\varepsilon}$ represents the part of $c_{\sigma,\varepsilon}$ that will determine its asymptotic behavior as we meet the critical points of the phase.

In the following we give the proof under the assumption that $|\xi| \geq 1$. In the case that $|\xi| \leq 1$ a likewise decomposition as in (3.14) will lead to the desired result. Again, this is similar as in the proof of [30, Theorem 2.6.5].

As a first step we check that the integrand of $d_{\sigma,\varepsilon}$ is in $L^1(\mathbb{R}_y^n \times \mathbb{R}_\eta^n)$ uniformly in $\sigma > 0$. Therefore, we introduce the operator

$$L_\varepsilon := \left(1 + \frac{|\xi - \eta|^2}{\varepsilon^2} + \frac{|x - y|^2}{\varepsilon^2} \right)^{-1} \left(1 + \frac{\xi - \eta}{\varepsilon} D_y + \frac{x - y}{\varepsilon} D_\eta \right)$$

satisfying the equality

$$L_\varepsilon e^{i(x-y)(\eta-\xi)/\varepsilon} = e^{i(x-y)(\eta-\xi)/\varepsilon}.$$

Using integration by parts we obtain

$$(2\pi\varepsilon)^n d_{\sigma,\varepsilon}(x, \xi) = \int e^{i(x-y)(\eta-\xi)/\varepsilon} ({}^tL_\varepsilon)^k \left\{ (1 - \chi(|\xi - \eta|/|\xi|)) e^{-\sigma\langle y \rangle - \sigma\langle \eta \rangle} a_\varepsilon(x, \eta) b_\varepsilon(y, \xi) \right\} dy d\eta.$$

Further, for every $k \in \mathbb{N}$ with $k > n + 1/2$ we obtain

$$\begin{aligned} (2\pi\varepsilon)^n d_{\sigma,\varepsilon}(x, \xi) &= \int_{|\xi-\eta| \geq |\xi|/4} \mathcal{O}\left(\frac{\omega_\varepsilon^{k\nu+l_1+l_2} \langle \eta \rangle^{m_1} \langle \xi \rangle^{m_2}}{(1 + |\xi - \eta|/\varepsilon + |x - y|/\varepsilon)^k}\right) dy d\eta \\ &= \int_{|\xi-\eta| \geq |\xi|/4} \mathcal{O}\left(\varepsilon^n \omega_\varepsilon^{k\nu+l_1+l_2} \langle \eta \rangle^{m_1} \langle \xi \rangle^{m_2} \left\{ 1 + \frac{|\xi - \eta|}{\varepsilon} \right\}^{n+1/2-k}\right) d\eta \\ &= \int_{|\xi-\eta| \geq |\xi|/4} \mathcal{O}\left(\varepsilon^{k-1/2} \omega_\varepsilon^{k\nu+l_1+l_2} \frac{\langle \eta \rangle^{m_1} \langle \xi \rangle^{m_2}}{(1 + |\xi| + |\eta|)^{k-n-1/2}}\right) d\eta \quad \text{as } \varepsilon \rightarrow 0. \end{aligned}$$

Then, by a straightforward calculation one shows that for every $k \geq |m_1| + 2n + 1$ we have

$$(2\pi\varepsilon)^n d_{\sigma,\varepsilon}(x, \xi) = \mathcal{O}\left(\varepsilon^{k-1/2} \omega_\varepsilon^{k\nu+l_1+l_2} \langle \xi \rangle^{m_1+m_2-k+2n+1+|m_1|}\right)$$

for ε sufficiently small. Since $(\omega_\varepsilon)_\varepsilon$ is a slow scale net we conclude that

$$(2\pi\varepsilon)^n d_{\sigma,\varepsilon}(x, \xi) = \mathcal{O}\left(\varepsilon^N \omega_\varepsilon^{l_1+l_2} \langle \xi \rangle^{m_1+m_2-N}\right) \quad \forall N \geq 0$$

uniformly for every $(x, \xi) \in \mathbb{R}^{2n}$ with $|\xi| \geq 1$ and $\sigma > 0$ as $\varepsilon \rightarrow 0$.

We next estimate $e_{\sigma,\varepsilon}(x, \xi)$. Using the coordinate change $z = y - x$ and integration by parts we can write

$$(2\pi\varepsilon)^n e_{\sigma,\varepsilon}(x, \xi) = \int_{\substack{|z| \geq 1/4 \\ |\xi-\eta| \leq |\xi|/2}} e^{-iz(\eta-\xi)/\varepsilon} ({}^tL_{1,\varepsilon})^k r_{\sigma,\varepsilon}(x, z, \xi, \eta) dz d\eta$$

where we have set $L_{1,\varepsilon} := -\varepsilon z/|z|^2 D_\eta$ and

$$r_{\sigma,\varepsilon}(x, z, \xi, \eta) := \chi(|\xi - \eta|/|\xi|) (1 - \chi(|z|)) e^{-\sigma\langle x+z \rangle - \sigma\langle \eta \rangle} a_\varepsilon(x, \eta) b_\varepsilon(x + z, \xi).$$

Note that the operator $L_{1,\varepsilon}$ satisfies

$${}^tL_{1,\varepsilon} = -L_{1,\varepsilon} \quad \text{and} \quad L_{1,\varepsilon} e^{-iz(\eta-\xi)/\varepsilon} = e^{-iz(\eta-\xi)/\varepsilon}.$$

Also, the integrand of $e_{\sigma,\varepsilon}(x, \xi)$ is integrable with respect to z if we take $k > n$. To check the integrability with respect to the η -variable we partition the domain of integration into the two regions

$$\Omega_1 := \{\eta \in \mathbb{R}^n : |\xi - \eta| \leq 1/4\}, \quad \Omega_2 := \{\eta \in \mathbb{R}^n : |\xi - \eta| \geq 1/4\}$$

and write

$$(2\pi\varepsilon)^n e_{\sigma,\varepsilon}(x, \xi) = \sum_{j=1}^2 \int_{\Omega_j} e^{-iz(\eta-\xi)/\varepsilon} ({}^tL_{1,\varepsilon})^k r_{\sigma,\varepsilon}(x, z, \xi, \eta) dz d\eta =: \sum_{j=1}^2 I_j.$$

Concerning I_1 we use the coordinate transformation $\zeta = \eta - \xi$ and by Peetre's inequality $\langle \xi + \zeta \rangle^s \leq 2^{|s|/2} \langle \xi \rangle^s \langle \zeta \rangle^{|s|}$ for every $s \in \mathbb{R}$ we obtain

$$I_1 = \int_{\substack{|z| \geq 1/4 \\ |\zeta| \leq 1/4}} \mathcal{O}\left(\varepsilon^k \omega_\varepsilon^{l_1+l_2} |z|^{-k} \langle \xi \rangle^{m_2} \langle \xi + \zeta \rangle^{m_1-k}\right) dz d\zeta = \mathcal{O}\left(\varepsilon^k \omega_\varepsilon^{l_1+l_2} \langle \xi \rangle^{m_1+m_2-k}\right)$$

for every $k > n$ as $\varepsilon \rightarrow 0$. For the estimation of the second part I_2 we introduce

$$L_{2,\varepsilon} := \frac{\varepsilon^2}{|\xi - \eta|^2} \frac{\xi - \eta}{\varepsilon} D_z$$

and let $j \in \mathbb{N}$. Again using integration by parts we obtain for $|\xi| \geq 1$

$$\begin{aligned} I_2 &= \int e^{-iz(\eta-\xi)/\varepsilon} ({}^tL_{2,\varepsilon})^j ({}^tL_{1,\varepsilon})^k r_{\sigma,\varepsilon}(x, z, \xi, \eta) dz d\eta \\ &= \int_{\substack{|z| \geq 1/4 \\ 1/4 \leq |\xi - \eta| \leq |\xi|/2}} \mathcal{O}(\varepsilon^{j+k} \omega_\varepsilon^{j\nu+l_1+l_2} |z|^{-k} \langle \xi \rangle^{m_2} \langle \eta \rangle^{m_1-k} |\xi - \eta|^{-j}) dz d\eta \end{aligned}$$

and is integrable with respect to z if we take $k > n$ for sufficiently small ε . Then using the coordinate change $\eta = \xi + \zeta$ and applying Peetre's inequality gives

$$\begin{aligned} I_2 &= \int_{1/4 \leq |\zeta| \leq |\xi|/2} \mathcal{O}(\varepsilon^{j+k} \omega_\varepsilon^{j\nu+l_1+l_2} \langle \xi \rangle^{m_1+m_2-k} \langle \zeta \rangle^{|m_1|+k-j}) d\zeta \\ &= \mathcal{O}(\varepsilon^{j+k} \omega_\varepsilon^{j\nu+l_1+l_2} \langle \xi \rangle^{m_1+m_2-k}) \end{aligned}$$

for every $j, k \in \mathbb{N}$ with $k > n$ and $j > |m_1| + k + n$ as $\varepsilon \rightarrow 0$. Combining the above yields to

$$(2\pi\varepsilon)^n e_{\sigma,\varepsilon}(x, \xi) = \mathcal{O}(\varepsilon^N \omega_\varepsilon^{l_1+l_2} \langle \xi \rangle^{m_1+m_2-N}) \quad \forall N \geq 0$$

uniformly for all $\sigma > 0$ and $(x, \xi) \in \mathbb{R}^{2n}$, $|\xi| \geq 1$ as $\varepsilon \rightarrow 0$. By the same arguments as above one can also show that for every $\alpha, \beta \in \mathbb{N}^n$, $|\xi| \geq 1$ and $N \in \mathbb{N}$ we have

$$|\partial_\xi^\alpha \partial_x^\beta d_{\sigma,\varepsilon}(x, \xi)| + |\partial_\xi^\alpha \partial_x^\beta e_{\sigma,\varepsilon}(x, \xi)| = \mathcal{O}(\varepsilon^N \omega_\varepsilon^{|\beta|+l_1+l_2} \langle \xi \rangle^{m_1+m_2-|\alpha|-N})$$

uniformly for $\sigma > 0$ as $\varepsilon \rightarrow 0$. So we deduce that $(d_{\sigma,\varepsilon})_\varepsilon$ and $(e_{\sigma,\varepsilon})_\varepsilon$ are contained in $\mathcal{M}_{\mathcal{S}^{m_1+m_2-N,N}}^{(\nu, l_1+l_2)}$ for every $N \in \mathbb{N}$ uniformly with respect to $\sigma > 0$ and $(x, \xi) \in \mathbb{R}^{2n}$, $|\xi| \geq 1$.

So it remains to study the term $(2\pi\varepsilon)^n f_{\sigma,\varepsilon}(x, \xi)$ which was given by

$$\int e^{i(x-y)(\eta-\xi)/\varepsilon} \chi(|\xi - \eta|/|\xi|) \chi_1(x, y) e^{-\sigma(\langle y \rangle + \langle \eta \rangle)} a_\varepsilon(x, \eta) b_\varepsilon(y, \xi) dy d\eta.$$

Now writing $\xi = \lambda\nu$ where $\lambda = |\xi|$ and using the coordinate transformations

$$\zeta = (\eta - \xi)/\lambda, \quad z = y - x$$

we obtain

$$f_{\sigma,\varepsilon}(x, \xi) = \frac{\lambda^n}{(2\pi\varepsilon)^n} \int_{|z| \leq 1/2, |\zeta| \leq 1/2} e^{-i\lambda z \zeta/\varepsilon} t_{\sigma,\varepsilon}(z, \zeta; x, \xi) dz d\zeta$$

where $t_{\sigma,\varepsilon}$ is a function in $\mathcal{C}_c^\infty(\mathbb{R}_z^n \times \mathbb{R}_\zeta^n)$ containing (x, ξ) as parameters and is given by

$$t_{\sigma,\varepsilon}(z, \zeta; x, \xi) = \chi(|\zeta|) \chi(|z|) e^{-\sigma\langle x+z \rangle - \sigma\langle \lambda(\nu+\zeta) \rangle} a_\varepsilon(x, \lambda(\nu + \zeta)) b_\varepsilon(x + z, \lambda\nu).$$

Then the method of stationary phase gives for every $N \in \mathbb{N}$, $N \geq 1$

$$f_{\sigma,\varepsilon}(x, \xi) = \sum_{|\alpha| \leq N-1} \frac{\varepsilon^{|\alpha|}}{\alpha! \lambda^{|\alpha|}} D_\zeta^\alpha \partial_z^\alpha t_{\sigma,\varepsilon}(z, \zeta; x, \xi) \Big|_{\substack{z=0 \\ \zeta=0}} + S_N(t_{\sigma,\varepsilon}; \lambda/\varepsilon)$$

and the remainder can be estimated as follows

$$|S_N(t_{\sigma,\varepsilon}; \lambda/\varepsilon)| \leq \frac{C\varepsilon^N}{N! \lambda^N} \sum_{|\alpha+\beta| \leq 2n+1} \|\partial_\zeta^\alpha \partial_z^\beta (\partial_\zeta \cdot \partial_z)^N t_{\sigma,\varepsilon}\|_{L^1(\mathbb{R}_z^n \times \mathbb{R}_\zeta^n)}$$

$$\begin{aligned} &= \mathcal{O}\left(\varepsilon^N \omega_\varepsilon^{(2n+1+N)\nu+l_1+l_2} \lambda^{m_1+m_2-N}\right) \\ &= \mathcal{O}\left(\varepsilon^{N-1} \omega_\varepsilon^{l_1+l_2} \langle \xi \rangle^{m_1+m_2-N}\right) \end{aligned}$$

for $|\xi| \geq 1$ and ε sufficiently small. Finally, since similar estimates hold true for the derivatives and we conclude that $(S_N(t_{\sigma,\varepsilon}))_\varepsilon$ is contained in $\mathcal{M}_{\mathcal{S}^{m_1+m_2-N, N-1}}^{(\nu, l_1+l_2)}$ for every $N \in \mathbb{N}$ uniformly with respect to $\sigma > 0$ for every $(x, \xi) \in \mathbb{R}^{2n}$ with $|\xi| \geq 1$.

