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EXISTENCE OF SOLUTIONS FOR A NONLINEAR FRACTIONAL BOUNDARY VALUE PROBLEM VIA A LOCAL MINIMUM THEOREM

CHUANZHI BAI

Abstract. This article concerns the existence of solutions to the nonlinear fractional boundary-value problem

$$\frac{d}{dt}\Big({}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t)) - {}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t))\Big) + \lambda f(u(t)) = 0, \quad \text{a.e. } t \in [0,T],$$
$$u(0) = u(T) = 0,$$

where $\alpha \in (1/2, 1]$, and λ is a positive real parameter. The approach is based on a local minimum theorem established by Bonanno.

1. Introduction

Fractional differential equations have been proved to be valuable tools in the modeling of many phenomena in various fields of physic, chemistry, biology, engineering and economics. There has been significant development in fractional differential equations, one can see the monographs of Miller and Ross [1], Samko et al [2], Podlubny [3], Hilfer [4], Kilbas et al [5] and the papers [7, 8, 9, 10, 11, 12, 13, 6, 14, 15, 16] and the references therein.

Critical point theory has been very useful in determining the existence of solution for integer order differential equations with some boundary conditions, for example [21, 6, 19, 18, 17, 20]. But until now, there are few results on the solution to fractional BVP which were established by the critical point theory, since it is often very difficult to establish a suitable space and variational functional for fractional BVP. Recently, Jiao and Zhou [22] investigated the fractional boundary-value problem

$$\frac{d}{dt}\Big(\frac{1}{2}{}_0D_t^{-\beta}(u'(t)) + \frac{1}{2}{}_tD_T^{-\beta}(u'(t))\Big) + \nabla F(t,u(t)) = 0, \quad \text{a.e. } t \in [0,T],$$
$$u(0) = u(T) = 0,$$

by using the critical point theory, where ${}_0D_t^{-\beta}$ and ${}_tD_T^{-\beta}$ are the left and right Riemann-Liouville fractional integrals of order $0 \le \beta < 1$ respectively, $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is a given function and $\nabla F(t,x)$ is the gradient of F at x.

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In this article, by using a local minimum theorem established by Bonanno in [23], a new approach is provided to investigate the existence of solutions to the following fractional boundary value problems

$$\begin{gathered} \frac{d}{dt}\Big({}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t)) - {}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t))\Big) + \lambda f(u(t)) = 0, \quad \text{a.e. } t \in [0,T], \\ u(0) = u(T) = 0, \end{gathered} \tag{1.1}$$

where $\alpha \in (1/2, 1]$, ${}_0D_t^{\alpha-1}$ and ${}_tD_T^{\alpha-1}$ are the left and right Riemann-Liouville fractional integrals of order $1-\alpha$ respectively, ${}_0^cD_t^{\alpha}$ and ${}_t^cD_T^{\alpha}$ are the left and right Caputo fractional derivatives of order $0 < \alpha \leq 1$ respectively, λ is a positive real parameter, and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function.

2. Preliminaries

In this section, we introduce some definitions and properties of the fractional calculus which are used in this article.

Definition 2.1 ([5]). Let f be a function defined on $[a, b]$. The left and right Riemann-Liouville fractional integrals of order α for a function f are defined by

$$\begin{gathered} {}_aD_t^{-\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)ds, \quad t \in [a, b],\ \alpha > 0, \\ {}_tD_b^{-\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1}f(s)ds, \quad t \in [a, b],\ \alpha > 0, \end{gathered}$$

provided the right-hand sides are pointwise defined on $[a, b]$, where $\Gamma(\alpha)$ is the standard gamma function.

Definition 2.2 ([5]). Let $\gamma \geq 0$ and $n \in \mathbb{N}$.

