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SING-CHANGING SOLUTIONS FOR NONLINEAR PROBLEMS WITH STRONG RESONANCE

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Abstract. Using critical point theory and index theory, we prove the existence and multiplicity of sign-changing solutions for some elliptic problems with strong resonance at infinity, under weaker conditions than in the references.

1. Introduction

In this article, we consider the equation

$$\begin{gathered}-\Delta u = f(u),\\ u \in H_0^1(\Omega).\end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. We assume that f is asymptotically linear; i.e., $\lim_{|u|\to\infty} \frac{f(u)}{u}$ exists. By setting

$$\alpha := \lim_{|u|\to\infty} \frac{f(u)}{u} \tag{1.2}$$

we can write $f(u) = \alpha u - g(u)$, where

$$\frac{g(u)}{u} \to 0 \quad \text{as } |u| \to \infty.$$

We Denote by $\lambda_1 < \lambda_2 < \dots < \lambda_j < \dots$ the distinct eigenvalues sequence of $-\Delta$ with the Dirichlet boundary conditions. We say that problem (1.1) is resonant at infinity if α in (1.2) is an eigenvalue λ_k. The situation when

$$\lim_{|u|\to\infty} g(u) = 0 \quad \text{and} \quad \lim_{|u|\to\infty} \int_0^u g(t)dt = \beta \in \mathbb{R}$$

is what we call strong resonance.

Now we present some results of this paper. We write (1.1) in the form

$$\begin{gathered}-\Delta u - \lambda_k u + g(u) = 0,\\ u \in H_0^1(\Omega).\end{gathered} \tag{1.3}$$

We assume that g is a smooth function satisfying the following conditions.

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(G1) $g(t)t \to 0$ as $|t| \to \infty$;
(G2) the real function $G(t) = \int_0^t g(s)ds$ is well defined and $G(t) \to 0$ as $t \to +\infty$.
(G3) $G(t) \geq 0$ for all $t \in \mathbb{R}$.

Theorem 1.1. *If* (G1)–(G3) *hold, then* (1.1) *has at least one solution.*

Since 0 is a particular point, we cannot ensure those solutions are nontrivial without additional conditions.

Theorem 1.2. *Let* $g(0) = 0$*, and suppose that* (G1)–(G3) *hold, and*
$$g'(0) = \sup\{g'(t) : t \in \mathbb{R}\}, \tag{1.4}$$
then (1.3) *has at least one sign-changing solution.*

Theorem 1.3. *Assume (G1)–(G3) hold and g is odd and $G(0) \geq 0$. Moreover suppose that there exists an eigenvalue $\lambda_h < \lambda_k$ such that*
$$g'(0) + \lambda_h - \lambda_k > 0.$$
Then (1.3) *possess at least $m = \dim(M_h \oplus \cdots \oplus M_k) - 1$ distinct pairs of sign-changing solutions (M_j denotes the eigenspace corresponding to λ_j).*

Remark 1.4. The references show only the existence of solutions to (1.3), while we obtain its sign-changing solutions under weaker conditions.

The resonance problem has been widely studied by many authors using various methods; see [1, 2, 4, 5, 8, 10] and the references therein. We will use critical point theory and pseudo-index theory to obtain sign-changing solutions for the strong resonant problem (1.3). We also allow the case in which resonance also occurs at zero.

In section 2, we give some preliminaries, which are fundamental in our paper. In section 3, we give some abstract critical point theorems, which are used to prove above theorems in this paper. In section 3, by using the above theorems, we prove the existence and multiplicity of sign-changing solutions.

2. Preliminaries

We denote by X a real Banach space. B_R denotes the closed ball in X centered at the origin and with radius $R > 0$. J is a continuously Frèchet differentiable map from X to $\mathbb{R}$; i.e., $J \in C^1(X, \mathbb{R})$.

In the literature, deformation theorems have been proved under the assumption that $J \in C^1(X, \mathbb{R})$ satisfies the well known Palais-Smale condition. In problems which do not have resonance at infinity, the (PS) condition is easy to verify. On the other hand, a weaker condition than the (PS) condition is needed to study problems with strong resonance at infinity.