As already mentioned in the beginning of the proof we now pass to the limit $\sigma \rightarrow 0_+$. Then by Lebesgue’s dominated convergence theorem we have

$$c_{\sigma,\varepsilon}(x, \xi) \rightarrow c_\varepsilon(x, \xi) \quad \text{as } \sigma \rightarrow 0_+$$

so that equation (3.13) reads

$$\begin{aligned} A_{\psi,\varepsilon} B_{\psi,\varepsilon} u_\varepsilon(x) &= (2\pi\varepsilon)^{-n} \lim_{\tau \rightarrow 0_+} \int e^{ix\xi/\varepsilon - \tau\langle \xi \rangle} c_\varepsilon(x, \xi) \hat{u}_\varepsilon(\xi/\varepsilon) d\xi \\ &= (2\pi\varepsilon)^{-n} \int e^{ix\xi/\varepsilon} c_\varepsilon(x, \xi) \hat{u}_\varepsilon(\xi/\varepsilon) d\xi \end{aligned}$$

where the last equation holds in the sense of oscillatory integrals. Moreover the generalized symbol $(c_\varepsilon)_\varepsilon$ is equal to $(a_\varepsilon \# b_\varepsilon)_\varepsilon$ as described in (3.12) and in particular $(c_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{S}^{m_1+m_2, 0}}^{(\nu, l_1+l_2)}$. This completes the proof. \square

4. A FACTORIZATION PROCEDURE FOR L_ψ

Concerning products of ψ -pseudodifferential operators that approximate L_ψ we make similar considerations as in the smooth setting which can be found in [29, Appendix II] and [25, Chapter 23]. First we write for $L_\psi = \varepsilon^2 L_\varepsilon$ with L_ε from the beginning

$$L_\psi(y, D_t, D_x, D_z) =: (\partial_z^2 + A(y, D_t, D_x))_\psi \tag{4.1}$$

where $A_\psi = A_\psi(y, D_t, D_x)$ is the ψ -pseudodifferential operator with generalized symbol $(a_\varepsilon)_\varepsilon$ and

$$a_\varepsilon(y, \tau, \xi) = c_\varepsilon(y) \tau^2 - \langle b_\varepsilon(y) \xi, \xi \rangle \quad \varepsilon \in (0, 1]. \tag{4.2}$$

To recall the requirements made on the coefficients $c_\varepsilon(y)$ and $b_{j,\varepsilon}(y), 1 \leq j \leq n-1$ we also refer to Section 1.

Before stating the main theorem of this section we give a few more details about notation. In the following we will study operators on \mathbb{R}^{n+1} of the form

$$S_\psi = \sum_{j=0}^2 S_{j,\psi}(y, D_t, D_x) (\partial_z^{2-j})_\psi$$

with coefficients $S_{j,\psi} \in \text{OP}_\psi \mathcal{M}_{\mathcal{S}^{k_j, 0}}^{(1, l_j)}$ for some $k_j, l_j \in \mathbb{R}, j = 0, 1, 2$. Further we write

$$S_\psi = \sum_{j=0}^2 S_{j,\psi}(y, D_t, D_x) (\partial_z^{2-j})_\psi \quad \text{on } I'$$

when the symbols of the coefficients $S_{j,\psi}$ are restricted to a set I' .

Concerning the factorization domain we make similar considerations as in [37] and introduce the set

$$I'_{\theta_1} := \{(y, \tau, \xi) \mid \tau \neq 0, \langle b(y) \xi, \xi \rangle < \sin^2(\theta_1) c(y) \tau^2\} \tag{4.3}$$

for some $\theta_1 \in (0, \pi/2)$. Then I'_{θ_1} is an open subset of $\mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$ and conic with respect to the second variable. Moreover we have chosen I'_{θ_1} independent of the parameter $\varepsilon \in (0, 1]$ for simplicity. Note that the main reason for restricting the analysis to the domain I'_{θ_1} is that $(a_\varepsilon)_\varepsilon$ is non-negative there.

Further, as already mentioned in Section 3 it is necessary to introduce the notion of polyhomogeneous generalized symbols that are smoothed off at the origin.

Definition 4.1. We say that a generalized symbol $(p_\varepsilon)_\varepsilon$ is polyhomogeneous in $\mathcal{M}_{\mathcal{S}^{m,k}}^{(\nu,l)}$, denoted by $(p_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{S}^{m,k}}^{(\nu,l)}$, if there exist a sequence of symbols $(\bar{p}_{m-j,\varepsilon})_\varepsilon$ in $\mathcal{M}_{\mathcal{S}^{m-j,k}}^{(\nu,\nu j+l)}$, $j \geq 0$ and a cut-off $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ equal to 1 near the origin so that $\forall N \geq 1$ we have

$$(p_\varepsilon - \sum_{j=0}^{N-1} \varepsilon^j (1 - \varphi) \bar{p}_{m-j,\varepsilon})_\varepsilon \in \mathcal{M}_{\mathcal{S}^{m-N,N+k}}^{(\nu,\nu N+l)} \quad \text{for } \xi \text{ sufficiently large.} \quad (4.4)$$

For this we will use the following notation

$$(p_\varepsilon)_\varepsilon \sim \sum_{j \geq 0} (\varepsilon^j \bar{p}_{m-j,\varepsilon})_\varepsilon \quad \text{in } \mathcal{M}_{\mathcal{S}^{m,k}}^{(\nu,l)}.$$

Here the homogeneous part $(\bar{p}_{m,\varepsilon})_\varepsilon$ is called the principal symbol of $(p_\varepsilon)_\varepsilon$ if there exist an $\eta \in (0, 1]$ and a constant $K > 0$ such that $\bar{p}_{m,\varepsilon} \neq 0$ for all $|\xi| \geq K$ and any fixed $\varepsilon \in (0, \eta]$. In particular a ψ -pseudodifferential operator is polyhomogeneous if its generalized symbol is polyhomogeneous.

Remark 4.2. First note that the principal symbol is uniquely determined if the dual variable is sufficiently large. Also (4.4) can be interpreted in the following sense: $\forall N \geq 1$ we have

$$(p_\varepsilon - \sum_{j=0}^{N-1} \varepsilon^j (1 - \varphi) \bar{p}_{m-j,\varepsilon})_\varepsilon = (r_\varepsilon)_\varepsilon + (s_\varepsilon)_\varepsilon \quad (4.5)$$

for some $(s_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{S}^{m-N,N+k}}^{(\nu,\nu N+l)}$ and $(r_\varepsilon)_\varepsilon$ is some compactly supported regular generalized symbol of order $-\infty$ in the following sense:

$$\begin{aligned} \exists \eta \in (0, 1] \exists k \in \mathbb{R} \forall m \in \mathbb{R} \forall \alpha, \beta \in \mathbb{N}^n \exists C > 0 \text{ such that} \\ |\partial_\xi^\alpha \partial_x^\beta r_\varepsilon(x, \xi)| \leq C \varepsilon^k \langle \xi \rangle^{m-|\alpha|} \quad \forall \varepsilon \in (0, \eta]. \end{aligned}$$

We refer to Section 5 where it is essential to distinguish between the two different types of errors on the right hand side of (4.5).

The aim of this section is now to factorize L_ψ on I'_{θ_1} in terms of two first-order operators of the form $L_{j,\psi} = (\partial_z + A_j)_\psi$ on I'_{θ_1} with $A_{j,\psi}$ a polyhomogeneous ψ -pseudodifferential operator with generalized symbol in $\mathcal{M}_{\mathcal{S}^{1,0}}^{(1,0)}$ on I'_{θ_1} , $j = 1, 2$. Here a first rough approximation would suggest $A_{j,\psi}$ to be the square root of the operator $A_\psi = \text{OP}_\psi((a_\varepsilon)_\varepsilon)$ given in (4.1) on the set I'_{θ_1} where $\pm(i\sqrt{a_\varepsilon})_\varepsilon$ is described explicitly, $j = 1, 2$.

We are now going to show the following result.

Theorem 4.3. Let $L_\psi = (\partial_z^2 + A)_\psi$ and I'_{θ_1} be as in (4.1) and (4.3) respectively. Then the operator L_ψ can be factorized into

$$L_\psi = L_{1,\psi} L_{2,\psi} + R_\psi \quad \text{on } I'_{\theta_1} \quad (4.6)$$

where $L_{j,\psi} = (\partial_z + A_j)_\psi$ and $A_{j,\psi} \in \text{OP}_\psi \mathcal{M}_{\mathcal{S}^{1,0}}^{(1,0)}_{\text{phg}}$ on I'_{θ_1} whose principal symbol is equal to $\pm(i\sqrt{a_\varepsilon})_\varepsilon$, $j = 1, 2$. Moreover the remainder is given by $R_\psi = \Gamma_{0,\psi} + \Gamma_{1,\psi}(\partial_z)_\psi$ for some ψ -pseudodifferential operators $\Gamma_{j,\psi}$ with generalized symbol $(\gamma_{j,\varepsilon})_\varepsilon$ in $\mathcal{N}_{\mathcal{S}^{2-j}}$ on I'_{θ_1} , $j = 0, 1$.

We remark that the product $L_{1,\psi}L_{2,\psi}$ in (4.6) is not a ψ -pseudodifferential operator on \mathbb{R}^{n+1} . But one can overcome this by introducing a generalized cut-off $g_\psi(D_t, D_y)$ such that the difference $L_{1,\psi}L_{2,\psi}g_\psi - L_{1,\psi}L_{2,\psi}$ is insignificant on some adequate subdomain of the phase space $T^*\mathbb{R}^{n+1} \setminus 0$ under a microlocal point of view. For the specification for microlocal analysis we refer to the next section.

Technical preliminaries. Note that a_ε as in (4.2) is a homogeneous function with respect to the dual variables (τ, ξ) and so that there exists $\eta \in (0, 1]$ such that

$$|a_\varepsilon(y, \tau, \xi)| \geq C|\tau, \xi|^2 \quad \text{on } I'_{\theta_2} \text{ for every } \varepsilon \in (0, \eta]$$

for some fixed $\theta_2 \in (0, \pi/2)$ with $\theta_1 < \theta_2$ and the constant $C > 0$ independent of ε . Moreover we set for some $\varepsilon_0 \in (0, 1]$ which will be specified later

$$\bar{b}_{j,\varepsilon}^{(0)}(y, \tau, \xi) := \begin{cases} \pm i\sqrt{a_\varepsilon(y, \tau, \xi)}, & \varepsilon \in (0, \varepsilon_0] \\ 0, & \varepsilon \in (\varepsilon_0, 1] \end{cases} \quad \text{on } I'_{\theta_2} \tag{4.7}$$

which in turn gives $(\bar{b}_{j,\varepsilon}^{(0)})_\varepsilon \in \mathcal{M}_{\mathcal{S}^{1,0}}^{(1,0)}$ on I'_{θ_2} . We now associate $(\bar{b}_{j,\varepsilon}^{(0)})_\varepsilon$ to a generalized symbol $(\bar{c}_{j,\varepsilon}^{(0)})_\varepsilon \in \mathcal{M}_{\mathcal{S}^{1,0}}^{(1,0)}(\mathbb{R}^n \times (\mathbb{R} \setminus 0) \times \mathbb{R}^{n-1})$ so that the difference $(\bar{c}_{j,\varepsilon}^{(0)} - \bar{b}_{j,\varepsilon}^{(0)})_\varepsilon$ is in $\mathcal{N}_{\mathcal{S}^1}$ on I'_{θ_1} in the sense of Definition 2.3.

Therefore, since $(c_\varepsilon(y))_\varepsilon$ is strictly non-zero we can define for fixed $\varepsilon \in (0, \varepsilon_1]$

$$f_\varepsilon(y, \tau, \xi) := \frac{\langle b_\varepsilon(y)\xi, \xi \rangle}{c_\varepsilon(y)\tau^2} \quad \text{on } \mathbb{R}^n \times (\mathbb{R} \setminus 0) \times \mathbb{R}^{n-1}.$$

Recall that $\varepsilon_1 \in (0, 1]$ is so that $\inf_{y \in \mathbb{R}^n} |c_\varepsilon(y)| \geq C$ for some $C > 0$ and every $\varepsilon \in (0, \varepsilon_1]$. Furthermore for every fixed $\varepsilon \in (0, \varepsilon_1]$ we let \tilde{h}_ε be the smooth function defined on $\mathbb{R}^n \times (\mathbb{R} \setminus 0) \times \mathbb{R}^{n-1}$, $0 \leq \tilde{h}_\varepsilon \leq 1$, which is given by

$$\tilde{h}_\varepsilon(y, \tau, \xi) := \begin{cases} 0, & |f_\varepsilon| \geq \sin^2 \gamma_2 \\ 1, & |f_\varepsilon| \leq \sin^2 \gamma_1 \\ \left(1 + e^{\frac{1}{|f_\varepsilon| - \sin^2 \gamma_1} - \frac{1}{\sin^2 \gamma_2 - |f_\varepsilon|}}\right)^{-1}, & \sin^2 \gamma_1 < |f_\varepsilon| < \sin^2 \gamma_2 \end{cases} \tag{4.8}$$

for some fixed γ_1 and γ_2 with $0 < \gamma_1 < \gamma_2 < \pi/2$. For all other $\varepsilon \in (0, 1]$ we may set $\tilde{h}_\varepsilon \equiv 0$ and we obtain that $(\tilde{h}_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{S}^{0,0}}^{(1,0)}(\mathbb{R}^n \times (\mathbb{R} \setminus 0) \times \mathbb{R}^{n-1})$ and also $(\tilde{h}_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{S}^{0,0}}^{(1,0)}$ on I'_{θ_2} .