(i) If $\gamma \in (n-1, n)$ and $f \in AC^n([a, b], \mathbb{R}^N)$, then the left and right Caputo fractional derivatives of order γ for function f denoted by ${}_a^cD_t^{\gamma}f(t)$ and ${}_t^cD_b^{\gamma}f(t)$, respectively, exist almost everywhere on $[a, b]$, ${}_a^cD_t^{\gamma}f(t)$ and ${}_t^cD_b^{\gamma}f(t)$ are represented by

$$\begin{gathered} {}_a^cD_t^{\gamma}f(t) = \frac{1}{\Gamma(n-\gamma)}\int_a^t (t-s)^{n-\gamma-1}f^{(n)}(s)ds, \quad t \in [a, b], \\ {}_t^cD_b^{\gamma}f(t) = \frac{(-1)^n}{\Gamma(n-\gamma)}\int_t^b (s-t)^{n-\gamma-1}f^{(n)}(s)ds, \quad t \in [a, b], \end{gathered}$$

respectively.

(ii) If $\gamma = n-1$ and $f \in AC^{n-1}([a, b], \mathbb{R}^N)$, then ${}_a^cD_t^{n-1}f(t)$ and ${}_t^cD_b^{n-1}f(t)$ are represented by

$${}_a^cD_t^{n-1}f(t) = f^{(n-1)}(t), \quad \text{and} \quad {}_t^cD_b^{n-1}f(t) = (-1)^{(n-1)}f^{(n-1)}(t), \quad t \in [a, b].$$

With these definitions, we have the rule for fractional integration by parts, and the composition of the Riemann-Liouville fractional integration operator with the Caputo fractional differentiation operator, which were proved in [5, 2].

Proposition 2.3 ([5, 2]). *We have the following property of fractional integration*

$$\int_a^b [{}_aD_t^{-\gamma}f(t)]g(t)dt = \int_a^b [{}_tD_b^{-\gamma}g(t)]f(t)dt, \quad \gamma > 0, \tag{2.1}$$

provided that $f \in L^p([a,b],\mathbb{R}^N)$, $g \in L^q([a,b],\mathbb{R}^N)$ *and* $p \geq 1$, $q \geq 1$, $1/p + 1/q \leq 1+\gamma$ *or* $p \neq 1$, $q \neq 1$, $1/p+1/q = 1+\gamma$.

Proposition 2.4 ([5]). *Let* $n \in \mathbb{N}$ *and* $n-1 < \gamma \leq n$. *If* $f \in AC^n([a,b],\mathbb{R}^N)$ *or* $f \in C^n([a,b],\mathbb{R}^N)$, *then*

$$
{}_aD_t^{-\gamma}({}_a^cD_t^{\gamma} f(t)) = f(t) - \sum_{j=0}^{n-1} \frac{f^{(j)}(a)}{j!}(t-a)^j\,,
$$

$$
{}_tD_b^{-\gamma}({}_t^cD_b^{\gamma} f(t)) = f(t) - \sum_{j=0}^{n-1} \frac{(-1)^j f^{(j)}(b)}{j!}(b-t)^j,
$$

for $t \in [a,b]$. *In particular, if* $0 < \gamma \leq 1$ *and* $f \in AC([a,b],\mathbb{R}^N)$ *or* $f \in C^1([a,b],\mathbb{R}^N)$, *then*

$$
{}_aD_t^{-\gamma}({}_a^cD_t^{\gamma} f(t)) = f(t) - f(a), \quad \textit{and} \quad {}_tD_b^{-\gamma}({}_t^cD_b^{\gamma} f(t)) = f(t) - f(b). \tag{2.2}
$$

Remark 2.5. In view of (2.1) and Definition 2.2, it is obvious that $u \in AC([0,T])$ is a solution of (1.1) if and only if u is a solution of the problem

$$
\begin{gathered}
\frac{d}{dt}\Big({}_0D_t^{-\beta}(u'(t)) + {}_tD_T^{-\beta}(u'(t))\Big) + \lambda f(u(t)) = 0, \quad \text{a.e. } t \in [0,T], \\
u(0) = u(T) = 0,
\end{gathered} \tag{2.3}
$$

where $\beta = 2(1-\alpha) \in [0,1)$.

To establish a variational structure for (1.1), it is necessary to construct appropriate function spaces. Denote by $C_0^\infty[0,T]$ the set of all functions $g \in C^\infty[0,T]$ with $g(0) = g(T) = 0$.