Definition 2.1. We say that $J \in C^1(X, \mathbb{R})$ satisfies the condition (C) in $]c_1, c_2[$ ($-\infty \leq c_1 < c_2 \leq +\infty$) if any sequence $\{u_k\} \subset J^{-1}(]c_1, c_2[)$, such that $\{J(u_k)\}$ is bounded and $J'(u_k) \to 0$, we have either

(i) $\{u_k\}$ is bounded and possesses a convergent subsequence, or
(ii) for all $c \in]c_1, c_2[$, there exists $\sigma, R, \alpha > 0$ such that $[c - \sigma, c + \sigma] \subset]c_1, c_2[$ and for all $u \in J^{-1}([c - \sigma, c + \sigma])$, $\|u\| \geq R : \|J'(u)\|\|u\| \geq \alpha$.

In [1, 3, 4], deformation theorems are obtained under the condition (C). For $c \in \mathbb{R}$, denote

$$A_c = \{u \in X : J(u) \le c\}, \quad K_c = \{u \in X : J'(u) = 0, J(u) = c\}.$$

Proposition 2.2. *Let X be a real Banach space, and let $J \in C^1(X, \mathbb{R})$ satisfy the condition (C) in $]c_1, c_2[$. If $c \in]c_1, c_2[$ and N is any neighborhood of K_c, there exists a bounded homeomorphism η of X onto X and constants $\bar{\varepsilon} > \varepsilon > 0$, such that $[c - \bar{\varepsilon}, c + \bar{\varepsilon}] \subset]c_1, c_2[$ satisfying the following properties:*

- (i) $\eta(A_{c+\varepsilon} \backslash N) \subset A_{c-\varepsilon}$.
- (ii) $\eta(A_{c+\varepsilon}) \subset A_{c-\varepsilon}$, *if* $K_c = \emptyset$.
- (iii) $\eta(x) = x$, *if* $x \notin J^{-1}([c - \bar{\varepsilon}, c + \bar{\varepsilon}])$.

Moreover, if G is a compact group of (linear) unitary transformation on a real Hilbert space H, then

- (vi) *η can be chosen to be G-equivariant, if the functional J is G-invariant. Particularly, η is odd if the functional J is even.*

3. Abstract critical point theorems

In this article, we shall obtain solutions to (1.3) by using the linking type theorem. Its different definitions can be seen in [9, 11] and references therein.

Definition 3.1. Let H be a real Hilbert space and A a closed set in H. Let B be an Hilbert manifold with boundary ∂B, we say A and ∂B link if

- (i) $A \cap \partial B = \emptyset$;
- (ii) If ϕ is a continuous map of H into itself such that $\phi(u) = u$ for all $u \in \partial B$, then $\phi(B) \cap A \neq \emptyset$.

Typical examples can be found in [1, 6, 7, 11].

Example 3.2. Let H_1, H_2 be two closed subspaces of H such that

$$H = H_1 \oplus H_2, \quad \dim H_2 < \infty.$$

Then if $A = H_1$, $B = B_R \cap H_2$, then A and ∂B link.

Example 3.3. Let H_1, H_2 be two closed subspaces of H such that $H = H_1 \oplus H_2$, $\dim H_2 < \infty$, and consider $e \in H_1$, $\|e\| = 1$, $0 < \rho < R_1, R_2$, set

$$A = H_1 \cap S_\rho, \quad B = \{u = v + te : v \in H_2 \cap B_{R_2}, 0 \le t \le R_1\}.$$

Then A and ∂B link.

Let $X \subset H$ be a Banach space densely embedded in H. Assume that H has a closed convex cone P_H and that $P := P_H \cap X$ has interior points in X. Let $J \in C^1(H, \mathbb{R})$. In [6], they construct the pseudo-gradient flow σ for J, and have the following definition.