Here and hereafter we let θ_1, θ_2 be fixed so that $0 < \theta_1 < \gamma_1 < \gamma_2 < \theta_2 < \pi/2$. Then there is an $\varepsilon_2 \in (0, 1]$ such that

$$I'_{\theta_1} \subset \{(y, \tau, \xi) \mid \tau \neq 0 \text{ and } |f_\varepsilon| \leq \sin^2 \gamma_1\} \quad \varepsilon \in (0, \varepsilon_2]$$

because of the assumptions made on the coefficients $c_\varepsilon(y), b_{j,\varepsilon}(y), y \in \mathbb{R}^n, 1 \leq j \leq n - 1$.

Similar observations yield to the following: there exists $\varepsilon_3 \in (0, 1]$ such that

$$\{(y, \tau, \xi) \mid \tau \neq 0, |f| \geq \sin^2 \theta_2\} \subset \{(y, \tau, \xi) \mid \tau \neq 0, |f_\varepsilon| \geq \sin^2 \gamma_2\}$$

for all $\varepsilon \in (0, \varepsilon_3]$ where we have set $f := \lim_{\varepsilon \rightarrow 0} f_\varepsilon$.

Then with $(\bar{b}_{j,\varepsilon}^{(0)})_\varepsilon$ as in (4.7) we set $\varepsilon_0 := \min_{1 \leq k \leq 3} \varepsilon_k$ and define its extension $(\bar{c}_{j,\varepsilon}^{(0)})_\varepsilon$ by

$$(\bar{c}_{j,\varepsilon}^{(0)})_\varepsilon := (\tilde{h}_\varepsilon \bar{b}_{j,\varepsilon}^{(0)})_\varepsilon \in \mathcal{M}_{\mathcal{S}^{1,0}}^{(1,0)}(\mathbb{R}^n \times (\mathbb{R} \setminus 0) \times \mathbb{R}^{n-1}).$$

Indeed we have that $(\bar{c}_{j,\varepsilon}^{(0)} - \bar{b}_{j,\varepsilon}^{(0)})_\varepsilon \in \mathcal{N}_{\mathcal{S}^1}$ on I'_{θ_1} since $\tilde{h}_\varepsilon \equiv 1$ on I'_{θ_1} .

Factorization procedure. Our aim here is to decompose the operator L_ψ as announced in (4.6). Therefore we give a construction scheme for the generalized symbols $(a_{j,\varepsilon})_\varepsilon$ of the operators $A_{j,\psi}$, $j = 1, 2$ by means of their polyhomogeneous asymptotic expansions; i.e., on the set I'_{θ_1} we have

$$(a_{j,\varepsilon}(y, \tau, \xi))_\varepsilon \sim \sum_{\mu \geq 0} (\varepsilon^\mu \bar{c}_{j,\varepsilon}^{(\mu)}(y, \tau, \xi))_\varepsilon \quad \text{in } \mathcal{M}_{\mathcal{S}^{1,0}}^{(1,0)} \tag{4.9}$$

and the sequence $\{(\bar{c}_{j,\varepsilon}^{(\mu)})_\varepsilon\}_{\mu \in \mathbb{N}}$ consists of elements $(\bar{c}_{j,\varepsilon}^{(\mu)})_\varepsilon \in \mathcal{M}_{\mathcal{S}^{1-\mu,0}}^{(1,\mu)}(\mathbb{R}^n \times (\mathbb{R} \setminus 0) \times \mathbb{R}^{n-1})$ and satisfies the uniformity condition:

$$\exists \eta \in (0, 1] \exists K > 0 \forall \mu \in \mathbb{N} \forall \alpha, \beta \in \mathbb{N}^n \exists C > 0 \text{ for which}$$

$$|\partial_{(\tau,\xi)}^\alpha \partial_y^\beta \bar{c}_{j,\varepsilon}^{(\mu)}(y, \tau, \xi)| \leq C \omega_\varepsilon^{|\beta|} \langle \xi \rangle^{1-\mu-|\alpha|} \quad \text{for } |\tau| \geq K, \varepsilon \in (0, \eta]$$

for $j = 1, 2$. More precisely we will recursively construct a sequence $\{(\bar{b}_{j,\varepsilon}^{(\mu)})_\varepsilon\}_\mu$ of symbols $(\bar{b}_{j,\varepsilon}^{(\mu)})_\varepsilon$ in $\mathcal{M}_{\mathcal{S}_{\text{hg}}^{1-\mu,\mu}}^{(1,\mu)}$ on I'_{θ_2} , $j = 1, 2$ such that

$$(\bar{c}_{j,\varepsilon}^{(\mu)}(y, \tau, \xi))_\varepsilon = (\varepsilon^{-\mu} \bar{b}_{j,\varepsilon}^{(\mu)}(y, \tau, \xi))_\varepsilon \quad \text{on } I'_{\theta_2}.$$

Recall that on I'_{θ_1} (4.9) is equivalent to the following: there exists a smooth cut-off equal to 1 near the origin such that for every $N \geq 1$ we have on I'_{θ_1}

$$(a_{j,\varepsilon} - \sum_{\mu=0}^{N-1} \varepsilon^\mu (1 - \varphi) \bar{c}_{j,\varepsilon}^{(\mu)})_\varepsilon \in \mathcal{M}_{\mathcal{S}^{1-N,N}}^{(1,N)} \quad \text{for } |(\tau, \xi)| \text{ sufficiently large.}$$

Proof of Theorem 4.3. To begin with, we set for $j = 1, 2$ and $\varepsilon \in (0, \varepsilon_0]$

$$\bar{a}_{j,\varepsilon}^{(1)} := \bar{b}_{j,\varepsilon}^{(0)}, \quad l_{j,\varepsilon}^{(1)} := i\zeta + a_{j,\varepsilon}^{(1)}$$

where

$$\bar{b}_{j,\varepsilon}^{(0)}(y, \tau, \xi) = \pm i \sqrt{a_\varepsilon(y, \tau, \xi)} \quad \text{on } I'_{\theta_2}, \quad j = 1, 2.$$

Moreover $(a_{j,\varepsilon}^{(1)})_\varepsilon \in \mathcal{M}_{\mathcal{S}_{\text{phg}}^{1,0}}^{(1,0)}$ with polyhomogeneous asymptotic expansion $(\tilde{h}_\varepsilon \bar{b}_{j,\varepsilon}^{(0)})_\varepsilon$ with \tilde{h}_ε as in (4.8) and $(\bar{b}_{j,\varepsilon}^{(0)})_\varepsilon$ only prescribed on I'_{θ_2} , $j = 1, 2$. As in (4.7) we set $\bar{b}_{j,\varepsilon}^{(0)}$ and $l_{j,\varepsilon}^{(1)}$ equal to zero for all other $\varepsilon \in (0, 1]$. Furthermore we define the first-order operator $L_{j,\psi}^{(1)}$ through $(\partial_z + A_j^{(1)})_\psi$ where $A_{j,\psi}^{(1)} = \text{OP}_\psi((a_{j,\varepsilon}^{(1)})_\varepsilon)$ for $j = 1, 2$.

Now taking $L_{1,\psi}^{(1)} L_{2,\psi}^{(1)}$ as a first approximation for L_ψ we can compute the error as follows:

$$\begin{aligned} L_{1,\psi}^{(1)} L_{2,\psi}^{(1)} - L_\psi &= (\partial_z + A_1^{(1)})_\psi (\partial_z + A_2^{(1)})_\psi - (\partial_z^2 + A)_\psi \\ &= (A_1^{(1)} + A_2^{(1)})_\psi (\partial_z)_\psi + \text{OP}_\psi((\varepsilon \partial_z a_{2,\varepsilon}^{(1)})_\varepsilon) + A_{1,\psi}^{(1)} A_{2,\psi}^{(1)} - A_\psi \\ &=: \Gamma_{0,\psi}^{(1)} + \Gamma_{1,\psi}^{(1)} (\partial_z)_\psi \end{aligned}$$

where we have used the Leibniz composition rule for ψ -pseudodifferential operators; that is, $(\partial_z)_\psi A_{2,\psi}^{(1)} = A_{2,\psi}^{(1)}(\partial_z)_\psi + \text{OP}_\psi((\varepsilon\partial_z a_{2,\varepsilon}^{(1)})_\varepsilon)$. Then $\Gamma_{1,\psi}^{(1)} = (A_1^{(1)} + A_2^{(1)})_\psi \in \text{OP}_\psi \mathcal{N}_{\mathcal{S}^1}$ on I'_{θ_1} . Moreover $\Gamma_{0,\psi}^{(1)}$ is equal to $\text{OP}_\psi((\varepsilon\partial_z a_{2,\varepsilon}^{(1)})_\varepsilon) + A_{1,\psi}^{(1)}A_{2,\psi}^{(1)} - A_\psi$ and is a polyhomogeneous ψ -pseudodifferential operator with generalized symbol $(\gamma_{0,\varepsilon}^{(1)})_\varepsilon$ in $\mathcal{M}_{\mathcal{S}^{0,1}}^{(1,1)}(I'_{\theta_1})$ by Proposition 3.8. In detail $(\gamma_{0,\varepsilon}^{(1)})_\varepsilon$ admits the asymptotic expansion

$$(\gamma_{0,\varepsilon}^{(1)})_\varepsilon \sim \left(\varepsilon\partial_z \bar{a}_{2,\varepsilon}^{(1)} + \sum_{|\alpha| \geq 1} \frac{\varepsilon^{|\alpha|}}{\alpha!} (D_\xi^\alpha \bar{a}_{1,\varepsilon}^{(1)})(\partial_x \bar{a}_{2,\varepsilon}^{(1)}) \right)_\varepsilon \quad \text{on } I'_{\theta_1}.$$

To improve this argument we will construct recursively the N -th approximation: $L_{1,\psi}^{(N)}L_{2,\psi}^{(N)} - L_\psi$. Before, for convenience of the reader, we compute the second order approximation for L_ψ .

Therefore we let $\varepsilon \in (0, \varepsilon_0]$ be fixed and define

$$\bar{a}_{j,\varepsilon}^{(2)} := \bar{a}_{j,\varepsilon}^{(1)} + \bar{b}_{j,\varepsilon}^{(1)}, \quad l_{j,\varepsilon}^{(2)} := i\zeta + a_{j,\varepsilon}^{(2)}$$

where $(a_{j,\varepsilon}^{(2)})_\varepsilon$ denotes the symbol with polyhomogeneous asymptotic expansion $(\tilde{h}_\varepsilon \bar{a}_{j,\varepsilon}^{(2)})_\varepsilon$ and the existence of the generalized symbol $(\bar{b}_{j,\varepsilon}^{(1)})_\varepsilon$ will be clarified immediately. Again we set $\bar{b}_{j,\varepsilon}^{(1)}$ and $l_{j,\varepsilon}^{(2)}$ equal zero for all other $\varepsilon \in (0, 1]$. From the preceding observations we have the approximation

$$L_{1,\psi}^{(1)}L_{2,\psi}^{(1)} - L_\psi = \Gamma_{0,\psi}^{(1)} + \Gamma_{1,\psi}^{(1)}(\partial_z)_\psi \quad \text{on } I'_{\theta_1}.$$

To obtain a suitable second order approximation we would have to keep the following expression small

$$\begin{aligned} L_{1,\psi}^{(2)}L_{2,\psi}^{(2)} - L_\psi &= (L_1^{(1)} + B_1^{(1)})_\psi(L_2^{(1)} + B_2^{(1)})_\psi - L_\psi \\ &= \Gamma_{0,\psi}^{(1)} + \Gamma_{1,\psi}^{(1)}(\partial_z)_\psi + (B_1^{(1)} + B_2^{(1)})_\psi(\partial_z)_\psi \\ &\quad + \text{OP}_\psi((\varepsilon\partial_z b_{2,\varepsilon}^{(1)})_\varepsilon) + A_{1,\psi}^{(1)}B_{2,\psi}^{(1)} + B_{1,\psi}^{(1)}A_{2,\psi}^{(1)} + B_{1,\psi}^{(1)}B_{2,\psi}^{(1)} \end{aligned} \tag{4.10}$$

where in the above step we have used the Leibniz rule.

We now specify $(\bar{b}_{1,\varepsilon}^{(1)})_\varepsilon, (\bar{b}_{2,\varepsilon}^{(1)})_\varepsilon$ on I'_{θ_2} as follows: because the the generalized symbol $(\bar{a}_{j,\varepsilon}^{(1)})_\varepsilon$ satisfies:

$$\exists \eta \in (0, 1] : |\bar{a}_{j,\varepsilon}^{(1)}(y, \tau, \xi)| \geq C|(\tau, \xi)| \quad \text{on } I'_{\theta_2}, \quad \forall \varepsilon \in (0, \eta] \tag{4.11}$$

for some constant $C > 0$ that is independent of $\varepsilon \in (0, \eta], j = 1, 2$, the matrix

$$\begin{pmatrix} 1 & 1 \\ \bar{a}_{2,\varepsilon}^{(1)} & \bar{a}_{1,\varepsilon}^{(1)} \end{pmatrix} \tag{4.12}$$

is invertible on I'_{θ_2} . We denote by $(\tilde{\gamma}_{0,\varepsilon}^{(1)})_\varepsilon$ and $(\tilde{\gamma}_{1,\varepsilon}^{(1)})_\varepsilon$ the naturally extended top order symbols on I'_{θ_2} of $(\gamma_{0,\varepsilon}^{(1)})_\varepsilon$ and $(\gamma_{1,\varepsilon}^{(1)})_\varepsilon$ on I'_{θ_1} respectively. Thus the system

$$\begin{aligned} -\tilde{\gamma}_{1,\varepsilon}^{(1)} &= \bar{b}_{1,\varepsilon}^{(1)} + \bar{b}_{2,\varepsilon}^{(1)} \\ -\tilde{\gamma}_{0,\varepsilon}^{(1)} &= \bar{b}_{1,\varepsilon}^{(1)}\bar{a}_{2,\varepsilon}^{(1)} + \bar{a}_{1,\varepsilon}^{(1)}\bar{b}_{2,\varepsilon}^{(1)} \end{aligned}$$

is uniquely solvable for $(\bar{b}_{j,\varepsilon}^{(1)})_\varepsilon$ in $\mathcal{M}_{\mathcal{S}^{0,1}}^{(1,1)}$ on $I'_{\theta_2}, j = 1, 2$.

