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FORCED OSCILLATION FOR HIGHER ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. We establish some oscillation criteria for the solutions to forced higher-order differential equations. We do not assume that the forcing term is the n-th derivative of an oscillatory function, and do not assume that the coefficients are of a definite sign. Our results are illustrated with examples.

1. Introduction

In the previous 50 years, there has been increasing interest in obtaining sufficient conditions for the oscillation/nonoscillation of solutions of different classes of differential equations; see for example [2, 3, 4, 5, 6, 7, 8, 11, 12, 13] and the references cited therein. Regarding forced higher-order differential equations, one can use a technique introduced by Kartsatos [12, 13], which assumes that the forcing term $f(t)$ is the n-th derivative of an oscillatory function $h(t)$ satisfying $\lim_{t\to\infty} h(t) = 0$. Under certain conditions, he found that the forced equation would remain oscillatory if the unforced equation is oscillatory. Agarwal and Grace [1] studied the superlinear differential equation

$$x^{(n)}(t) + p(t)x(t)|^{\alpha-1}x(t) = f(t), \quad t \in [t_0, \infty), \tag{1.1}$$

where $p(t) < 0$ and $\alpha > 1$, using general means without imposing the Kartsatos condition. Ou and Wong [15] investigated the equation

$$x^{(n)}(t) + p(t)g(x(t)) = f(t), \quad t \in [t_0, \infty),$$

assuming that $p(t) \geq 0$ (< 0) on $[t_0, \infty)$, $xg(x) > 0$ for $x \neq 0$, and there exists a constant $c > 0$ such that $|g(x)| \geq c|x|^\alpha$, for $\alpha > 1$, or $|g(x)| \leq c|x|^\alpha$, for $0 < \alpha < 1$. Sun and Wong [18] studied (1.1) when $0 < \alpha < 1$, and they do not assume that $p(t)$ is of definite sign. Recently, Çakmak and Tiryaki [6] established some oscillation criteria for the forced higher-order differential equation

$$x^{(n)}(t) + \sum_{i=1}^{n-1} a_i x^{(i)}(t) + q(t)f(x(g(t))) = e(t), \tag{1.2}$$

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where a_i are real constants, $q(t)$, $f(t)$, $e(t)$ and $g(t)$ are real continuous functions, $xf(x) > 0$ whenever $x \neq 0$ and $\lim_{t\to\infty} g(t) = \infty$. We refer the reader for more oscillation results to [6, 17, 20], and to [9, 14, 19] for oscillatory potentials.

The purpose of this paper is to extend the oscillation criteria to higher order differential equations

$$\sum_{i=1}^{n} a_i x^{(i)}(t) + \Phi\Big(t, x(h(t)), x(g(t)), x(l(t))\Big) = f(t) \tag{1.3}$$

and

$$\sum_{i=1}^{n} a_i x^{(i)}(t) - \Phi(t, x(h(t)), x(g(t)), x(l(t))) = f(t), \tag{1.4}$$

where a_i are real numbers with $a_n \equiv 1$, and h, g, l and f are real continuous functions satisfying

$$\lim_{t\to\infty} h(t) = \lim_{t\to\infty} g(t) = \lim_{t\to\infty} l(t) = \infty\,.$$

Here $\Phi : [t_0,\infty)\times\mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a continuous function satisfying conditions (2.1) and (2.4) below. It is easy to see that when $\Phi = p(t)|x(t)|^{\alpha-1}x(t)$, $\Phi = p(t)g(x(t))$ and $\Phi = q(t)f(x(g(t)))$, Equation (1.3) reduces to (1.1) and (1.2), respectively.

A solution is said to be oscillatory if it has arbitrarily large zeros; i.e., for any $T > 0$ there exists a $t \geq T$ such that $x(t) = 0$.

To the best of our knowledge (1.3) and (1.4) have not been considered earlier. We hope to kindle the reader's interest in further research on the oscillation of these equations that arise, for example, in population growth with competitive species. Also, we want to present interesting examples that illustrate the importance of our results.

2. Main results

In the following, we consider a nonnegative kernel $H(t,s)$ defined on the set $\mathbb{D} := \{(t,s) : t \geq s \geq t_0\}$. We shall assume that $H(t,s)$ is sufficiently smooth in the variable s, so that the following conditions are satisfied:

(H1) $H(t,t) = 0$, $H(t,s) \geq 0$ for $t \geq s \geq t_0$,

(H2) The partial derivatives satisfy:

$$H_i(t,s) = (-1)^i \frac{\partial^i H}{\partial s^i}, \quad i = 0,1,\dots,n \text{ for} t > s \geq t_0,$$

H3) $H_i(t,t) = 0$, $i = 0,1,\dots,n-1$,

(H4) $H^{-1}(t,t_0)H_i(t,t_0) = O(1)$ as $t \to \infty$ for $i = 1,2,\dots,n-1$.