Definition 2.6 ([22]). Let $0 < \alpha \leq 1$. The fractional derivative space E_0^α is defined by the closure of $C_0^\infty[0,T]$ with respect to the norm

$$
\|u\|_\alpha = \Big(\int_0^T |{}_0^cD_t^\alpha u(t)|^2 dt + \int_0^T |u(t)|^2 dt\Big)^{1/2}, \quad \forall u \in E^\alpha.
$$

Remark 2.7. It is obvious that the fractional derivative space E_0^α is the space of functions $u \in L^2[0,T]$ having an α-order Caputo fractional derivative ${}_0^cD_t^\alpha u \in L^2[0,T]$ and $u(0) = u(T) = 0$.

Proposition 2.8 ([22]). *Let* $0 < \alpha \leq 1$. *The fractional derivative space* E_0^α *is reflexive and separable Banach space.*

Lemma 2.9 ([22]). *Let* $0 < \alpha \leq 1$. *For all* $u \in E_0^\alpha$, *we have*

$$
\|u\|_{L^2} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \|{}_0^cD_t^\alpha u\|_{L^2}, \tag{2.4}
$$

$$
\|u\|_\infty \leq \frac{T^{\alpha-1/2}}{\Gamma(\alpha)(2(\alpha-1)+1)^{1/2}} \|{}_0^cD_t^\alpha u\|_{L^2}. \tag{2.5}
$$

By (2.4), we can consider E_0^α with respect to the norm

$$
\|u\|_\alpha = \Big(\int_0^T |{}_0^cD_t^\alpha u(t)|^2 dt\Big)^{1/2} = \|{}_0^cD_t^\alpha u\|_{L^2}, \quad \forall u \in E_0^\alpha \tag{2.6}
$$

in the following analysis.

Lemma 2.10 ([22]). *Let $1/2 < \alpha \le 1$, then for all any $u \in E_0^\alpha$, we have*

$$|\cos(\pi\alpha)|\|u\|_\alpha^2 \le -\int_0^T {}_0^cD_t^\alpha u(t) \cdot {}_t^cD_T^\alpha u(t)dt \le \frac{1}{|\cos(\pi\alpha)|}\|u\|_\alpha^2. \tag{2.7}$$

Our main tools is the local minimum theorem [23] which is recalled below. Given a set X and two functionals $\Phi, \Psi : X \to \mathbb{R}$, let

$$\beta(r_1, r_2) = \inf_{v\in\Phi^{-1}(]r_1,r_2[)} \frac{\sup_{u\in\Phi^{-1}(]r_1,r_2[)} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)}, \tag{2.8}$$

$$\rho_2(r_1, r_2) = \sup_{v\in\Phi^{-1}(]r_1,r_2[)} \frac{\Psi(v) - \sup_{u\in\Phi^{-1}(]-\infty,r_1])} \Psi(u)}{\Phi(v) - r_1}, \tag{2.9}$$

for all $r_1, r_2 \in R$, with $r_1 < r_2$.

Theorem 2.11 ([23]). *Let X be a reflexive real Banach space; $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gateaux differential function whose Gateaux derivative admits a continuous inverse on X^*; $\Psi : X \to \mathbb{R}$ be a continuously Gateaux differentiable function whose Gateaux derivative is compact. Put $I_\lambda = \Phi - \lambda\Psi$ and assume that there are $r_1, r_2 \in \mathbb{R}$, with $r_1 < r_2$, such that*

$$\beta(r_1, r_2) < \rho_2(r_1, r_2),$$

where β and ρ_2 are given by (2.8) and (2.9). Then, for each $\lambda \in \big(\frac{1}{\rho_2(r_1,r_2)}, \frac{1}{\beta(r_1,r_2)}\big)$ there is $u_{0,\lambda} \in \Phi^{-1}(]r_1, r_2[)$ such that $I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$ for all $u \in \Phi^{-1}(]r_1, r_2[)$ and $I'_\lambda(u_{0,\lambda}) = 0$.