With this choice of $(\bar{b}_{j,\varepsilon}^{(1)})_\varepsilon$ the second-order approximation for the operator L_ψ in (4.10) now reads

$$L_{1,\psi}^{(2)}L_{2,\psi}^{(2)} - L_\psi = \Gamma_{0,\psi}^{(2)} + \Gamma_{1,\psi}^{(2)}(\partial_z)_\psi \quad \text{on } I'_{\theta_1}$$

and $\Gamma_{0,\psi}^{(2)} = \Gamma_{0,\psi}^{(2)}(y, D_t, D_x)$ and $\Gamma_{1,\psi}^{(2)} = \Gamma_{1,\psi}^{(2)}(y, D_t, D_x)$ are the ψ -pseudodifferential operators with generalized symbols $(\gamma_{0,\varepsilon}^{(2)})_\varepsilon$ in $\mathcal{M}_{\mathcal{S}_{\text{phg}}^{0,2}}^{(1,2)}$ on I'_{θ_1} and $(\gamma_{1,\varepsilon}^{(2)})_\varepsilon$ is in $\mathcal{N}_{\mathcal{S}^1}$ on I'_{θ_1} , respectively.

To continue the proof, we assume for $N \geq 1$ that $(\bar{b}_{j,\varepsilon}^{(\nu)})_\varepsilon \in \mathcal{M}_{\mathcal{S}_{\text{hg}}^{1-\nu,\nu}}^{(1,\nu)}$ on I'_{θ_2} is determined for all $\nu \leq N - 1$ and $j = 1, 2$. For fixed $\varepsilon \in (0, \varepsilon_0]$ and $j = 1, 2$ we set

$$\bar{a}_{j,\varepsilon}^{(N)} := \sum_{\nu=0}^{N-1} \bar{b}_{j,\varepsilon}^{(\nu)}, \quad L_{j,\psi}^{(N)} = (\partial_z + A_j^{(N)})_\psi$$

with $A_{j,\psi}^{(N)}(y, D_t, D_x) = \text{OP}_\psi((a_{j,\varepsilon}^{(N)})_\varepsilon)$ with $(a_{j,\varepsilon}^{(N)})_\varepsilon$ the symbol with polyhomogeneous expansion $(\tilde{h}_\varepsilon \bar{a}_{j,\varepsilon}^{(N)})_\varepsilon$. Again for all other $\varepsilon \in (0, 1]$ we set $a_{j,\varepsilon}^{(N)}$ and $L_{j,\psi}^{(N)}$ equal to zero, $j = 1, 2$. Furthermore, we suppose an N -th order approximation for L_ψ of the form

$$L_{1,\psi}^{(N)}L_{2,\psi}^{(N)} - L_\psi = \Gamma_{0,\psi}^{(N)} + \Gamma_{1,\psi}^{(N)}(\partial_z)_\psi \quad \text{on } I'_{\theta_1}$$

where $\Gamma_{0,\psi}^{(N)}$ and $\Gamma_{1,\psi}^{(N)}$ are the ψ -pseudodifferential operators with generalized symbols $(\gamma_{0,\varepsilon}^{(N)})_\varepsilon$ in $\mathcal{M}_{\mathcal{S}_{\text{phg}}^{2-N,N}}^{(1,N)}$ on I'_{θ_1} and $(\gamma_{1,\varepsilon}^{(N)})_\varepsilon$ in $\mathcal{N}_{\mathcal{S}^1}$ on I'_{θ_1} , respectively.

Again we denote by $(\tilde{\gamma}_{0,\varepsilon}^{(N)})_\varepsilon$ and $(\tilde{\gamma}_{1,\varepsilon}^{(N)})_\varepsilon$ the naturally extended top order terms on I'_{θ_2} of $(\gamma_{0,\varepsilon}^{(N)})_\varepsilon$ and $(\gamma_{1,\varepsilon}^{(N)})_\varepsilon$ on I'_{θ_1} . For the induction step we specify $(b_{1,\varepsilon}^{(N)})_\varepsilon, (b_{2,\varepsilon}^{(N)})_\varepsilon$ on I'_{θ_2} as follows: since $(\bar{a}_{j,\varepsilon}^{(1)})_\varepsilon$ satisfies (4.11) the matrix in (4.12) is invertible on I'_{θ_2} and thus the system

$$\begin{aligned} -\tilde{\gamma}_{1,\varepsilon}^{(N)} &= \bar{b}_{1,\varepsilon}^{(N)} + \bar{b}_{2,\varepsilon}^{(N)} \\ -\tilde{\gamma}_{0,\varepsilon}^{(N)} &= \bar{b}_{1,\varepsilon}^{(N)} \bar{a}_{2,\varepsilon}^{(1)} + \bar{a}_{1,\varepsilon}^{(1)} \bar{b}_{2,\varepsilon}^{(N)} \end{aligned}$$

is uniquely solvable for $(\bar{b}_{j,\varepsilon}^{(N)})_\varepsilon$ in $\mathcal{M}_{\mathcal{S}_{\text{hg}}^{1-N,N}}^{(1,N)}$ on I'_{θ_2} , $j = 1, 2$.

We write $B_{j,\psi}^{(N)}$ for the ψ -pseudodifferential operator with polyhomogeneous generalized symbol $(b_{j,\varepsilon}^{(N)})_\varepsilon := (\tilde{h}_\varepsilon \bar{b}_{j,\varepsilon}^{(N)})_\varepsilon$ and set $L_{j,\psi}^{(N+1)} := (L_j^{(N)} + B_j^{(N)})_\psi$. Then the following is valid

$$\begin{aligned} L_{1,\psi}^{(N+1)}L_{2,\psi}^{(N+1)} - L_\psi &= (L_1^{(N)} + B_1^{(N)})_\psi (L_2^{(N)} + B_2^{(N)})_\psi - L_\psi \\ &= \Gamma_{0,\psi}^{(N)} + \Gamma_{1,\psi}^{(N)}(\partial_z)_\psi + (B_1^{(N)} + B_2^{(N)})_\psi (\partial_z)_\psi \\ &\quad + \text{OP}_\psi((\varepsilon \partial_z \bar{b}_{2,\varepsilon}^{(N)})_\varepsilon) + A_{1,\psi}^{(N)} B_{2,\psi}^{(N)} + B_{1,\psi}^{(N)} A_{2,\psi}^{(N)} + B_{1,\psi}^{(N)} B_{2,\psi}^{(N)}. \end{aligned}$$

Indeed we have an $(N+1)$ -th order approximation of the form

$$L_{1,\psi}^{(N+1)}L_{2,\psi}^{(N+1)} - L_\psi = \Gamma_{0,\psi}^{(N+1)} + \Gamma_{1,\psi}^{(N+1)}(\partial_z)_\psi$$

and $\Gamma_{0,\psi}^{(N+1)}$ and $\Gamma_{1,\psi}^{(N+1)}$ are the ψ -pseudodifferential operators with generalized symbols $(\gamma_{0,\varepsilon}^{(N+1)})_\varepsilon \in \mathcal{M}_{\text{phg}}^{(1,N+1)}$ and $(\gamma_{1,\varepsilon}^{(N+1)})_\varepsilon \in \mathcal{N}_{\mathcal{S}^1}$ on I'_{θ_1} , respectively. This completes the induction step. \square

5. THE GENERALIZED INFINITE WAVE FRONT SET

In this section we will discuss an alternative description of microlocality and regularity theory in Colombeau algebras of generalized functions. We refer to [31, 17, 18] for more details on the commonly used notion of a generalized wave front set. Therein the generalized wave front set is explained by replacing the standard \mathcal{C}^∞ -regularity by \mathcal{G}^∞ -regularity. In the same manner we can reformulate local H^∞ -regularity used in the Sobolev based wave front set in terms of \mathcal{G}^∞ -regularity.

In the case of generalized pseudodifferential operators that satisfies logarithmic slow scale estimates this is a straightforward modification of the regularity results in [17] within the Colombeau theory in a Sobolev based context as in [42]. But the situation changes dramatically when working with ψ -pseudodifferential operators with symbols of log-type.

For this reason we will introduce the notion of generalized infinite wave front set describing negligibility of a generalized function at infinite points. For further studies about the infinite wave front set and the more refined semiclassical wave front set we refer to [8, 1] and [30, 11, 23].

5.1. Microlocal behavior at infinity. Here we introduce a suitable notion of asymptotic negligibility of a generalized function $u \in \mathcal{G}_{2,2}(\mathbb{R}^n)$ with respect to certain regularities on the phase space.

To do so we use the following notation which is similar to [8]. For a non-zero vector $\xi_0 \in \mathbb{R}^n$ we write for the projection onto the unit sphere $\xi_0/|\xi_0|$. Moreover, given such ξ_0 we say that $\Gamma_{\infty\xi_0} \subset \mathbb{R}^n$ is a conic neighborhood of the direction $\xi_0/|\xi_0|$ if $\Gamma_{\infty\xi_0}$ is the intersection of the complement of some open ball centered at the origin with an open cone containing the direction.

Furthermore, for (x_0, ξ_0) in $T^*\mathbb{R}^n \setminus 0$ we say that a generalized symbol $p := (p_\varepsilon)_\varepsilon \in \mathcal{M}_{\text{phg}}^{(\nu,0)}$ is elliptic at $(x_0, \infty\xi_0)$ or also elliptic at infinity at (x_0, ξ_0) , if there is an open neighborhood U of x_0 and a constant $C > 0$ such that

$$|p_\varepsilon(x, \xi)| \geq C \langle \xi \rangle^m \quad \forall (x, \xi) \in U \times \Gamma_{\infty\xi_0} \text{ as } \varepsilon \rightarrow 0.$$

We denote by $\text{Ell}^i(p)$ the set of all points $(x_0, \xi_0) \in T^*\mathbb{R}^n \setminus 0$ where p is elliptic at infinity.

Then the generalized infinite wave front set is defined as follows.

Definition 5.1. For $u \in \mathcal{G}_{2,2}(\mathbb{R}^n)$ we denote by $\text{WF}^i(u) \subset T^*\mathbb{R}^n \setminus 0$ the generalized infinite wave front set of u which is characterized as follows. We say that $(x_0, \xi_0) \notin \text{WF}^i(u)$ at infinity, denoted by $(x_0, \infty\xi_0) \notin \text{WF}^i(u)$, if there exists $\chi := (\chi_\varepsilon)_\varepsilon \in \mathcal{M}_{\text{phg}}^{(0,0)}$ elliptic at $(x_0, \infty\xi_0)$ so that

$$\text{OP}_\psi(\chi)u = 0 \quad \text{in } \mathcal{G}_{2,2}(\mathbb{R}^n).$$

Remark 5.2. Let $(u_\varepsilon)_\varepsilon \in \mathcal{M}_{H^\infty}$ and suppose that there exists $\chi_m := (\chi_{m,\varepsilon})_\varepsilon \in \mathcal{M}_{\text{phg}}^{(0,0)}$ for some $m \in \mathbb{R}$ elliptic at $(x_0, \infty\xi_0)$ with

$$\forall q \in \mathbb{N} : \|\text{OP}_{\psi,\varepsilon}(\chi_{m,\varepsilon})u_\varepsilon\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(\varepsilon^q) \quad \text{as } \varepsilon \rightarrow 0.$$

Using [12, Proposition 3.4] then the above line is equivalent to $\text{OP}_\psi(\chi_m)u = 0$ in $\mathcal{G}_{2,2}$.

Furthermore we introduce the notion of an infinite conic support to describe singularities of ψ -pseudodifferential operators. Therefore let $p := (p_\varepsilon)_\varepsilon \in \mathcal{M}_{\text{phg}}^{(v,l),k}$ be the generalized symbol of P_ψ . Then the infinite conic support of P_ψ is the set $\text{conesupp}^i(p) \subset T^*\mathbb{R}^n \setminus 0$ and is defined as the complement (in $T^*\mathbb{R}^n \setminus 0$) of the set of points $(x_0, \xi_0) \in T^*\mathbb{R}^n \setminus 0$ such that there exist an open neighborhood U of x_0 , a conic open neighborhood Γ_{ξ_0} of the direction ξ_0 and a constant $K > 0$ such that the following is satisfied:

$$\forall \alpha, \beta \in \mathbb{N}^n \quad \forall N \in \mathbb{N} : |\partial_\xi^\alpha \partial_x^\beta p_\varepsilon(x, \xi)| \langle \xi \rangle^{-m+N+|\alpha|} = \mathcal{O}(\varepsilon^N) \quad \text{as } \varepsilon \rightarrow 0, \quad (5.1)$$

uniformly in $(x, \xi) \in U \times (\Gamma_{\xi_0} \cap \{|\xi| \geq K\})$. This is again a condition on the behavior for a generalized symbol at infinity and we therefore say that $(x_0, \xi_0) \notin \text{conesupp}^i(p)$ at infinity and write $(x_0, \infty\xi_0) \notin \text{conesupp}^i(p)$ whenever (5.1) is fulfilled.