Throughout the paper, for $t \geq s \geq t_0$, we let

$$d_+ := \max\{0,d\}, \quad d_- := \max\{0,-d\},$$
$$h(t,s) := \sum_{i=1}^{n} a_i \frac{H_i(t,s)}{H^{1/\gamma}(t,s)}.$$

Theorem 2.1. *Assume that there exists a nonnegative function $G_1(t,s)$ defined on $[t_0,\infty)\times[t_0,\infty)$ such that*

$$\sum_{i=1}^{n} a_i H_i(t,s)x + H(t,s)\Phi(s,y,z,w) \begin{cases} \geq G_1(t,s), & \text{if } x,y,z,w > 0 \\ \leq -G_1(t,s), & \text{if } x,y,z,w < 0 \end{cases} \tag{2.1}$$

for $t > s \geq t_0$,

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} (H(t,s)f(s) + G_1(t,s))ds = \infty \tag{2.2}$$

and

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} (H(t,s)f(s) - G_1(t,s))ds = -\infty. \tag{2.3}$$

Then every solution of (1.3) *is oscillatory.*

Proof. Let x be a non-oscillatory solution of (1.3) on $[t_0,\infty)$. First assume that $x(t) > 0$ on some interval $[T,\infty)$, $t \geq t_0$. Multiplying both sides of (1.3) by $H(t,s)$, with t replaced by s, for $t \geq s \geq 0$ integrating with respect to s from T to t, we have

$$\begin{aligned}
&\int_T^t H(t,s)f(s)ds \\
&= \int_T^t H(t,s)\sum_{i=1}^{n} a_i x^{(i)}(s)ds + \int_T^t H(t,s)\Phi(s,x(h(s)),x(g(s)),x(l(s)))ds \\
&= \sum_{i=1}^{n} a_i \int_T^t H(t,s)x^{(i)}(s)ds + \int_T^t H(t,s)\Phi(s,x(h(s)),x(g(s)),x(l(s)))ds.
\end{aligned}$$

Integrating by parts and using (H1), (H2) and (H3). For $i = 1,2,3,\dots,n$, we obtain

$$\begin{aligned}
&\int_T^t H(t,s)x^{(i)}(s)ds \\
&= -H(t,T)x^{(i-1)}(T) - \sum_{j=1}^{i-1} H_j(t,T)x^{(i-j-1)}(T) + \int_T^t H_i(t,s)x(s)ds,
\end{aligned}$$

where $\sum_{j=1}^{0} = 0$. This implies

$$\begin{aligned}
&\int_T^t H(t,s)f(s)ds \\
&= -\sum_{i=1}^{n} a_i\Big[H(t,T)x^{(i-1)}(T) + \sum_{j=1}^{i-1} H_j(t,T)x^{(i-j-1)}(T)\Big] \\
&\quad + \int_T^t \Big[\sum_{i=1}^{n} a_i H_i(t,s)x(s) + H(t,s)\Phi(s,x(h(s)),x(g(s)),x(l(s)))\Big]ds.
\end{aligned}$$

In view of (H4), there exists a constant M such that

$$-\sum_{i=1}^{n} a_i\Big[H(t,T)x^{(i-1)}(T) + \sum_{j=1}^{i-1} H_j(t,T)x^{(i-j-1)}(T)\Big] \geq MH(t,T),$$

which implies

$$\begin{aligned}
&\int_T^t H(t,s)f(s)ds \\
&\geq MH(t,T) + \int_T^t \Big[\sum_{i=1}^{n} a_i H_i(t,s)x(s) + H(t,s)\Phi(s,x(h(s)),x(g(s)),x(l(s)))\Big]ds.
\end{aligned}$$

Then (2.1) implies

$$\frac{1}{H(t,T)}\int_T^t (H(t,s)f(s)-G_1(t,s))ds \geq M,$$

which leads to a contradiction to (2.3).

Next, we assume that x satisfies (1.3) and is eventually negative. The same argument as above leads to a contradiction, provided

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t (H(t,s)f(s)+G_1(t,s))ds = \infty,$$

which indeed holds due to condition (2.2). $\square$

Theorem 2.2. *Assume there exists a nonnegative function $G_2(t,s)$ defined on $[t_0,\infty)\times[t_0,\infty)$ such that*

$$\sum_{i=1}^n a_i H_i(t,s)x - H(t,s)\Phi(s,y,z,w)\begin{cases}\leq G_2(t,s), & \text{if } x,y,z,w>0\\ \geq -G_2(t,s), & \text{if } x,y,z,w<0\end{cases} \tag{2.4}$$

for $t>s\geq t_0$,

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^t (H(t,s)f(s)-G_2(t,s))ds = \infty \tag{2.5}$$

and

$$\liminf_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^t (H(t,s)f(s)+G_2(t,s))ds = -\infty. \tag{2.6}$$

Then every solution of (1.4) is oscillatory.