Definition 3.4. *Let $W \subset X$ be an invariant set under σ. W is said to be an admissible invariant set for J if*

- (a) *W is the closure of an open set in X;*
- (b) *if $u_n = \sigma(t_n, v) \to u$ in H as $t_n \to \infty$ for some $v \notin W$ and $u \in K$, then $u_n \to u$ in X;*
- (c) *If $u_n \in K \cap W$ is such that $u_n \to u$ in H, then $u_n \to u$ in X; (d) For any $u \in \partial W \backslash K$, we have $\sigma(t, u) \in \mathring{W}$ for $t > 0$.*

Now let $S = X\backslash W$, $W = P \cup (-P)$. As the similar proof to that in [6], the W is an admissible invariant set for J in the following section 4. We define

$$\phi^* = \{\Gamma(t,x) : [0,1] \times X \to X \text{ continuous in the } X\text{-topology and } \Gamma(t,W) \subset W\}.$$

In [11], a new linking theorem is given under the condition (PS). Since the deformation is still hold under the condition (C), the following theorem holds.

Theorem 3.5. *Suppose that W is an admissible invariant set of J and that J is in $C^1(H,\mathbb{R})$ such that*

1. *J satisfies condition (C) in $]0,+\infty[$;*
2. *There exist a closed subset $A \subset H$ and a Hilbert manifold $B \subset H$ with boundary ∂B satisfying*
   - (a) *there exist two constants $\beta > \alpha \geq 0$ such that*
   $$J(u) \leq \alpha,\ \forall u \in \partial B; \quad J(u) \geq \beta,\ \forall u \in A;$$
   i.e., $a_0 := \sup_{\partial B} J \leq b_0 := \inf_A J$.
   - (b) *A and ∂B link;*
   - (c) $\sup_{u\in B} J(u) < +\infty$.

Then a critical value of J is given by

$$a^* = \inf_{\Gamma\in\phi^*} \sup_{\Gamma([0,1],A)\cap S} J(u).$$

Furthermore, assuming that $0 \notin K_{a^*}$, we have $K_{a^*} \cap S \neq \emptyset$ if $a^* > b_0$, and $K_{a^*} \cap A \neq \emptyset$ if $a^* = b_0$.

In this article, we shall consider the symmetry given by a $\mathbb{Z}_2$ action, more precisely even functionals.

Theorem 3.6. *Suppose $J \in C^1(H,\mathbb{R})$ and the positive cone P is an admissible invariant for J, $K_c \cap \partial P = \emptyset$ for $c > 0$, such that*

1. *J satisfies condition (C) in $]0,+\infty[$, and $J(0) \geq 0$;*
2. *There exist two closed subspace H^+, H^- of H, with $\operatorname{codim} H^+ < +\infty$ and two constants $c_\infty > c_0 > J(0)$ satisfying*
$$J(u) \geq c_0, \forall u \in S_\rho \cap H^+; \quad J(u) < c_\infty, \forall u \in H^-.$$
3. *J is even.*

Then if $\dim H^- > 1 + \operatorname{codim} H^+$, J possesses at least $m := \dim H^- - \operatorname{codim} H^+ - 1$ ($m := \dim H^- - 1$ resp.) distinct pairs of critical points in $X\backslash P \cup (-P)$ with critical values belong to $[c_0, c_\infty]$.

The above theorem locates the critical points more precisely than [1, 6, Theorem 3.3] and the references therein.

We shall use pseudo-index theory to prove Theorem 3.6. First, we need the notation of genus and its properties, see [6, 7]. Let

$$\Sigma_X = \{A \subset X : A \text{ is closed in } X, A = -A\}.$$

We denote by $i_X(A)$ the genus of A in X.