The idea here is that $\text{conesupp}^i(p)^c$ are the directions on the phase space in which P_ψ annihilates singularities as they are contained in $\mathcal{N}_{\mathcal{G}^m}(U \times \Gamma)$. To give a connection to these two notions of wave front sets we state the following theorem.

Theorem 5.3. *Given $u \in \mathcal{G}_{2,2}$ and P_ψ a ψ -pseudodifferential operator with generalized symbol $p = (p_\varepsilon)_\varepsilon \in \mathcal{M}_{\text{phg}}^{(v,l),k}$. Then the following statement is valid*

$$\text{WF}^i(P_\psi u) \subset \text{WF}^i(u) \cap \text{conesupp}^i(p)$$

and we say that the ψ -pseudodifferential operator P_ψ is microlocal at infinity.

We remark that most of the properties of the infinite wave front set of a generalized function in $\mathcal{G}_{2,2}$ can be derived from the theorem of Calderón-Vaillancourt for the class pseudodifferential operators with symbols in $S_{0,0}^0$.

Proof. We first show the inclusion relation $\text{WF}^i(P_\psi u) \subset \text{conesupp}^i(p)$. Therefore we let $(x_0, \xi_0) \in T^*\mathbb{R}^n \setminus 0$ such that $(x_0, \infty\xi_0) \notin \text{conesupp}^i(p)$ which in turn implies that $(p_\varepsilon)_\varepsilon$ is in $\mathcal{N}_{\mathcal{G}^m}(U \times \Gamma_{\xi_0})$ for some open neighborhood U of x_0 and some conic open neighborhood Γ_{ξ_0} of ξ_0 . More precisely, we have that $\exists K_1 > 0$ such that for all $\alpha, \beta \in \mathbb{N}^n$ and all $N \in \mathbb{N}$

$$|\partial_\xi^\alpha \partial_x^\beta p_\varepsilon(x, \xi)| \langle \xi \rangle^{-m+N+|\alpha|} = \mathcal{O}(\varepsilon^N) \quad \text{as } \varepsilon \rightarrow 0,$$

uniformly in $(x, \xi) \in U \times \Gamma_{\infty\xi_0}$ with $\Gamma_{\infty\xi_0} = \Gamma_{\xi_0} \cap \{|\xi| \geq K_1\}$. We will now construct a symbol $(\chi_{J,K}^{(-m)})_\varepsilon \in \mathcal{M}_{\text{phg}}^{(0,0),-m,0}$ elliptic at $(x_0, \infty\xi_0)$ such that

$$\forall q \in \mathbb{N} : \|\text{OP}_{\psi,\varepsilon}(\chi_{J,K}^{(-m)})P_{\psi,\varepsilon}u_\varepsilon\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(\varepsilon^q) \quad \text{as } \varepsilon \rightarrow 0.$$

For that reason let $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ such that $\phi(z) = 1$ for $|z| \leq 1/2$ and $\phi(z) = 0$ for $|z| \geq 1$. Now, given $(x_0, \xi_0) \in T^*\mathbb{R}^n \setminus 0$ we define for fixed $J > 0$ the function

$$\lambda_J^{(-m)}(x, \xi) := \phi\left(\frac{x - x_0}{J}\right) \phi\left(\left\{\frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|}\right\} \frac{1}{J}\right) |\xi|^{-m} \quad \text{for } \xi \neq 0.$$

Further for some fixed $K > 0$ let

$$\chi_{J,K}^{(-m)}(x, \xi) := (1 - \phi)\left(\frac{\xi}{2K}\right) \lambda_J^{(-m)}(x, \xi). \quad (5.2)$$

Then for $J, K > 0$ fixed $(\chi_{J,K}^{(-m)}(x, \xi))_\varepsilon \in \mathcal{M}_{\text{phg}}^{(0,0)}$ is elliptic at $(x_0, \infty\xi_0)$ and supported in

$$|x - x_0| \leq J, \quad \left| \frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|} \right| \leq J, \quad |\xi| \geq K.$$

So $\chi_{J,K}^{(-m)}$ is a cut-off in a conic neighborhood of ξ_0 and is supported in a cone of directions near ξ_0 . In particular we have $\chi_{J,K}^{(-m)} \# p_\varepsilon = 0$ for $|\xi| \leq K$. We now choose $J, K > 0$ such that

$$\text{supp}(\chi_{J,K}^{(-m)}) \subset U \times \Gamma_{\infty\xi_0}.$$

Then by the L^2 -boundedness theorem of Calderón-Vaillancourt for symbols of class $S_{0,0}^0$, see [41, Chapter 13, Theorem 1.3], there are constants $j_0, j_1 \in \mathbb{N}$ and $C > 0$ each of which depend on n but independent of $\varepsilon > 0$ such that

$$\begin{aligned} \|\text{OP}_{\psi,\varepsilon}(\chi_{J,K}^{(-m)})P_{\psi,\varepsilon}u_\varepsilon\|_{L^2} &= \|\text{OP}_{\psi,\varepsilon}(\chi_{J,K}^{(-m)} \#_2 p_\varepsilon)u_\varepsilon\|_{L^2} \\ &\leq C \sup_{|\alpha| \leq j_0, |\beta| \leq j_1} \|\partial_\xi^\alpha \partial_x^\beta (\chi_{J,K}^{(-m)} \#_2 p_\varepsilon)\|_{L^\infty(\mathbb{R}^{2n})} \|u_\varepsilon\|_{L^2} \end{aligned}$$

where in the last step we performed the rescaling by means of the asymptotic expansion of the second kind for the composition formula of the pseudodifferential operators expressed by using the notation $\#_2$, see Subsection 3.2. Moreover we used the following estimation

$$\begin{aligned} |\partial_\xi^\alpha \partial_x^\beta (\chi_{J,K}^{(-m)} \#_2 p_\varepsilon(x, \varepsilon\xi))| &= |\varepsilon^{|\alpha|} (\partial_\xi^\alpha \partial_x^\beta (\chi_{J,K}^{(-m)} \#_2 p_\varepsilon)(x, \varepsilon\xi)| \\ &\leq \varepsilon^{|\alpha|} \sup_{(x,\xi) \in \mathbb{R}^{2n}} |(\partial_\xi^\alpha \partial_x^\beta (\chi_{J,K}^{(-m)} \#_2 p_\varepsilon)(x, \varepsilon\xi)| \\ &= \varepsilon^{|\alpha|} \|\partial_\xi^\alpha \partial_x^\beta (\chi_{J,K}^{(-m)} \#_2 p_\varepsilon)\|_{L^\infty(\mathbb{R}^{2n})}. \end{aligned}$$

Now using the fact that $(p_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{S}^m}(U \times \Gamma_{\xi_0})$ one has for every $q \in \mathbb{N}$ that

$$\sup_{|\alpha| \leq j_0, |\beta| \leq j_1} \|\partial_\xi^\alpha \partial_x^\beta (\chi_{J,K}^{(-m)} \#_2 p_\varepsilon)\|_{L^\infty(\mathbb{R}^{2n})} = \mathcal{O}(\varepsilon^q) \quad \text{as } \varepsilon \rightarrow 0$$

by Proposition 3.8 and we deduce that $(x_0, \infty\xi_0) \notin \text{WF}^i(P_\psi u)$ which completes the first part of the proof.

We continue the proof by showing the following inclusion:

Claim. $\text{WF}^i(P_\psi u) \subset \text{WF}^i(u)$.

Proof of the Claim. We take $(x_0, \infty\xi_0) \notin \text{WF}^i(u)$ and let U be some open neighborhood of x_0 and Γ_{ξ_0} a conic neighborhood of ξ_0 . Then, for some $j \in \mathbb{N}$ there exists $\chi_j := (\chi_{j,\varepsilon})_\varepsilon \in \mathcal{M}_{\text{phg}}^{(0,0)}$ elliptic at $(x_0, \infty\xi_0)$ such that for some constant $K_1 > 0$ we have

$$\exists C > 0 \exists \eta \in (0, 1] : |\chi_{j,\varepsilon}(x, \xi)| \geq C \langle \xi \rangle^j \quad \forall \varepsilon \in (0, \eta] \tag{5.3}$$

uniformly on $U \times \Gamma_{\infty\xi_0}$ with $\Gamma_{\infty\xi_0} = \Gamma_{\xi_0} \cap \{(x, \xi) : |\xi| \geq K_1\}$ and for all $q \in \mathbb{N}$,

$$\|\text{OP}_{\psi,\varepsilon}(\chi_{j,\varepsilon})u_\varepsilon\|_{L^2} = \mathcal{O}(\varepsilon^q) \quad \text{as } \varepsilon \rightarrow 0.$$

Then the assumptions in Lemma 6.1 from the next section are satisfied on $U \times \Gamma_{\xi_0}$ and there is a $\tilde{\chi}_{-j} := (\tilde{\chi}_{-j,\varepsilon})_\varepsilon \in \mathcal{M}_{\text{phg}}^{(0,0)}$ such that

$$\text{OP}_\psi(\tilde{\chi}_{-j}) \text{OP}_\psi(\chi_j) = I_\psi + R_\psi \quad \text{on } U \times \Gamma_{\xi_0}$$

where the generalized symbol $(r_\varepsilon)_\varepsilon$ of R_ψ is in $\mathcal{M}_{\text{phg}}^{(0,0)}$ on $U \times \Gamma_{\xi_0}$ for every $N \in \mathbb{N}$.

As in (5.2) one constructs a generalized symbol $(\kappa_{j-m})_\varepsilon \in \mathcal{M}_{\mathcal{S}_{\text{phg}}^{j-m,0}}^{(0,0)}$ elliptic at $(x_0, \infty\xi_0)$ such that the $\text{supp}(\kappa_{j-m})$ is contained in the set where (5.3) is valid. We then write

$$\begin{aligned} & \text{OP}_{\psi,\varepsilon}(\kappa_{j-m}\#_2 p_\varepsilon) \\ &= \text{OP}_{\psi,\varepsilon}(\kappa_{j-m}\#_2 p_\varepsilon\#_2 \tilde{\chi}_{-j,\varepsilon}) \text{OP}_{\psi,\varepsilon}(\chi_{j,\varepsilon}) - \text{OP}_{\psi,\varepsilon}(\kappa_{j-m}\#_2 p_\varepsilon\#_2 r_\varepsilon). \end{aligned}$$

Concerning the first term on the right hand side we rescale and obtain by the Calderón-Vaillancourt theorem that there are constants $j_0, j_1 \in \mathbb{N}$ and $C > 0$ such that

$$\begin{aligned} & \|\text{OP}_{\psi,\varepsilon}(\kappa_{j-m}\#_2 p_\varepsilon\#_2 \tilde{\chi}_{-j,\varepsilon}) \text{OP}_{\psi,\varepsilon}(\chi_{j,\varepsilon}) u_\varepsilon\|_{L^2} \\ & \leq C \sup_{|\alpha| \leq j_0, |\beta| \leq j_1} \|\partial_\xi^\alpha \partial_x^\beta \kappa_{j-m}\#_2 p_\varepsilon\#_2 \tilde{\chi}_{-j,\varepsilon}\|_{L^\infty(\mathbb{R}^{2n})} \|\text{OP}_{\psi,\varepsilon}(\chi_{j,\varepsilon}) u_\varepsilon\|_{L^2} \end{aligned}$$

and the latter expression is $\mathcal{O}(\varepsilon^q)$ as $\varepsilon \rightarrow 0$ for every $q \in \mathbb{N}$ by assumption. Similarly we obtain for every $q \in \mathbb{N}$:

$$\|\text{OP}_{\psi,\varepsilon}(\kappa_{j-m}\#_2 p_\varepsilon\#_2 r_\varepsilon) u_\varepsilon\|_{L^2} = \mathcal{O}(\varepsilon^q) \quad \text{as } \varepsilon \rightarrow 0$$

since $(r_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{S}^0}$ on the support of κ_{j-m} . Therefore $(x_0, \infty\xi_0) \notin \text{WF}^i(P_\psi u)$. \square

As a next result we reformulate the infinite wave front set in terms of the Fourier transform of a localized function.

For this purpose we introduce the space $\mathcal{G}_c(\mathbb{R}^n)$ of compactly supported generalized functions consisting of those $u \in \mathcal{G}(\mathbb{R}^n)$ such that for some $K \Subset \mathbb{R}^n$ the restriction u to $\mathbb{R}^n \setminus K$ is equal to 0 as an element of $\mathcal{G}(\mathbb{R}^n \setminus K)$. Note that for an open subset Ω of \mathbb{R}^n the space $\mathcal{G}(\Omega)$ is defined by the space \mathcal{G}_E by setting $E = \mathcal{E}(\Omega)$ the space of smooth functions on Ω topologized through the family of seminorms $p_{K_n,j}(f) := \sup_{x \in K_n, |\alpha| \leq j} |\partial^\alpha f(x)|$ where $(K_n)_n$ is an exhausting sequence of compact subsets of Ω .

Proposition 5.4. *Let $u \in \mathcal{G}_c(\mathbb{R}^n)$. If $(x_0, \infty\xi_0) \notin \text{WF}^i(u)$ then there exists $\gamma \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ with $\gamma(x_0) \neq 0$ and a conic neighborhood Γ of ξ_0 such that for all $N, q \in \mathbb{N}$ there exists $C > 0$ satisfying*

$$|\mathcal{F}(\gamma u_\varepsilon)(\xi)| \leq C \varepsilon^q \langle \xi \rangle^{-N} \quad \text{as } \varepsilon \rightarrow 0 \tag{5.4}$$

for all $\xi \in \Gamma$ with $|\xi| \geq K/\varepsilon$ for some $K > 0$ independent of ε .