Proof. Let x be a non-oscillatory solution of (1.4) on $[t_0,\infty)$. First assume that $x(t)>0$ on some interval $[T,\infty)$, $T\geq t_0$. As in the proof of Theorem 2.1, we obtain

$$\begin{aligned}&\int_T^t H(t,s)f(s)ds\\ &= -\sum_{i=1}^n a_i\Big[H(t,T)x^{(i-1)}(T)+\sum_{j=1}^{i-1}H_j(t,T)x^{(i-j-1)}(T)\Big]\\ &\quad+\int_T^t\Big[\sum_{i=1}^n a_iH_i(t,s)x(s)-H(t,s)\Phi(s,x(h(s)),x(g(s)),x(l(s)))\Big]ds.\end{aligned}$$

Again, by (H4), there exists a constant C such that

$$-\sum_{i=1}^n a_i\Big[H(t,T)x^{(i-1)}(T)+\sum_{j=1}^{i-1}H_j(t,T)x^{(i-j-1)}(T)\Big]\leq CH(t,T),$$

which implies

$$\begin{aligned}&\int_T^t H(t,s)f(s)ds\\ &\leq CH(t,T)+\int_T^t\Big[\sum_{i=1}^n a_iH_i(t,s)x(s)-H(t,s)\Phi(s,x(h(s)),x(g(s)),x(l(s)))\Big]ds.\end{aligned}$$

From this inequality and (2.4) we obtain
$$\frac{1}{H(t,T)}\int_T^t [H(t,s)f(s) - G_2(t,s)]ds \le C,$$
which leads to a contradiction to (2.5).

The proof for $x(t) < 0$ is similar to the first part. $\square$

As particular cases, we present oscillation criteria for the differential equations
$$\sum_{i=1}^{n} a_i x^{(i)}(t) + r(t)\psi_\gamma(x(h(t))) + p(t)\psi_\alpha(x(g(t))) + q(t)\psi_\beta(x(l(t))) = f(t) \quad (2.7)$$
and
$$\sum_{i=1}^{n} a_i x^{(i)}(t) - r(t)\psi_\gamma(x(h(t))) + p(t)\psi_\alpha(x(g(t))) + q(t)\psi_\beta(x(l(t))) = f(t) \quad (2.8)$$
which satisfy conditions (2.1) and (2.4) respectively, where $\psi_\gamma(u) := |u|^{\gamma-1}u$, $\gamma > 0$, and where r and q are real continuous functions and p is a positive function. Our interest is to establish oscillation criteria for (2.7) and (2.8) without assuming that r, q, f are of definite sign and without assuming that $f(t)$ is the n-th derivative of an oscillatory function.

Corollary 2.3. *Let $h(t) \equiv g(t) \equiv l(t) \equiv t$ on $[t_0, \infty)$. Assume that $0 < \gamma < 1$ and $\alpha > \beta > \gamma$ hold and $\sum_{i=1}^{n} a_i H_i(t,s) > 0$, for $t \ge s \ge t_0$. If (2.2) and (2.3) are satisfied, where*
$$\begin{gathered}
G_1(t,s) := (\gamma - 1)\gamma^{\gamma/(1-\gamma)}|Q_1(s)|^{1/(1-\gamma)}|h(t,s)|^{\gamma/(\gamma-1)},\\
Q_1(s) := r_-(s) + \sigma_1 p^{(\gamma-\beta)/(\alpha-\beta)}(s) q_-^{(\alpha-\gamma)/(\alpha-\beta)}(s),\\
\sigma_1 := (\alpha-\beta)(\alpha-\gamma)^{(\gamma-\alpha)/(\alpha-\beta)}(\beta-\gamma)^{(\beta-\gamma)/(\alpha-\beta)},
\end{gathered}$$
then every solution of (2.7) is oscillatory.