Proposition 3.7. *Assume that $A, B \in \Sigma_X$, $h \in C(X,X)$ is an odd homeomorphism, then*

- (i) *$i_X(A) = 0$ if and only if $A = \emptyset$;*
- (ii) *$A \subset B \Rightarrow i_X(A) \leq i_X(B)$ (monotonicity);*

(iii) $i_X(A \cup B) \le i_X(A) + i_X(B)$ *(subadditivity);*
(iv) $i_X(A) \le i_X(\overline{h(A)})$ *(supervariancy);*
(v) *if A is a compact set, then $i_X(A) < +\infty$ and there exists $\delta > 0$ such that $i_X(N_\delta(A)) = i_X(A)$, where $N_\delta(A)$ denotes the closed δ-neighborhood of A (continuity);*
(vi) *if $i_X(A) > k, V$ is a k-dimensional subspace of X, then $A \cap V^\perp \neq \emptyset$;*
(vii) *if W is a finite dimensional subspace of X, then $i_X(h(S_\rho) \cap W) = \dim W$.*
(viii) *Let V, W be two closed subspace of X with $\operatorname{codim} V < +\infty, \dim W < +\infty$. Then if h is bounded odd homeomorphism on X, we have*
$$i_X(W \cap h(S_\rho \cap V)) \ge \dim W - \operatorname{codim} V.$$

The above proposition is still true when we replace Σ_X by Σ_H with obvious modifications.

Proposition 3.8 ([6])**.** *If $A \in \Sigma_X$ with $2 \le i_X(A) < \infty$, then $A \cap S \neq \emptyset$.*

Proposition 3.9. *Let $A \in \Sigma_H$, then $A \cap X \in \Sigma_X$ and $i_H(A) \ge i_X(A \cap X)$.*

Definition 3.10 ([1, 6])**.** Let $I = (\Sigma, \mathcal{H}, i)$ be an index theory on H related to a group G, and $B \in \Sigma$. We call a pseudo-index theory (related to B and I) a triplet
$$I^* = (B, \mathcal{H}^*, i^*)$$
where $\mathcal{H}^* \subset \mathcal{H}$ is a group of homeomorphism on H, and $i^* : \Sigma \to \mathbb{N} \cup \{+\infty\}$ is the map defined by
$$i^*(A) = \min_{h \in \mathcal{H}^*} i(h(A) \cap B).$$

Proof of Theorem 3.6. Consider the genus $I = (\Sigma, \mathcal{H}, i)$ and the pseudo-index theory relate to I and $B = S_\rho \cap H^+$, $I^* = (S_\rho \cap H^+, \mathcal{H}^*, i^*)$, where
$$\begin{aligned}\mathcal{H}^* = \{&h \text{ is an odd bounded homeomorphism on } H \text{ and}\\ &h(u) = u \text{ if } u \notin J^{-1}(]0, +\infty[)\}.\end{aligned}$$

Obviously conditions [1, $(a_1), (a_2)$ of theorem 2.9] are satisfied with $a = 0$, $b = +\infty$ and $b = S_\rho \cap H^+$. Now we prove that condition (a_3) is satisfied with $\bar{A} = H^-$. It is obvious that $\bar{A} \subset J^{-1}(]-\infty, c_\infty])$, and by property (iv) of genus, we have
$$\begin{aligned}i^*(\bar{A}) = i^*(H^-) &= \min_{h \in \mathcal{H}^*} i(h(H^-) \cap S_\rho \cap H^+)\\ &= \min_{h \in \mathcal{H}^*} i(H^- \cap h^{-1}(S_\rho \cap H^+)).\end{aligned}$$
Now by (viii) of Proposition 3.7, we have
$$i(H^- \cap h^{-1}(S_\rho \cap H^+)) \ge \dim H^- - \operatorname{codim} H^+.$$
Therefore, we have
$$i^*(\bar{A}) \ge \dim H^- - \operatorname{codim} H^+.$$
Then by [6, Theorem 2.9] and Proposition 3.8 above, the numbers
$$c_k = \inf_{A \in \Sigma_k} \sup_{u \in A \cap S} J(u), \quad k = 2, \dots, \dim H^- - \operatorname{codim} H^+.$$
are critical values of J and
$$J(0) < c_0 \le c_k \le c_\infty, \quad k = 2, \dots, \dim H^- - \operatorname{codim} H^+. \tag{3.1}$$