Note that our notion of regularity is derived from the infinite wave front set and therefore differs from [28, Section 6] and [17, Sections 2,3]. It is also different to [31, Definitions 3.13 and 3.14, Section 3.2.2] if one replaces there the condition of \mathcal{G}^∞ rapid decrease by rapid decrease. We remark that a generalized function $u \in \mathcal{G}(\mathbb{R}^n)$ is of rapid decrease if it has a representative $(u_\varepsilon)_\varepsilon$ with the property:

$$\begin{aligned} & \exists N \in \mathbb{N} \forall p \in \mathbb{N} \exists C > 0 \exists \eta \in (0, 1] : \\ & |u_\varepsilon(x)| \leq C \varepsilon^{-N} \langle x \rangle^{-p} \quad \forall x \in \mathbb{R}^n, \varepsilon \in (0, \eta]. \end{aligned}$$

Proof. The following proof is similar to [21, Proposition 7.4]. First note that $\mathcal{G}_c \subset \mathcal{G}_{2,2}$. Since $(x_0, \infty\xi_0) \notin \text{WF}^i(u)$ we can find a $\chi := (\chi_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{S}_{\text{phg}}^{-m,0}}^{(0,0)}$ elliptic at $(x_0, \infty\xi_0)$ such that $\text{OP}_\psi(\chi)u = 0$ in $\mathcal{G}_{2,2}$ for some $m \in \mathbb{R}$. Furthermore we let $\gamma \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ with $\gamma(x_0) \neq 0$. Then there exists a symbol $\phi \in \mathcal{S}^0(\mathbb{R}^n)$, $\text{supp}(\phi) \subset$

$\Gamma_{\infty\xi_0} = \Gamma_{\xi_0} \cap \{\xi : |\xi| \geq K\}$ for some $K > 0$ and $\phi(t\xi) = 1$ for $t \geq 1, \xi \in \Gamma_{\infty\xi_0}$ so that

$$\phi_\psi(D)\gamma(x) = A_\psi \text{OP}_\psi(\chi) + R_\psi$$

where A_ψ and R_ψ is a ψ -pseudodifferential operator with generalized symbol in $\mathcal{M}_{\mathcal{S}^{m,0}}^{(0,0)}$ and $\mathcal{N}_{\mathcal{S}^0}$, respectively. Then by assumption we deduce that $\phi_\psi(D)\gamma u = 0$ in $\mathcal{G}_{2,2}$. In particular we deduce that $\phi_\psi(D)\gamma u = 0$ in $\mathcal{G}_{\mathcal{S}}$ since $\mathcal{G}_{\mathcal{S}} \hookrightarrow \mathcal{G}_{2,2}$, see [12, Proposition 3.5]. Using the fact that the Fourier transform is an isomorphism on $\mathcal{G}_{\mathcal{S}}$ we obtain $\xi \mapsto (\phi(\varepsilon\xi)\widehat{\gamma u}_\varepsilon(\xi))_\varepsilon$ is in $\mathcal{N}_{\mathcal{S}}$ under consideration of the scaling. Hence we also have that

$$\forall N \in \mathbb{N} \forall q \in \mathbb{N} : \|\langle \xi/\varepsilon \rangle^N \phi(\xi)\widehat{\gamma u}_\varepsilon(\xi/\varepsilon)\|_{L^\infty} = \mathcal{O}(\varepsilon^q)$$

showing (5.4). □

We also study the behavior of the infinite wave front set of a function under the action of ψ -pseudodifferential operators that are elliptic at infinity.

Theorem 5.5. *Let P_ψ be a ψ -pseudodifferential operator with symbol $p := (p_\varepsilon)_\varepsilon$ in $\mathcal{M}_{\mathcal{S}^{m,0}}^{(\nu,0)}$ that is elliptic at $(x_0, \infty\xi_0) \in T^*\mathbb{R}^n \setminus 0$. Then the following holds:*

$$\text{WF}^i(u) \subset \text{WF}^i(P_\psi u) \cup \text{Ell}^i(p)^c.$$

Proof. Suppose $(x_0, \infty\xi_0)$ is not contained in the right hand side of the claimed inclusion relation. Since $(x_0, \infty\xi_0) \in \text{Ell}^i(p)$ the symbol $(p_\varepsilon)_\varepsilon$ is elliptic at $(x_0, \infty\xi_0)$ and therefore there is an open neighborhood U of x_0 and conic neighborhood Γ_{ξ_0} containing ξ_0 such that

$$|p_\varepsilon(x, \xi)| \geq C\langle \xi \rangle^m \quad \forall (x, \xi) \in U \times \Gamma_{\infty\xi_0} \text{ as } \varepsilon \rightarrow 0.$$

By Lemma 6.1 from the next section there exists an approximative inverse Q_ψ so that

$$I_\psi = Q_\psi P_\psi + R_\psi \quad \text{on } U \times \Gamma_{\xi_0} \tag{5.5}$$

and $(x_0, \infty\xi_0) \notin \text{cone supp}^i(r)$ and $r := (r_\varepsilon)_\varepsilon$ is the generalized symbol of R_ψ . Hence $(x_0, \infty\xi_0) \notin \text{WF}^i(R_\psi u)$ by Theorem 5.3.

Furthermore, since $(x_0, \infty\xi_0) \notin \text{WF}^i(P_\psi u)$ we obtain that

$$(x_0, \infty\xi_0) \notin \text{WF}^i(Q_\psi P_\psi u) \subset \text{WF}^i(P_\psi u)$$

and by (5.5) we deduce that $(x_0, \infty\xi_0) \notin \text{WF}^i(u)$. □

5.2. Microlocal factorization. In this section we use the notion of microlocal behavior at infinite points of a given generalized function in $\mathcal{G}_{2,2}(\mathbb{R}^n)$ to give a microlocal interpretation of Theorem 4.3. To do so let I be a conic subset of $\mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$. We then say that two generalized functions $u, v \in \mathcal{G}_{2,2}$ are microlocally equivalent at infinity on I if and only if there exists $(\chi_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{S}^{\infty,0}}^{(0,0)}$ elliptic at $(x_0, \infty\xi_0)$ for every $(x_0, \xi_0) \in I$ such that $\chi_\psi(u - v) = 0$ in $\mathcal{G}_{2,2}$.

Similar to [37] we introduce a subset of the phase space associated to I'_{θ_1} which is given by

$$I_{\theta_1} := \{(t, y, \tau, \eta) \mid (y, \tau, \xi) \in I'_{\theta_1}, |\zeta| \leq \sqrt{c_1}|\tau|\} \tag{5.6}$$

and $c_1 > 0$ is the upper bound of the Hölder continuous coefficient $c(y)$. Note that in the following I_{θ_1} will serve as the adequate space-frequency domain on which we establish the microlocal factorization theorem at infinite points.

As already mentioned in the preceding section $L_{j,\psi} = (\partial_z + A_j(y, D_t, D_x))_\psi$, $j = 1, 2$ is not a ψ -pseudodifferential operator on \mathbb{R}^{n+1} . To overcome this we are going to introduce a microlocal cut-off $g_\psi(D_t, D_y)$ for I_{θ_1} such that

$$g_\psi L_{j,\psi} u = L_{j,\psi} u \quad \text{microlocally at infinity on } I_{\theta_1} \tag{5.7}$$

where $u \in \mathcal{G}_{2,2}$ and $g_\psi L_{j,\psi}$ is a ψ -pseudodifferential operator with generalized symbol in $\mathcal{M}_{\mathcal{S}^{1,0}}^{(1,0)}$. Here the microlocal cut-off g_ψ is constructed in the following way.

First let K_0, K_1 be some fixed constants so that $0 < K_0 < K_1 < \infty$. Then $g \in \mathcal{C}^\infty(\mathbb{R}^{n+1})$, $0 \leq g \leq 1$ is defined by

$$g(\tau, \eta) := \begin{cases} 0, & |\zeta| \geq 3\sqrt{c_1}|(\tau, \xi)| \text{ or } |(\tau, \xi)| \leq K_0 \\ \tilde{\sigma}(\tau, \xi), & |\zeta| \leq 2\sqrt{c_1}|(\tau, \xi)| \\ \frac{\tilde{\sigma}(\tau, \xi)}{1 + e^{\frac{|\zeta|/|(\tau, \xi)| - 2\sqrt{c_1}|(\tau, \xi)|}{3\sqrt{c_1}|(\tau, \xi)| - |\zeta|/|(\tau, \xi)|}}}, & 2\sqrt{c_1}|(\tau, \xi)| < |\zeta| < 3\sqrt{c_1}|(\tau, \xi)| \end{cases}$$

and the function $\tilde{\sigma}$ is a cut-off near the origin given by

$$\tilde{\sigma}(\tau, \xi) := \begin{cases} 0, & |(\tau, \xi)| \leq K_0 \\ 1, & |(\tau, \xi)| \geq K_1 \\ \frac{1}{1 + e^{\frac{K_1 - K_0}{|(\tau, \xi)| - K_0} - \frac{K_1 - K_0}{K_1 - |(\tau, \xi)|}}}, & K_0 < |(\tau, \xi)| < K_1. \end{cases}$$

Then $g \in S^0(\mathbb{R}^{n+1})$ has the form shown in Figure 2:

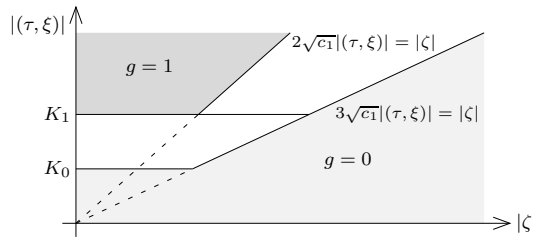


FIGURE 2.

Moreover, $g_\psi L_{j,\psi}$ acts as a ψ -pseudodifferential operator in (t, x, z) with generalized symbol in $\mathcal{M}_{\mathcal{S}^{1,0}}^{(1,0)}$ which can be shown by an adaption of [25, Theorem 18.1.35].

Furthermore (5.7) is satisfied, since $\text{conesupp}^i(g - 1) \cap I_{\theta_1} = \emptyset$. In the same manner one shows that $g_\psi L_{1,\psi} L_{2,\psi} = L_{1,\psi} L_{2,\psi}$ microlocally at infinity on I_{θ_1} .

Summarizing the observations from above we obtain the following main theorem:

Theorem 5.6. *Let L_ψ and I_{θ_1} be as in (4.1) and (5.6). Then the operator L_ψ can be factorized into a product of two first-order ψ -pseudodifferential operators as follows*

$$L_\psi = L_{1,\psi} L_{2,\psi} \quad \text{microlocally at infinity on } I_{\theta_1}$$

where $L_{j,\psi} = (\partial_z + A_j)_\psi$ and $A_{j,\psi}$ is the ψ -pseudodifferential operator as in Theorem 4.3.

6. MICROLOCAL DIAGONALIZATION FOR L_ψ

The main issue in this section is to diagonalize the microlocal equation $L_\psi U = F$ using the refined factorization theorem 5.6. In detail, we will rewrite the equation

$$L_\psi U = F \quad \text{microlocally at infinity on } I_{\theta_1}$$

into an equivalent system of the form

$$(\partial_z - iB_\pm(x, z, D_t, D_x))_\psi u_\pm = f_\pm \quad \text{microlocally at infinity on } I_{\theta_1}.$$

To show this we will discuss a different approach to the one given by Stolk in [37] as it turns out that the factorization theorem already contains all the ingredients for the diagonalization. Let us begin with the following lemma which was already used in the previous section.

Lemma 6.1. *Let $m \in \mathbb{R}$ and P_ψ be a polyhomogeneous ψ -pseudodifferential operator whose generalized symbol is given by $(p_\varepsilon)_\varepsilon \sim \sum_{j=0}^\infty (\varepsilon^j \bar{p}_{m-j,\varepsilon})_\varepsilon$ in $\mathcal{M}_{\mathcal{S}^{m,0}}^{(\nu,0)}$ on I'_{θ_1} for some $(\bar{p}_{m-j,\varepsilon})_\varepsilon \in \mathcal{M}_{\mathcal{S}^{m-j,0}}^{(\nu,\nu j)}$ on I'_{θ_1} , $j \geq 0$. Furthermore suppose that the principal symbol $(\bar{p}_{m,\varepsilon})_\varepsilon$ satisfies an estimate of the form*

$$\exists C > 0 \exists \eta \in (0, 1] : |\bar{p}_{m,\varepsilon}(y, \tau, \xi)| \geq C |(\tau, \xi)|^m \quad \text{on } I'_{\theta_1}, \varepsilon \in (0, \eta]. \tag{6.1}$$

Then there exists a ψ -pseudodifferential operator Q_ψ with generalized symbol in $\mathcal{M}_{\mathcal{S}^{-m,0}}^{(\nu,0)}$ such that the following is valid

$$Q_\psi P_\psi = I_\psi + R_\psi \quad \text{on } I'_{\theta_1}, \tag{6.2}$$

where I is the identity and the generalized symbol of R_ψ is in $\mathcal{N}_{\mathcal{S}^0}$ on I'_{θ_1} . More precisely, the polyhomogeneous generalized symbol $(q_\varepsilon)_\varepsilon$ of Q_ψ is written in terms of its asymptotic expansion as

$$(q_\varepsilon)_\varepsilon \sim \sum_{k \geq 0} (\varepsilon^k \bar{q}_{-m-k,\varepsilon})_\varepsilon \quad \text{in } \mathcal{M}_{\mathcal{S}^{-m,0}}^{(\nu,0)} \quad \text{on } I'_{\theta_1} \tag{6.3}$$

for some $(\bar{q}_{-m-k,\varepsilon})_\varepsilon \in \mathcal{M}_{\mathcal{S}^{-m-k,0}}^{(\nu,\nu k)}$ on I'_{θ_1} , $k \geq 0$.