Proof. Let x be non-oscillatory solution of (2.7) on $[t_0, \infty)$. First assume that $x(t) > 0$ on some interval $[T, \infty)$. We claim that (2.1) is satisfied with $x = x(s)$, $y = x(h(s))$, $z = x(g(s))$ and $w = x(l(s))$. As in the proof of Theorem 2.1, we obtain
$$\begin{aligned}
&\sum_{i=1}^{n} a_i H_i(t,s)x(s) + H(t,s)\Phi(s, x(h(s)), x(g(s)), x(l(s)))\\
&= \sum_{i=1}^{n} a_i H_i(t,s)x(s) + H(t,s)\Big[r(s)x^\gamma(s) + (p(s)x^\alpha(s) + q(s)x^\beta(s))\Big]\\
&\ge \sum_{i=1}^{n} a_i H_i(t,s)x(s) - H(t,s)r_-(s)x^\gamma(s) + H(t,s)x^\gamma(s)(p(s)x^{\alpha-\gamma}(s)\\
&\quad - q_-(s)x^{\beta-\gamma}(s))
\end{aligned}$$
For a given s, set $F(x) := px^{\alpha-\gamma} - q_- x^{\beta-\gamma}$, for $x > 0$ and $\alpha > \beta > \gamma > 0$. Thus F obtains its minimum at
$$x = (\alpha-\gamma)^{1/(\beta-\alpha)}(\beta-\gamma)^{1/(\alpha-\beta)}p^{(\gamma-\beta)/((\alpha-\beta)(\alpha-\gamma))}(q_-)^{1/(\alpha-\beta)},$$
and
$$F_{\min} = -\sigma_1 p^{(\gamma-\beta)/(\alpha-\beta)}(q_-)^{(\alpha-\gamma)/(\alpha-\beta)}.$$

Then

$$
\begin{aligned}
&\sum_{i=1}^{n} a_i H_i(t,s)x(s) + H(t,s)\Phi(s, x(h(s)), x(g(s)), x(l(s))) \\
&\ge \sum_{i=1}^{n} a_i H_i(t,s)x(s) - H(t,s)Q_1(s)x^{\gamma}(s).
\end{aligned}
$$

Define $X \ge 0$ and $Y > 0$ by

$$
X^{\gamma} := HQ_1 x^{\gamma}, \quad Y^{\gamma-1} := \gamma^{-1} Q_1^{-1/\gamma} h.
$$

Then, using the inequality (see [10])

$$
\gamma X Y^{\gamma-1} - X^{\gamma} \ge (\gamma-1))Y^{\gamma}, \qquad 0 < \gamma < 1, \tag{2.9}
$$

we obtain

$$
\begin{aligned}
&\sum_{i=1}^{n} a_i H_i(t,s)x(s) + H(t,s)\Phi(s, x(h(s)), x(g(s)), x(l(s))) \\
&\ge \sum_{i=1}^{n} a_i H_i(t,s)x(s) - H(t,s)Q_1(s)x^{\gamma}(s) \\
&\ge (\gamma-1)\gamma^{\gamma/(1-\gamma)} Q_1^{1/(1-\gamma)}(s) h^{\gamma/(\gamma-1)}(t,s) = G_1(t,s).
\end{aligned}
$$

Then from Theorem 2.1, we obtain the desired result. The proof for $x(t) < 0$ similar to the first part of this proof. □

Corollary 2.4. *Let $h(t) \equiv t$ and $l(t) \equiv g(t)$ on $[t_0, \infty)$. Assume that $0 < \gamma < 1$ and $\alpha > \beta > 0$ hold and $\sum_{i=1}^{n} a_i H_i(t,s) > 0$, for $t \ge s \ge t_0$. If* (2.2) *and* (2.3) *are satisfied, where*

$$
\begin{aligned}
G_1(t,s) &:= (\gamma-1)\gamma^{\gamma/(1-\gamma)} r_-^{1/(1-\gamma)}(s) h^{\gamma/(\gamma-1)}(t,s) \\
&\quad + \delta H(t,s) p^{\beta/(\beta-\alpha)}(s) q_-^{\alpha/(\alpha-\beta)}(s), \\
\delta &:= (\beta-\alpha)\alpha^{\alpha/(\beta-\alpha)}\beta^{\beta/(\alpha-\beta)},
\end{aligned}
$$

then every solution of (2.7) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (2.7) on $[t_0, \infty)$. First assume that $x(t) > 0$ on some interval $[T, \infty)$. We claim that (2.1) holds for $x = x(s)$, $y = x(h(s))$, $z = x(g(s))$ and $w = x(l(s))$. As in the proof of Theorem 2.1, we obtain

$$
\begin{aligned}
&\sum_{i=1}^{n} a_i H_i(t,s)x(s) + H(t,s)\Phi(s, x(h(s)), x(g(s)), x(l(s))) \\
&= \sum_{i=1}^{n} a_i H_i(t,s)x(s) + H(t,s)\Big[r(s)x^{\gamma}(s) + (p(s)x^{\alpha}(s) + q(s)x^{\beta}(s))\Big] \\
&\ge \sum_{i=1}^{n} a_i H_i(t,s)x(s) - H(t,s)r_-(s)x^{\gamma}(s) + H(t,s)(p(s)x^{\alpha}(g(s)) \\
&\quad - q_-(s)x^{\beta}(g(s)))
\end{aligned} \tag{2.10}
$$