3. Main result

For given $u \in E_0^\alpha$, we define functionals $\Phi, \Psi : E^\alpha \to \mathbb{R}$ as follows:

$$\Phi(u) := -\int_0^T {}_0^cD_t^\alpha u(t) \cdot {}_t^cD_T^\alpha u(t)dt, \quad \Psi(u) := \int_0^T F(u(t))dt,$$

where $F(u) = \int_0^u f(s)ds$. Clearly, Φ and Ψ are Gateaux differentiable functional whose Gateaux derivative at the point $u \in E_0^\alpha$ are given by

$$\Phi'(u)v = -\int_0^T ({}_0^cD_t^\alpha u(t) \cdot {}_t^cD_T^\alpha v(t) + {}_t^cD_T^\alpha u(t) \cdot {}_0^cD_t^\alpha v(t))dt,$$

$$\Psi'(u)v = \int_0^T f(u(t))v(t)dt = -\int_0^T \int_0^t f(u(s))ds \cdot v'(t)dt,$$

for every $v \in E_0^\alpha$. By Definition 2.2 and (2.2), we have

$$\Phi'(u)v = \int_0^T ({}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u(t)) - {}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u(t))) \cdot v'(t)dt.$$

Hence, $I_\lambda = \Phi - \lambda\Psi \in C^1(E_0^\alpha, \mathbb{R})$. If $u_* \in E_0^\alpha$ is a critical point of I_λ, then

$$\begin{aligned} 0 = I'_\lambda(u_*)v = \int_0^T &\Big({}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u_*(t)) - {}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u_*(t)) \\ &+ \lambda\int_0^t f(u_*(s))ds\Big) \cdot v'(t)dt, \end{aligned} \tag{3.1}$$

for $v \in E_0^\alpha$. We can choose $v \in E_0^\alpha$ such that

$$v(t) = \sin\frac{2k\pi t}{T} \quad \text{or} \quad v(t) = 1 - \cos\frac{2k\pi t}{T}, \quad k = 1, 2, \dots.$$

The theory of Fourier series and (3.1) imply

$$ {}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u_*(t)) - {}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u_*(t)) + \lambda \int_0^t f(u_*(s))ds = C \tag{3.2}$$

a.e. on $[0,T]$ for some $C \in \mathbb{R}$. By (3.2), it is easy to show that $u_* \in E_0^\alpha$ is a solution of (1.1).

By Lemma 2.9, when $\alpha > 1/2$, for each $u \in E_0^\alpha$ we have

$$\|u\|_\infty \le \Omega\Big(\int_0^T |{}_0^cD_t^\alpha u(t)|^2 dt\Big)^{1/2} = \Omega\|u\|_\alpha, \tag{3.3}$$

where

$$\Omega = \frac{T^{\alpha-\frac12}}{\Gamma(\alpha)\sqrt{2(\alpha-1)+1}}. \tag{3.4}$$

Given two constants $c \ge 0$ and $d \ne 0$, with $c \ne \sqrt{\frac{\omega_{\alpha,d}}{|\cos(\pi\alpha)|}} \cdot \Omega$, where Ω as in (3.4). Put

$$\omega_{\alpha,d} := \frac{4\Gamma^2(2-\alpha)}{\Gamma(4-2\alpha)} T^{1-2\alpha} d^2 (2^{2\alpha-1} - 1).$$

Theorem 3.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and $\frac12 < \alpha \le 1$. Assume that there exist a positive constant c and a constant $d \ne 0$ with*

$$\sqrt{\frac{\omega_{\alpha,d}}{|\cos(\pi\alpha)|}}\,\Omega < c, \tag{3.5}$$

such that

$$0 < \frac{\max_{|\eta|\le c} F(\eta)}{c^2|\cos(\pi\alpha)|} < \frac{\frac{1}{\Gamma(2-\alpha)|d|}\int_0^{\Gamma(2-\alpha)|d|} F(x)dx}{\omega_{\alpha,d}\Omega^2}. \tag{3.6}$$

Then, for each

$$\lambda \in \Big(\frac{\omega_{\alpha,d}\Gamma(2-\alpha)|d|}{T\int_0^{\Gamma(2-\alpha)|d|} F(x)dx}, \frac{c^2|\cos(\pi\alpha)|}{T\Omega^2 \max_{|\eta|\le c} F(\eta)}\Big),$$

problem (1.1) admits at least one solution $\bar u$ such that $\|\bar u\|_\alpha < c/\Omega$.