If for every k, $c_k \neq c_{k+1}$, we obtain the conclusion of Theorem 3.6. Assume now $c = c_k = \cdots = c_{k+r}$ with $r \geq 1$ and $k + r \leq \dim H^- - \operatorname{codim} H^+$. Then as in the proof to [6, Theorem 2.9], we have

$$i(K_c \cap S) \geq r + 1 \geq 2 \tag{3.2}$$

Now from proposition 3.8 and (3.1), we deduce that

$$0 \notin K_c \cap S. \tag{3.3}$$

Since a finite set (not containing 0) has genus 1, we deduce from (3.2) and (3.3) that K_c above contains infinitely many sign-changing critical points. Therefore, J has at least $m := \dim H^- - \operatorname{codim} H^+ - 1$ distinct pairs of sign-changing critical points in $X \backslash P \cup (-P)$ with critical values belong to $[c_0, c_\infty]$.

If $\operatorname{codim} H^+ = 0$, we consider c_j for $j \geq 2$. As above arguments $J(0) < c_0 \leq c_2 \leq c_3 \leq \cdots \leq c_{\dim H^-} \leq c_\infty$ and if $c := c_j = \cdots = c_{j+l}$ for $2 \leq j \leq j+l \leq \dim H^-$ with $l \geq 1$, then $i(K_c \cap S) \geq l + 1 \geq 2$. Therefore, J has at least $\dim H^- - 1$ pairs of sign-changing critical points with values belong to $[c_0, c_\infty]$. □

We remark that Theorem 3.5 above can also be proved by the pseudo-index theory as Theorem 3.6.

4. Proof of Theorems 1.1–1.3

We shall apply the abstract results of section 3 to problem (1.3). Let $H := H_0^1(\Omega), X := C_0^1(\Omega)$. Clearly the solutions of (1.3) are the critical points of the functional

$$J(u) = \frac{1}{2}(\|u\|^2 - \lambda_k |u|^2) + \int_\Omega G(u)dx, \tag{4.1}$$

where $|\cdot|$ denotes the norm in $L^2(\Omega)$, then $J \in C^1(H, \mathbb{R})$. We denote by M_j the eigenspace corresponding to the eigenvalue λ_j. If $m \geq 0$ is an integer number, set

$$H^-(m) = \oplus_{j \leq m} M_j,$$

and $H^+(m)$ the closure in $H_0^1(\Omega)$ of the linear space spanned by $\{M_j\}_{j \geq m}$. Clearly $H^+(m) \cap H^-(m) = M_m$.

Proposition 4.1. *If* (G1), (G2) *hold, then the functional* J *defined by* (4.1) *satisfies the condition (C) in* $]0, +\infty[$.

Proof of Theorem 1.1. If $G(0) = 0$, then by (G3), G takes its minimum at 0, so $g(0) = 0$ and 0 is a solution of (1.3). We assume that $G(0) > 0$. Similar proof to that in [1], there exist $R, \gamma > 0$ such that

$$J(u) \geq \gamma, \quad u \in H^+(k+1);$$
$$J(u) \leq \frac{\gamma}{2}, \quad u \in H^-(k) \cap S_R.$$

Let $\partial B = H^-(k) \cap S_R$, $A = H^+(k+1)$, then by Example 3.2 we have that ∂B and A link, and J is bounded on $B = H^-(k) \cap B_R$. Moreover by Proposition 4.1, J satisfies condition (C) in $]0, +\infty[$. So the conclusion of Theorem 1.1 follows by Theorem 3.5. □

Note that if $J(0) = 0$, then the solutions obtained in Theorem 1.1 are sign-changing solutions.