As in the proof of Corollary 2.3, we obtain

$$
px^{\alpha} - q_- x^{\beta} \ge \delta p^{\beta/(\beta-\alpha)} q_-^{\alpha/(\alpha-\beta)}, \tag{2.11}
$$

where $\delta=(\beta-\alpha)\alpha^{\alpha/(\beta-\alpha)}\beta^{\beta/(\alpha-\beta)}$ and

$$\sum_{i=1}^{n}a_iH_i(t,s)x(s)-H(t,s)r_-(s)x^{\gamma}(s)\geq(\gamma-1)\gamma^{\gamma/(1-\gamma)}r_-^{1/(1-\gamma)}(s)h^{\gamma/(\gamma-1)}(t,s). \tag{2.12}$$

Using (2.11) and (2.12) in (2.10), we have

$$\begin{aligned}&\sum_{i=1}^{n}a_iH_i(t,s)x(s)+H(t,s)\Phi(s,x(h(s)),x(g(s)),x(l(s)))\\&\geq(\gamma-1)\gamma^{\gamma/(1-\gamma)}r_-^{1/(1-\gamma)}(s)h^{\gamma/(\gamma-1)}(t,s)+\delta H(t,s)p^{\beta/(\beta-\alpha)}q_-^{\alpha/(\alpha-\beta)}\\&=G_1(t,s).\end{aligned}$$

Then from Theorem 2.1, we obtain that every solution of equation (2.7) is oscillatory. □

Example 2.5. Consider the equation

$$\begin{aligned}&\sum_{i=1}^{n}a_ix^{(i)}(t)+\phi_1(t)\psi_\gamma(x(t))+t^{\frac{\mu}{3}}\psi_3(x(g(t)))+t^{\frac{8\mu}{27}}\cos^{1/9}(t)\psi_{\frac{8}{3}}(x(g(t)))\\&=t^{\mu}\cos(s),\end{aligned} \tag{2.13}$$

where $0<\gamma<1$, $\phi_1(t)\geq0$, for $t\geq t_0$, $a_i\geq0$, $i=1,2,\dots,n-1$ and $a_n\equiv1$. By taking $H(t,s)=(t-s)^n$. It is easy to see that (H1)-(H4) are satisfied and $\sum_{i=1}^{n}a_iH_i(t,s)>0$. Applying Corollary 2.4, every solution of (2.13) is oscillatory if $\mu>n$.

Corollary 2.6. *Let $q(t)$ is a nonnegative function and $h(t)\equiv t$ on $[t_0,\infty)$. Assume that $0<\gamma<1$ and $\sum_{i=1}^{n}a_iH_i(t,s)>0$, for $t\geq s\geq t_0$ hold. If (2.2) and (2.3) are satisfied, where*

$$G_1(t,s):=(\gamma-1)\gamma^{\gamma/(1-\gamma)}r_-^{1/(1-\gamma)}(s)h^{\gamma/(\gamma-1)}(t,s)$$

then every solution of (2.7) is oscillatory.

Corollary 2.7. *Let $r(t)$ and $q(t)$ be nonnegative functions on $[t_0,\infty)$. Assume that $\sum_{i=1}^{n}a_iH_i(t,s)\geq0$, for $t\geq s\geq t_0$ holds. If (2.2) and (2.3) are satisfied, then every solution of (2.7) is oscillatory.*

Example 2.8. Consider the equation (2.7) with $f(t)=t^{\mu}\sin(t)$, where $n\geq2$, μ, $a_i\geq0$, $i=1,2,\dots,n-1$ and $a_n\equiv1$. Also assume $r(t)$ and $q(t)$ be nonnegative functions on $[t_0,\infty)$. By taking $H(t,s)=(t-s)^{\lambda}$, $\lambda>n-1$. It is easy to see that $H(t,s)$ satisfies all of (H1)–(H4) and $\sum_{i=1}^{n}a_iH_i(t,s)\geq0$. Then, by Corollary 2.7, every solution of equation (2.7) is oscillatory if $\mu>\lambda$.