Proof. The proof follows the classical arguments given in [29, Chapter 2] for ε -dependent symbols. We construct the generalized symbol of Q_ψ by means of its asymptotic expansion. Therefore we will recursively define symbols $(\bar{q}_{-m-k,\varepsilon})_\varepsilon \in \mathcal{M}_{\mathcal{S}^{-m-k,0}}^{(\nu,\nu k)}$ on I'_{θ_1} , $k \geq 0$, so that the symbol of Q_ψ is given by (6.3) and satisfies (6.2).

Because of (6.1) we may define for $(y, \tau, \xi) \in I'_{\theta_1}$ and some $\eta \in (0, 1]$,

$$\bar{q}_{-m,\varepsilon}(y, \tau, \xi) := \begin{cases} \bar{p}_{m,\varepsilon}(y, \tau, \xi)^{-1}, & \varepsilon \in (0, \eta] \\ 0, & \text{otherwise.} \end{cases}$$

Concerning the asymptotic behavior of the derivatives of $\bar{q}_{-m,\varepsilon}$ we use the Leibniz rule and obtain that there exists $C > 0$ such that for every $\varepsilon > 0$ sufficiently small,

$$\begin{aligned} |\partial_{(\tau,\xi)}^\alpha \partial_y^\beta \bar{q}_{-m,\varepsilon}| &= \left| \sum_{\substack{\alpha_1 + \dots + \alpha_\mu = \alpha \\ \beta_1 + \dots + \beta_\mu = \beta}} \left(\partial_{(\tau,\xi)}^{\alpha_1} \partial_y^{\beta_1} \bar{p}_{m,\varepsilon} \right) \cdots \left(\partial_{(\tau,\xi)}^{\alpha_\mu} \partial_y^{\beta_\mu} \bar{p}_{m,\varepsilon} \right) \frac{1}{\bar{p}_{m,\varepsilon}^{1+\mu}} \right| \\ &\leq C \omega_\varepsilon^{|\beta|} |(\tau, \xi)|^{-m-|\alpha|} \quad \text{on } I'_{\theta_1} \end{aligned}$$

from which we deduce that $(\bar{q}_{-m,\varepsilon})_\varepsilon \in \mathcal{M}_{\mathcal{S}_{\text{hg}}^{-m,0}}^{(\nu,0)}$ on I'_{θ_1} . We now proceed by induction. Therefore we define $(\bar{q}_{-m-k,\varepsilon})_\varepsilon$ on I'_{θ_1} by

$$\bar{q}_{-m-k,\varepsilon} := -\left\{ \sum_{|\gamma|+j+l=k, l < k} \frac{1}{\gamma!} D_\xi^\gamma \bar{q}_{-m-l,\varepsilon} \partial_x^\gamma \bar{p}_{m-j,\varepsilon} \right\} \frac{1}{\bar{p}_{m,\varepsilon}} \quad k \geq 1 \text{ on } I'_{\theta_1}. \tag{6.4}$$

Assuming that $(\bar{q}_{-m-N,\varepsilon})_\varepsilon \in \mathcal{M}_{\mathcal{S}_{\text{hg}}^{-m-N,0}}^{(\nu,\nu N)}(I'_{\theta_1})$ for $N < k$ then from the construction given in (6.4) one easily verifies that $(\bar{q}_{-m-k,\varepsilon})_\varepsilon \in \mathcal{M}_{\mathcal{S}_{\text{hg}}^{-m-k,0}}^{(\nu,\nu k)}(I'_{\theta_1})$.

Moreover by Lemma 3.4 we get the existence of a polyhomogeneous generalized symbol $(q_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{S}_{\text{phg}}^{-m,0}}^{(\nu,0)}$ having the following asymptotic expansion

$$(q_\varepsilon)_\varepsilon \dot{\sim} \sum_{k \geq 0} (\varepsilon^k \bar{q}_{-m-k,\varepsilon})_\varepsilon \quad \text{in } \mathcal{M}_{\mathcal{S}^{-m,0}}^{(\nu,0)} \text{ on } I'_{\theta_1}.$$

Let Q_ψ be the ψ -pseudodifferential operator with generalized symbol $(q_\varepsilon)_\varepsilon$. Then (6.3) is satisfied and it remains to show equation (6.2). Concerning the asymptotic expansion of $Q_\psi P_\psi$ one has for every $N \geq 1$

$$\begin{aligned} \sum_{|\gamma| < N} \frac{\varepsilon^{|\gamma|}}{\gamma!} D_\xi^\gamma q_\varepsilon \partial_x^\gamma p_\varepsilon &= \bar{q}_{-m,\varepsilon} \bar{p}_{m,\varepsilon} + \sum_{k=1}^{N-1} \sum_{|\gamma|+l+j=k} \frac{\varepsilon^k}{\gamma!} D_\xi^\gamma \bar{q}_{-m-l,\varepsilon} \partial_x^\gamma \bar{p}_{m-j,\varepsilon} \\ &+ \sum_{k \geq N} \sum_{\substack{|\gamma|+l+j=k \\ |\gamma| < N}} \frac{\varepsilon^k}{\gamma!} D_\xi^\gamma \bar{q}_{-m-l,\varepsilon} \partial_x^\gamma \bar{p}_{m-j,\varepsilon} \end{aligned} \tag{6.5}$$

on $V_K := I'_{\theta_1} \cap (\mathbb{R}^n \times \{(\tau, \xi) : |(\tau, \xi)| \geq K\})$ for some $K > 0$ independent of ε . Here the second term on the right hand side vanishes by (6.4) on V_K . Furthermore the last expression of (6.5) is in $\mathcal{N}_{\mathcal{S}^0}(I'_{\theta_1})$ and $\bar{q}_{-m,\varepsilon} \bar{p}_{m,\varepsilon} = 1$ on V_K which establishes the statement made in (6.2). \square

Similarly one can construct a ψ -pseudodifferential operator \tilde{Q}_ψ with generalized symbol in $\mathcal{M}_{\mathcal{S}_{\text{phg}}^{-m,0}}^{(\nu,0)}$ having a representation of the form (6.3) and satisfies

$$P_\psi \tilde{Q}_\psi = I_\psi + \tilde{R}_\psi \quad \text{on } I'_{\theta_1}$$

where the generalized symbol of \tilde{R}_ψ is in $\mathcal{N}_{\mathcal{S}^0}$ on I'_{θ_1} . Furthermore Q_ψ is related to \tilde{Q}_ψ in the following way

$$Q_\psi = Q_\psi(P_\psi \tilde{Q}_\psi) = (Q_\psi P_\psi) \tilde{Q}_\psi = \tilde{Q}_\psi \quad \text{mod } \text{OP}_\psi \mathcal{N}_{\mathcal{S}^m} \text{ on } I'_{\theta_1}.$$

Diagonalization. To get an idea we start rewriting the inhomogeneous equation $L_\psi U = F$ into a first-order system with respect to the parameter z :

$$\left[(\text{Id } \partial_z)_\psi - \begin{pmatrix} 0 & 1 \\ -A & 0 \end{pmatrix}_\psi \right] \begin{pmatrix} U \\ (\partial_z)_\psi U \end{pmatrix} = \begin{pmatrix} 0 \\ F \end{pmatrix} \tag{6.6}$$

with $U, F \in \mathcal{G}_{2,2}(\mathbb{R}^{n+1})$, Id the 2×2 identity matrix and A_ψ the ψ -pseudodifferential operator with generalized symbol $(a_\varepsilon)_\varepsilon$ as in (4.2). For brevity we will drop the identity matrix Id in the equations from now on. Hereafter we are going to reduce the operator in (6.6) to diagonal form. To make this notion rigorous we make the following arrangements.

First let

$$Q_\psi = \begin{pmatrix} Q_\psi^{11} & Q_\psi^{12} \\ Q_\psi^{21} & Q_\psi^{22} \end{pmatrix}_\psi, \quad P_\psi = \begin{pmatrix} P_\psi^{11} & P_\psi^{12} \\ P_\psi^{21} & P_\psi^{22} \end{pmatrix}_\psi$$

be 2×2 matrices of ψ -pseudodifferential operators whose entries satisfy the requirements of Lemma 6.1. Also we choose the top order of the symbols of Q_ψ^{11}, Q_ψ^{21} equal to $1 - m$ and those of Q_ψ^{12}, Q_ψ^{22} equal to $-m$ for some fixed $m \in \mathbb{R}$.

Using Lemma 6.1 we choose P_ψ to be the approximative inverse of Q_ψ on I'_{θ_1} in the following sense: with P_ψ^{11}, P_ψ^{12} being operators of order $m - 1$ and P_ψ^{21}, P_ψ^{22} of order m and one has $P_\psi Q_\psi = \text{Id}_\psi + E_\psi$ on I'_{θ_1} for some 2×2 ψ -pseudodifferential operator matrix E_ψ of the form

$$E_\psi = \begin{pmatrix} E_\psi^{11} & E_\psi^{12} \\ E_\psi^{21} & E_\psi^{22} \end{pmatrix}_\psi$$

and $E_\psi^{11}, E_\psi^{22} \in \text{OP}_\psi \mathcal{N}_{\mathcal{S}^0}, E_\psi^{12} \in \text{OP}_\psi \mathcal{N}_{\mathcal{S}^{-1}}$ and $E_\psi^{21} \in \text{OP}_\psi \mathcal{N}_{\mathcal{S}^1}$ on I'_{θ_1} .

Then by (5.7) and (6.6) the equation

$$L_\psi U = F \quad \text{microlocally at infinity on } I_{\theta_1}$$

holds if and only if

$$g_\psi Q_\psi \begin{pmatrix} \partial_z & -1 \\ A & \partial_z \end{pmatrix}_\psi P_\psi Q_\psi \begin{pmatrix} U \\ (\partial_z)_\psi U \end{pmatrix} = g_\psi Q_\psi \begin{pmatrix} 0 \\ F \end{pmatrix} \quad \text{microlocally at infinity on } I_{\theta_1}$$

where g_ψ is the microlocal cut-off function from the previous section.

Furthermore we will use the following notation: for $l = 1, 2$ and $k \in \mathbb{N}$ we denote by $\mathcal{E}_\psi^{(k,l)}$ operators of the form

$$\sum_{k \leq j \leq k+l} R_{2-j,\psi}(y, D_t, D_x) (\partial_z^{j-k})_\psi$$

and the generalized symbol of $R_{2-j,\psi} = R_{2-j,\psi}(y, D_t, D_x)$ is in $\mathcal{N}_{\mathcal{S}^{2-j}}$ for $k \leq j \leq k + l$.

In the following we assume that R_ψ is a ψ -pseudodifferential operator valued 2×2 error matrix with entries in $\mathcal{E}_\psi^{(1,1)}$ on I'_{θ_1} . Also, we let $B_{\pm,\psi} = B_{\pm,\psi}(y, D_t, D_x) \in \text{OP}_\psi \mathcal{M}_{\mathcal{S}^{1,0}}^{\text{phg}}$ on I'_{θ_1} .

To obtain a diagonalization for the operator in (6.6) we will search for operators P_ψ, Q_ψ, R_ψ and $B_{\pm,\psi}$ as above such that the following adapted formulation of the problem is valid in the region I'_{θ_1} :

$$Q_\psi \left[(\partial_z)_\psi - \begin{pmatrix} 0 & 1 \\ -A & 0 \end{pmatrix}_\psi \right] P_\psi = \begin{pmatrix} \partial_z - iB_+ & 0 \\ 0 & \partial_z - iB_- \end{pmatrix}_\psi + R_\psi. \quad (6.7)$$

To show the existence of these operators we will construct them explicitly. Therefore we presume that (6.7) can be solved for some P_ψ, Q_ψ, R_ψ and $B_{\pm,\psi}$ with the above conditions.

Multiplying equation (6.7) with $Q_\psi \begin{pmatrix} 1 \\ (\partial_z)_\psi \end{pmatrix}$ from the right we obtain for the left hand side an expression of the form

$$Q_\psi \begin{pmatrix} \partial_z & -1 \\ A & \partial_z \end{pmatrix}_\psi P_\psi Q_\psi \begin{pmatrix} 1 \\ (\partial_z)_\psi \end{pmatrix} = \begin{pmatrix} Q_\psi^{12}(\partial_z^2 + A)_\psi + S_\psi^{(1,+)} \\ Q_\psi^{22}(\partial_z^2 + A)_\psi + S_\psi^{(1,-)} \end{pmatrix} \quad \text{on } I'_{\theta_1} \quad (6.8)$$

with $S_\psi^{(1,\pm)} \in \mathcal{E}_\psi^{(m,2)}$ on I'_{θ_1} . Similarly we compute for the right hand side

$$\begin{aligned} & \left[\begin{pmatrix} \partial_z - iB_+ & 0 \\ 0 & \partial_z - iB_- \end{pmatrix}_\psi + R_\psi \right] Q_\psi \begin{pmatrix} 1 \\ \partial_z \end{pmatrix}_\psi \\ &= \begin{pmatrix} (\partial_z - iB_+)_\psi (Q_\psi^{12}(\partial_z)_\psi + Q_\psi^{11}) + S_\psi^{(2,+)} \\ (\partial_z - iB_-)_\psi (Q_\psi^{22}(\partial_z)_\psi + Q_\psi^{21}) + S_\psi^{(2,-)} \end{pmatrix} \quad \text{on } I'_{\theta_1} \end{aligned} \tag{6.9}$$

where $S_\psi^{(2,\pm)} \in \mathcal{E}_\psi^{(m,2)}$ on I'_{θ_1} . Combining (6.8) and (6.9) we get an improved formulation of (6.7) which reads

$$\begin{aligned} Q_\psi^{12}(\partial_z^2 + A)_\psi &= (\partial_z - iB_+)_\psi (Q_\psi^{12}(\partial_z)_\psi + Q_\psi^{11}) + R_\psi^{(+)} \quad \text{on } I'_{\theta_1} \\ Q_\psi^{22}(\partial_z^2 + A)_\psi &= (\partial_z - iB_-)_\psi (Q_\psi^{22}(\partial_z)_\psi + Q_\psi^{21}) + R_\psi^{(-)} \quad \text{on } I'_{\theta_1}, \end{aligned} \tag{6.10}$$

with $R_\psi^{(\pm)} \in \mathcal{E}_\psi^{(m,2)}$ on I'_{θ_1} .