Proof of Theorem 1.2. Since $g(0) = 0, u(x) = 0$ is a solution of (1.3). In this case, we are interested in finding the existence of sign-changing solutions to (1.3). The case $g(t) = 0$ for all $t \in \mathbb{R}$ is trivial. We assume that $g(t) \neq 0$ for some t. Then it is easy to see that (G2), (G3) and (1.4) imply $g'(0) > 0$. Similar proof to that in [1, Theorem 5.1], each of the following holds:

$$\lambda_1 - \lambda_k + g'(0) > 0, \tag{4.2}$$

$\lambda_k \neq \lambda_1$ and there exists $\lambda_h \in \sigma(-\Delta)$ with $\lambda_2 \leq \lambda_h \leq \lambda_k$ such that

$$\lambda_h - \lambda_k + g'(0) > 0, \quad \frac{1}{2}(\lambda_{h-1} - \lambda_k)t^2 + G(t) \leq G(0) \quad \forall t \in \mathbb{R}. \tag{4.3}$$

Under (4.1), there exist three positive constants $\rho < R, \gamma$ such that

$$J(u) \geq J(0) + \gamma, \quad u \in S_\rho; J(e) \leq J(0) + \frac{\gamma}{2}, \quad e \in M_1 \cap S_\rho.$$

Since $J(0) = G(0) \cdot |\Omega| \geq 0$ ($|\Omega|$ is the Lebesgue measure of Ω), we have

$$0 < J(0) + \frac{\gamma}{2} < J(0) + \gamma.$$

Fix $e \in M_1 \cap S_\rho$, set

$$A = S_\rho, \quad B = \{te : t \in [0, R]\}.$$

Then by Example 3.2, A and ∂B link and J is bounded on B. Moreover by Proposition 4.1, J satisfies condition (C) in $]0, +\infty[$. Then by Theorem 3.5, J possesses a critical point u_0 such that $J(u_0) \geq J(0) + \gamma$. So u_0 is a sign-changing solution to (1.3).

Under (4.3) similar arguments to that above, we get

$$J(u) \geq J(0) + \gamma, \quad u \in H^+(h) \cap S_\rho;\ J(u) \leq J(0) + \frac{\gamma}{2}, \quad u \in \partial B(h, R).$$

where $B(h, R) = \{u + te : u \in H^-(h-1) \cap B_R,\ e \in M_h \cap S_1, \quad 0 \leq t \leq R\}$. Set

$$A = H^+(h) \cap S_\rho, \quad B = B(h, R).$$

Then by Example 3.3, A and ∂B link and J is bounded on B. Moreover by Proposition 4.1, J satisfies condition (C). Using Theorem 3.5, we can conclude that J possesses a sign-changing critical point u_0 with $J(u_0) \geq J(0) + \gamma$. $\square$

Remark 4.2. If $g'(0) = 0$; i.e., resonance at 0 is allowed, then by using an argument similar to that in the proof of Theorem 1.2, problem (1.3) still has at least a sign-changing solution under these conditions: Let $g(0) = 0$. Assume that (G1), (G2) hold and

$$G(t) > 0, \quad \forall t \neq 0, \quad G(0) = 0.$$

Moreover suppose that either of the following holds

$$\lambda_k = \lambda_1;$$

$$\lambda_k \neq \lambda_1 \text{ and } \frac{1}{2}(\lambda_{k-1} - \lambda_k)t^2 + G(t) \leq 0 \quad \text{for all } t \in \mathbb{R}.$$

Proof of Theorem 1.3. By [1, Proposition 3.1 and Lemma 5.3], the assumptions of Theorem 3.6 are satisfied with

$$H^+ = H^+(h), \quad H^- = H^-(k).$$

Thus there exist at least $\dim H^- - \operatorname{codim} H^+ - 1 = \dim\{M_h \oplus \dots M_k\} - 1$ distinct pairs of sign-changing solutions of (1.3). $\square$

Remark 4.3. We also allow resonance at zero in problem (1.3). By using [1, Theorem 3.2 and Lemma 5.4], we have: Assume that g is odd and (G1) (G2) are satisfied. Suppose moreover $G(t) > 0$ for all $t \neq 0$ and $G(0) = 0$. Then (1.3) possesses at least $\dim M_k - 1$ distinct pairs of sign-changing solutions. (M_k denotes the eigenspace corresponding to λ_k with $k \geq 2$)

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