Corollary 2.9. *Let $q(t)$ be a nonnegative function and $h(t)\equiv g(t)\equiv l(t)\equiv t$ on $[t_0,\infty)$. Assume that $\gamma>1$, $\alpha>\gamma>\beta>0$ and $Q_2(t)<0$, for $t\geq t_0$. If (2.5) and (2.6) are satisfied, where*

$$\begin{aligned}G_2(t,s)&:=(\gamma-1)\gamma^{\gamma/(1-\gamma)}|Q_2(s)|^{1/(1-\gamma)}|h(t,s)|^{\gamma/(\gamma-1)},\\Q_2(s)&:=-r(s)-\sigma_2p^{(\gamma-\beta)/(\alpha-\beta)}(s)q^{(\alpha-\gamma)/(\alpha-\beta)}(s),\\\sigma_2&:=(\alpha-\beta)(\alpha-\gamma)^{(\gamma-\alpha)/(\alpha-\beta)}(\gamma-\beta)^{(\beta-\gamma)/(\alpha-\beta)},\end{aligned}$$

then every solution of (2.8) is oscillatory.

Proof. Let x a non-oscillatory solution of (2.8) on $[t_0,\infty)$. First assume that $x(t) > 0$ on some interval $[T,\infty)$. We claim that (2.4) is satisfied with $x = x(s)$, $y = x(h(s))$, $z = x(g(s))$ and $w = x(l(s))$. As in the proof of Theorem 2.2, we obtain

$$\begin{aligned}
&\sum_{i=1}^{n} a_i H_i(t,s)x(s) - H(t,s)\Phi(s,x(h(s)),x(g(s)),x(l(s)))\\
&= \sum_{i=1}^{n} a_i H_i(t,s)x(s) - H(t,s)\big[r(s)x^{\gamma}(s) + (p(s)x^{\alpha}(s) + q(s)x^{\beta}(s))\big]\\
&\le \big|\sum_{i=1}^{n} a_i H_i(t,s)\big|x(s) - H(t,s)r(s)x^{\gamma}(s)\\
&\quad - H(t,s)x^{\gamma}(s)(p(s)x^{\alpha-\gamma}(s) + q(s)x^{\beta-\gamma}(s)).
\end{aligned} \tag{2.14}$$

For a given s, set $K(x) := px^{\alpha-\gamma} + qx^{\beta-\gamma}$, for $x > 0$ and $\alpha > \gamma > \beta > 0$. Thus K obtains its minimum at

$$x = (\alpha-\gamma)^{1/(\beta-\alpha)}(\gamma-\beta)^{1/(\alpha-\beta)}p^{(\gamma-\beta)/((\alpha-\beta)(\alpha-\gamma))}q^{1/(\alpha-\beta)}$$

and

$$K_{\min} = \sigma_2 p^{(\gamma-\beta)/(\alpha-\beta)}q^{(\alpha-\gamma)/(\alpha-\beta)},$$

where $\sigma_2 = (\alpha-\beta)(\alpha-\gamma)^{(\gamma-\alpha)/(\alpha-\beta)}(\gamma-\beta)^{(\beta-\gamma)/(\alpha-\beta)}$. Then, from this and (2.14), we obtain

$$\begin{aligned}
&\sum_{i=1}^{n} a_i H_i(t,s)x(s) - H(t,s)\Phi(s,x(h(s)),x(g(s)),x(l(s)))\\
&\le |\sum_{i=1}^{n} a_i H_i(t,s)|x(s) - H(t,s)|Q_2(s)|x^{\gamma}(s).
\end{aligned} \tag{2.15}$$

Define $X \ge 0$ and $Y \ge 0$ by

$$X^{\gamma} := H|Q_2|x^{\gamma}, \quad Y^{\gamma-1} := \gamma^{-1}|Q_2|^{-1/\gamma}|h|.$$

Then, using the inequality (see [10])

$$\gamma XY^{\gamma-1} - X^{\gamma} \le (\gamma-1)Y^{\gamma}, \qquad \gamma > 1, \tag{2.16}$$

we obtain

$$\begin{aligned}
&|\sum_{i=1}^{n} a_i H_i(t,s)|x(s) - H(t,s)|Q_2(s)|x^{\gamma}(s)\\
&\le (\gamma-1)\gamma^{\gamma/(1-\gamma)}|Q_2(s)|^{1/(1-\gamma)}|h(t,s)|^{\gamma/(\gamma-1)} = G_2(t,s).
\end{aligned}$$

Then from Theorem 2.2, we obtain the desired result. The proof for $x < 0$ is similar to the case above. □