We note that (6.10) is a coupled system of two equations each of which stating a factorization similar to (4.6) in Theorem 4.3. In the following we relate (4.6) to (6.10) which will guarantee existence of the basic approach made in (6.7). Now since Theorem 4.3 allows two different factorizations (depending on the sign of the principal symbol) we will modify both of them to deduce the refined reformulation in (6.10).

Thus, with a view to Theorem 4.3, we first write $L_\psi = (\partial_z^2 + A)_\psi$ in the form

$$L_\psi = (\partial_z + A_{11})_\psi (\partial_z + A_{12})_\psi + \Gamma_{1,\psi} \quad \text{on } I'_{\theta_1} \tag{6.11}$$

and for $j = 1, 2$, $A_{1j,\psi} = A_{1j,\psi}(y, D_t, D_x)$ is a polyhomogeneous ψ -pseudodifferential operator with generalized symbol $(a_{1j,\varepsilon})_\varepsilon$ as in Theorem 4.3. Moreover we choose $A_{11,\psi}$ and $-A_{12,\psi}$ such that their principal symbols are equal to $(-i\sqrt{a_\varepsilon})_\varepsilon$. Furthermore $\Gamma_{1,\psi}$ is in $\mathcal{E}_\psi^{(0,1)}$ on I'_{θ_1} . Likewise we obtain

$$L_\psi = (\partial_z + A_{21})_\psi (\partial_z + A_{22})_\psi + \Gamma_{2,\psi} \quad \text{on } I'_{\theta_1} \tag{6.12}$$

where $A_{2j,\psi} = A_{2j,\psi}(y, D_t, D_x)$, $j = 1, 2$ are polyhomogeneous ψ -pseudodifferential operators but at this point the top order symbols of $A_{21,\psi}$ and $-A_{22,\psi}$ equal to $(i\sqrt{a_\varepsilon})_\varepsilon$. Again $\Gamma_{2,\psi}$ is in $\mathcal{E}_\psi^{(0,1)}$ on I'_{θ_1} .

An expansion in (6.11) and (6.12) then gives the following for $j = 1, 2$

$$(\partial_z^2 + A)_\psi = (\partial_z^2)_\psi + (A_{j1} + A_{j2})_\psi (\partial_z)_\psi + \text{OP}_\psi((\varepsilon \partial_z a_{j2})_\varepsilon) + A_{j1,\psi} A_{j2,\psi} + \Gamma_{j,\psi}$$

on I'_{θ_1} where we have used the Leibniz rule. By construction of the generalized symbols of $A_{j1,\psi}$ and $A_{j2,\psi}$ we observe that $(a_{j1,\varepsilon})_\varepsilon = (-a_{j2,\varepsilon})_\varepsilon$ modulo $\mathcal{N}_{\mathcal{S}^1}$ on I'_{θ_1} , $j = 1, 2$. Using this (6.11) and (6.12) then read

$$\begin{aligned} (\partial_z^2 + A)_\psi &= (\partial_z + A_{11})_\psi (\partial_z - A_{11})_\psi \quad \text{mod } \mathcal{E}_\psi^{(0,1)} \quad \text{on } I'_{\theta_1} \\ &= (\partial_z + A_{21})_\psi (\partial_z - A_{21})_\psi \quad \text{mod } \mathcal{E}_\psi^{(0,1)} \quad \text{on } I'_{\theta_1}. \end{aligned} \tag{6.13}$$

With m being the fixed real number from the beginning of this section we now choose $\tilde{Q}_\psi^{(\pm)} \in \text{OP}_\psi \mathcal{M}_{\mathcal{S}^{\text{phg}}}^{(1,0)}$ and so that the requirements of Lemma 6.1 are fulfilled.

Using the same result we obtain the existence of two ψ -pseudodifferential operators $Q_\psi^{(\pm)}$ with generalized symbols in $\mathcal{M}_{\mathcal{S}^{\text{phg}}}^{(1,0)}$ satisfying (6.3) and

$$\tilde{Q}_\psi^{(\pm)} Q_\psi^{(\pm)} = Q_\psi^{(\pm)} \tilde{Q}_\psi^{(\pm)} = I_\psi \quad \text{mod } \text{OP}_\psi \mathcal{N}_{\mathcal{S}^0} \quad \text{on } I'_{\theta_1}.$$

Inserting $\tilde{Q}_\psi^{(+)}Q_\psi^{(+)}$ into the first line of (6.13) and $\tilde{Q}_\psi^{(-)}Q_\psi^{(-)}$ into the second line yields

$$\begin{aligned} (\partial_z^2 + A)_\psi &= (\partial_z + A_{11})_\psi \tilde{Q}_\psi^{(+)} (Q_\psi^{(+)}(\partial_z)_\psi - Q_\psi^{(+)}A_{11,\psi}) \quad \text{mod } \mathcal{E}_\psi^{(0,2)} \text{ on } I'_{\theta_1} \\ &= (\partial_z + A_{21})_\psi \tilde{Q}_\psi^{(-)} (Q_\psi^{(-)}(\partial_z)_\psi - Q_\psi^{(-)}A_{21,\psi}) \quad \text{mod } \mathcal{E}_\psi^{(0,2)} \text{ on } I'_{\theta_1}. \end{aligned}$$

Here we define

$$\begin{aligned} -iB_{+,\psi} &:= Q_\psi^{(+)}A_{11,\psi}\tilde{Q}_\psi^{(+)} - \text{OP}_\psi((\varepsilon\partial_z q_\varepsilon^{(+)})_\varepsilon)\tilde{Q}_\psi^{(+)} \\ -iB_{-,\psi} &:= Q_\psi^{(-)}A_{21,\psi}\tilde{Q}_\psi^{(-)} - \text{OP}_\psi((\varepsilon\partial_z q_\varepsilon^{(-)})_\varepsilon)\tilde{Q}_\psi^{(-)}. \end{aligned} \tag{6.14}$$

Consequently, $B_{\pm,\psi}$ are polyhomogeneous ψ -pseudodifferential operators with real-valued top order symbol in $\mathcal{M}_{\mathcal{S}_{\text{hg}}}^{(1,0)}$ on I'_{θ_1} . Furthermore a straightforward calculation shows that with this choices for $B_{\pm,\psi}$ we have

$$\begin{aligned} (\partial_z + A_{11})_\psi \tilde{Q}_\psi^{(+)} &= \tilde{Q}_\psi^{(+)}(\partial_z - iB_+)_\psi \quad \text{mod } \mathcal{E}_\psi^{(1-m,0)} \text{ on } I'_{\theta_1} \\ (\partial_z + A_{21})_\psi \tilde{Q}_\psi^{(-)} &= \tilde{Q}_\psi^{(-)}(\partial_z - iB_-)_\psi \quad \text{mod } \mathcal{E}_\psi^{(1-m,0)} \text{ on } I'_{\theta_1}. \end{aligned}$$

leading to the following decomposition on I'_{θ_1} ,

$$\begin{aligned} (\partial_z^2 + A)_\psi &= \tilde{Q}_\psi^{(+)}(\partial_z - iB_+)_\psi (Q_\psi^{(+)}(\partial_z)_\psi - Q_\psi^{(+)}A_{11,\psi}) \quad \text{mod } \mathcal{E}_\psi^{(0,2)} \\ &= \tilde{Q}_\psi^{(-)}(\partial_z - iB_-)_\psi (Q_\psi^{(-)}(\partial_z)_\psi - Q_\psi^{(-)}A_{21,\psi}) \quad \text{mod } \mathcal{E}_\psi^{(0,2)}. \end{aligned}$$

Comparing this with the refined approach (6.10) we make the following choice for the matrix Q_ψ from the beginning

$$\begin{aligned} Q_\psi^{11} &:= -Q_\psi^{(+)}A_{11,\psi}, & Q_\psi^{12} &:= Q_\psi^{(+)}, \\ Q_\psi^{21} &:= -Q_\psi^{(-)}A_{21,\psi}, & Q_\psi^{22} &:= Q_\psi^{(-)}. \end{aligned}$$

Hence

$$Q_\psi = \begin{pmatrix} -Q_\psi^{(+)}A_{11,\psi} & Q_\psi^{(+)} \\ -Q_\psi^{(-)}A_{21,\psi} & Q_\psi^{(-)} \end{pmatrix} = \begin{pmatrix} Q^{(+)} & 0 \\ 0 & Q^{(-)} \end{pmatrix}_\psi \begin{pmatrix} -A_{11} & 1 \\ -A_{21} & 1 \end{pmatrix}_\psi \tag{6.15}$$

and $Q_\psi^{(\pm)}$ are polyhomogeneous ψ -pseudodifferential operators of order $-m$ on I'_{θ_1} and elliptic in the sense of (6.1).

Since the top order symbol of $A_{j1,\psi}$ is given by $(\pm i\sqrt{a_\varepsilon})_\varepsilon$, $j = 1, 2$ it follows from (6.15) that Q_ψ is elliptic in the sense of (6.1). Furthermore, the approximative inverse matrix P_ψ of Q_ψ is given by

$$P_\psi = \begin{pmatrix} -C_\psi & C_\psi \\ -A_{21,\psi}C_\psi & 1_\psi + A_{21,\psi}C_\psi \end{pmatrix} \begin{pmatrix} \tilde{Q}_\psi^{(+)} & 0 \\ 0 & \tilde{Q}_\psi^{(-)} \end{pmatrix}_\psi \quad \text{on } I'_{\theta_1}$$

where C_ψ is the generalized parametrix of $(A_{11} - A_{21})_\psi$ in the sense of Lemma 6.1. We have therefore found an appropriate operator-valued matrix P_ψ, Q_ψ, R_ψ and operators $B_{\pm,\psi}$ solving (6.7). We have therefore proved the following theorem.

Theorem 6.2. *Let L_ψ as in (4.1) and $U, F \in \mathcal{G}_{2,2}(\mathbb{R}^{n+1})$. Then there are operators Q_ψ as in (6.15) and $B_{\pm,\psi}$ as in (6.14) such that the equation*

$$L_\psi U = F \quad \text{microlocally at infinity on } I_{\theta_1} \tag{6.16}$$

holds if and only if

$$\begin{aligned} (\partial_z - iB_+(y, D_t, D_x))_\psi u_+ &= f_+ \quad \text{microlocally at infinity on } I_{\theta_1} \text{ and} \\ (\partial_z - iB_-(y, D_t, D_x))_\psi u_- &= f_- \quad \text{microlocally at infinity on } I_{\theta_1}. \end{aligned} \quad (6.17)$$

Furthermore, the coupling effect is computed as follows

$$\begin{pmatrix} u_+ \\ u_- \end{pmatrix} := Q_\psi \begin{pmatrix} U \\ (\partial_z)_\psi U \end{pmatrix}, \quad \begin{pmatrix} f_+ \\ f_- \end{pmatrix} := Q_\psi \begin{pmatrix} 0 \\ F \end{pmatrix}. \quad (6.18)$$

Closing remarks. Let us briefly summarize the above. First we explained a factorization procedure for the semiclassical operator L_ψ in the non-trivial case of generalized coefficients of log-type when acting on $\mathcal{G}_{2,2}$. To overcome the error made in this factorization we further introduced an adapted notion of microlocal regularity.

Concerning the motivating part of this paper we want to make the following remarks. Because of the lack of an adequate description of propagation of singularities in this setting it is not clear so far if and how one can derive approximated solutions to (6.16) from solutions of a perturbation of the problem (6.17) as we have seen in the smooth case in Subsection 1.1. Even if we allow the operator L_ε given in (1.2) to have logarithmic slow scale regular coefficients this problem remains unsolved. Once again, we want to mention that in the case of logarithmic slow scale case no semiclassical interpretation of the situation is necessary and microlocal regularity is based on local $\mathcal{G}_{2,2}^\infty$ -regularity.

Moreover we note that apart from the microlocal restrictions in the equations of (6.17) the operators $L_{j,\psi}$ meet the conditions of Theorem 3.1 of [26], $j = 1, 2$, if the coefficients in (1.2) from the beginning are of log-type with an appropriately chosen exponent $r \in \mathbb{N}$ which depends only on the dimension n . More precisely r plays the same role as k does in [26, Remark 3.2]. Therefore if it is possible to associate to (6.17) a global description of the same by only slight manipulations of the operators $L_{j,\psi} = (\partial_z - iB_\pm(y, D_t, D_x))_\psi$, $j = 1, 2$ in a microlocal sense (as in the smooth setting) one can derive well-posedness to the corresponding Cauchy problems which in turn approximate (6.16) on I_{θ_1} .

However, as already pointed out in Subsection 2.1 we so far only handled the case where the semiclassical asymptotic regime was restricted to $\hbar(\varepsilon) = \varepsilon$ as $\varepsilon \rightarrow 0$. This suggests itself to ask for general criteria of other possible choices for the semiclassical scale $\hbar(\varepsilon)$. Under the viewpoint of parameter-dependent representation theory for generalized pseudodifferential operators this will hopefully also give more insight in the notion of microlocal regularity at infinity. Here a future aim is to obtain a refined notion of microlocalization which is capable to give a global characterization and hence has to include regularity results for Colombeau generalized objects also for finite points.

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