Corollary 2.10. *Let $h(t) \equiv g(t) \equiv l(t) \equiv t$ on $[t_0,\infty)$. Assume that $\gamma > 1$, $\alpha > \beta > \gamma > 0$ and $Q_3(t) < 0$, for $t \ge t_0$. If* (2.5) *and* (2.6) *are satisfied, where*

$$\begin{aligned}
G_2(t,s) &:= (\gamma-1)\gamma^{\gamma/(1-\gamma)}|Q_3(s)|^{1/(1-\gamma)}|h(t,s)|^{\gamma/(\gamma-1)},\\
Q_3(s) &:= -r(s) + \sigma_1 p^{(\gamma-\beta)/(\alpha-\beta)}(s)q_-^{(\alpha-\gamma)/(\alpha-\beta)}(s),\\
\sigma_1 &:= (\alpha-\beta)(\alpha-\gamma)^{(\gamma-\alpha)/(\alpha-\beta)}(\beta-\gamma)^{(\beta-\gamma)/(\alpha-\beta)},
\end{aligned}$$

then every solution of (2.8) *is oscillatory.*

Proof. Let x be a non-oscillatory solution of (2.8) on some interval $[t_0, \infty)$. We claim that (2.4) holds for $x = x(s)$, $y = x(h(s))$, $z = x(g(s))$ and $w = x(l(s))$. As in the proof of Theorem 2.2, we obtain

$$\begin{aligned}
&\sum_{i=1}^{n} a_i H_i(t,s)x(s) - H(t,s)\Phi(s, x(h(s)), x(g(s)), x(l(s))) \\
&= \sum_{i=1}^{n} a_i H_i(t,s)x(s) - H(t,s)\big[r(s)x^{\gamma}(s) + (p(s)x^{\alpha}(s) + q(s)x^{\beta}(s))\big] \\
&\le |\sum_{i=1}^{n} a_i H_i(t,s)|x(s) - H(t,s)r(s)x^{\gamma}(s) \\
&\quad - H(t,s)x^{\gamma}(s)(p(s)x^{\alpha-\gamma}(s) - q_-(s)x^{\beta-\gamma}(s)).
\end{aligned}$$

As in the proof of Corollary 2.9, we have

$$px^{\alpha-\gamma} - p_- x^{\beta-\gamma} \ge -\sigma_1 p^{(\gamma-\beta)/(\alpha-\beta)} q_-^{(\alpha-\gamma)/(\alpha-\beta)},$$

which implies

$$\begin{aligned}
&\sum_{i=1}^{n} a_i H_i(t,s)x(s) - H(t,s)\Phi(s, x(h(s)), x(g(s)), x(l(s)) \\
&\le |\sum_{i=1}^{n} a_i H_i(t,s)|x(s)ds - H(t,s)|Q_3(s)|x^{\gamma}(s).
\end{aligned}$$

The rest of the proof is the same as the proof of Corollary 2.9 with Q_2 replaced by Q_3. ◻

Corollary 2.11. *Let $r(t)$ be a positive function, and $h(t) \equiv t$ and $l(t) \equiv g(t)$ on $[t_0, \infty)$. Assume that $\gamma > 1$ and $\alpha > \beta > 0$. If* (2.5) *and* (2.6) *are satisfied, where*

$$\begin{aligned}
G_2(t,s) &:= (\gamma-1)\gamma^{\gamma/(1-\gamma)}|r(s)|^{1/(1-\gamma)}|h(t,s)|^{\gamma/(\gamma-1)} \\
&\quad - \delta H(t,s)p^{\beta/(\beta-\alpha)}(s)q_-^{\alpha/(\alpha-\beta)}(s), \\
\delta &:= (\beta-\alpha)\alpha^{\alpha/(\beta-\alpha)}\beta^{\beta/(\alpha-\beta)},
\end{aligned}$$

then every solution of (2.8) *is oscillatory.*

Proof. Let x a nonoscillatory solution of (2.8) on some interval $[t_0, \infty)$. As in the proof of Theorem 2.2, we obtain

$$\begin{aligned}
&\sum_{i=1}^{n} a_i H_i(t,s)x(s) - H(t,s)\Phi(s, x(h(s)), x(g(s)), x(l(s))) \\
&= \sum_{i=1}^{n} a_i H_i(t,s)x(s) - H(t,s)\big[r(s)x^{\gamma}(s) + (p(s)x^{\alpha}(s) + q(s)x^{\beta}(s))\big] \\
&\le |\sum_{i=1}^{n} a_i H_i(t,s)|x(s) - H(t,s)|r(s)|x^{\gamma}(s) \\
&\quad - H(t,s)(p(s)x^{\alpha}(g(s)) - q_-(s)x^{\beta}(g(s))).
\end{aligned}$$

As in the proof of Corollary 2.4, we have

$$px^{\alpha} - q_- x^{\beta} \ge \delta p^{\beta/(\beta-\alpha)} q_-^{\alpha/(\alpha-\beta)},$$

and as in the proof of Corollary 2.9, we obtain

$$|\sum_{i=1}^{n} a_i H_i(t,s)|x(s) - H(t,s)|r(s)|x^{\gamma}(s) \le (\gamma-1)\gamma^{\gamma/(1-\gamma)}|r(s)|^{1/(1-\gamma)}|h(t,s)|^{\gamma/\gamma-1}.$$

Then

$$\begin{aligned}&\sum_{i=1}^{n} a_i H_i(t,s)x(s) - H(t,s)\Phi(s, x(h(s)), x(g(s)), x(l(s)))\\ &\le (\gamma-1)\gamma^{\gamma/(1-\gamma)}|r(s)|^{1/(1-\gamma)}|h(t,s)|^{\gamma/\gamma-1} - H(t,s)\delta p^{\beta/(\beta-\alpha)} q_-^{\alpha/(\alpha-\beta)} = G_2(t,s),\end{aligned}$$

which implies that every solution of (2.8) is oscillatory. □

Example 2.12. Consider the equation

$$\begin{aligned}&x'(t) + \phi_2(t)\psi_{\gamma}(x(t))\\ &= e^{\frac{5t}{3}}\psi_{\frac{7}{3}}(x(g(t))) + e^{\frac{10}{21}t}\sin^{\frac{5}{7}}(t)\psi_{\frac{2}{3}}(x(g(t))) + e^t\sin(t),\end{aligned} \tag{2.17}$$

where $\gamma > 1$ and $\phi_2(t) < 0$, for $t \ge t_0$. By taking $H(t,s) = 1$ and $t > s \ge t_0$, it is easy to see that $H(t,s)$ satisfies all of (H1)–(H4). Applying Corollary 2.11, every solution of (2.17) is oscillatory.

Corollary 2.13. *Let $r(t)$ be a nonnegative function and $h(t) \equiv t$ on $[t_0,\infty)$. Assume that $\gamma > 1$ holds. If (2.5) and (2.6) are satisfied, where*

$$G_2(t,s) := (\gamma-1)\gamma^{\gamma/(1-\gamma)} r^{1/(1-\gamma)}(s)|h(t,s)|^{\gamma/\gamma-1},$$

then every solution of (2.8) is oscillatory.

Corollary 2.14. *Let $r(t)$ and $q(t)$ be nonnegative functions , and assume that $\sum_{i=1}^{n} a_i H_i(t,s) \le 0$, for $t \ge s \ge t_0$ holds. If (2.5) and (2.6) are satisfied, then every solution of (2.8) is oscillatory.*

Example 2.15. Consider equation (2.8) with $f(t) = e^t\sin(t)$, where $n \ge 2$, $a_i \le 0$, $i = 1, 2, \dots, n-1$ and $a_n \equiv 1$. Also, we assume that $r(t)$ and $q(t)$ be nonnegative functions. By taking $H(t,s) = (t-s)^{\lambda}$ and $\lambda > n-1$, it is easy to see that $H(t,s)$ satisfies (H1)–(H4) and $\sum_{i=1}^{n} a_i H_i(t,s) \le 0$. Then, by Corollary 2.14, every solution of (2.8) is oscillatory.

The above results are extendable to neutral equation in this from

$$\begin{aligned}&\sum_{i=1}^{n} a_i y^{(i)}(t) + r(t)\Phi_{\gamma}(x(h(t))) + \sum_{i=1}^{n}\Big[p_i(t)\Phi_{\alpha_i}(x(g_i(t))) + q_i(t)\Phi_{\beta_i}(x(l_i(t)))\Big]\\ &\quad + \overline{r}(t)\Phi_{\overline{\gamma}}(x(\overline{h}(t))) + \sum_{j=1}^{m}\Big[\overline{p}_j(t)\Phi_{\overline{\alpha}_j}(x(\overline{g}_j(t))) + \overline{q}_j(t)\Phi_{\overline{\beta}_j}(x(\overline{l}_j(t)))\Big]\\ &= a(t)x(t) + b(t)x(t-\tau) + f(t)\end{aligned}$$

where $y(t) := x(t) + \delta(t)x(\tau(t))$ with $\Phi_{\eta}(u) := |u|^{\eta-1}u$, $\eta > 0$, and where a_i are real numbers with $a_n \equiv 1$ and r, $\overline{r}$ p_i, $\overline{p}_j$, q_j, $\overline{q}_j$, h, $\overline{h}$, g_i, $\overline{g}_j$, l_i, $\overline{l}_j$, a, b and f are real continuous functions such that $a, b, p_i, \overline{p}_j$ are positive, and and $\lim_{t\to\infty} h(t) = \lim_{t\to\infty} \overline{h}(t) = \lim_{t\to\infty} g_i(t) = \lim_{t\to\infty} \overline{g}_j(t) = \lim_{t\to\infty} l_i(t) = \lim_{t\to\infty} \overline{l}_j(t) = \infty$. The details are left to the reader to check them.

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