Proof. Let Φ, Ψ be the functionals defined above. It is well known that they satisfy all regularity assumptions requested in Theorem 2.11 and that the critical point of the functional $\Phi - \lambda\Psi$ in E_0^α is exactly the solution of (1.1). Put

$$\begin{gathered} r = \frac{|\cos(\pi\alpha)|}{\Omega^2}c^2, \\ u_0(t) = \begin{cases} \frac{2\Gamma(2-\alpha)d}{T}t, & t \in [0, T/2), \\ \frac{2\Gamma(2-\alpha)d}{T}(T-t), & t \in [T/2, T] \end{cases} \end{gathered} \tag{3.7}$$

It is easy to check that $u_0(0) = u_0(T) = 0$ and $u_0 \in L^2[0,T]$. The direct calculation shows that

$${}_0^cD_t^\alpha u_0(t) = \begin{cases} \frac{2d}{T}t^{1-\alpha}, & t \in [0, T/2), \\ \frac{2d}{T}(t^{1-\alpha} - 2(t-\frac{T}{2})^{1-\alpha}), & t \in [T/2, T] \end{cases}$$

and

$$\begin{aligned}
\|u_0\|_\alpha^2 &= \int_0^T ({}_0^cD_t^\alpha u_0(t))^2 dt = \int_0^{\frac{T}{2}} + \int_{T/2}^T ({}_0^cD_t^\alpha u_0(t))^2 dt \\
&= \frac{4d^2}{T^2}\Big[\int_0^T t^{2(1-\alpha)}dt - 4\int_{T/2}^T t^{1-\alpha}(t-\frac{T}{2})^{1-\alpha}dt + 4\int_{T/2}^T (t-\frac{T}{2})^{2(1-\alpha)}dt\Big] \\
&= \frac{4(1+2^{2\alpha-1})d^2}{3-2\alpha}T^{1-2\alpha} - \frac{16d^2}{T^2}\int_{T/2}^T t^{1-\alpha}(t-\frac{T}{2})^{1-\alpha}dt < \infty.
\end{aligned}$$

That is, ${}_0^cD_t^\alpha u_0 \in L^2[0,T]$. Thus, $u_0 \in E_0^\alpha$. Moreover, the direct calculation shows

$$ {}_t^cD_T^\alpha u_0(t) = \begin{cases} \frac{2d}{T}((T-t)^{1-\alpha} - 2(\frac{T}{2}-t)^{1-\alpha}), & t \in [0, T/2), \\ \frac{2d}{T}(T-t)^{1-\alpha}, & t \in [T/2, T] \end{cases}$$

and

$$\begin{aligned}
\Phi(u_0) &= -\int_0^T {}_0^cD_t^\alpha u_0(t) \cdot {}_t^cD_T^\alpha u_0(t) dt \\
&= -(\frac{2d}{T})^2\Big[\int_0^{\frac{T}{2}} t^{1-\alpha}\Big((T-t)^{1-\alpha} - 2\big(\frac{T}{2}-t\big)^{1-\alpha}\Big)dt \\
&\quad + \int_{T/2}^T (T-t)^{1-\alpha}\cdot\Big(t^{1-\alpha} - 2(t-\frac{T}{2})^{1-\alpha}\Big)dt\Big] \\
&= -(\frac{2d}{T})^2\Big[\int_0^T t^{1-\alpha}(T-t)^{1-\alpha}dt - 4\int_0^{\frac{T}{2}} t^{1-\alpha}\big(\frac{T}{2}-t\big)^{1-\alpha}dt\Big] \\
&= -(\frac{2d}{T})^2\Big[\frac{\Gamma^2(2-\alpha)}{\Gamma(4-2\alpha)}T^{3-2\alpha} - 4\frac{\Gamma^2(2-\alpha)}{\Gamma(4-2\alpha)}(\frac{T}{2})^{3-2\alpha}\Big] \\
&= \frac{4\Gamma^2(2-\alpha)}{\Gamma(4-2\alpha)}T^{1-2\alpha}(2^{2\alpha-1}-1)d^2 = \omega_{\alpha,d},
\end{aligned}$$

and

$$\Psi(u_0) = \int_0^T F(u_0(t))dt = \frac{T}{\Gamma(2-\alpha)|d|}\int_0^{\Gamma(2-\alpha)|d|} F(x)dx.$$

Hence, from (3.5), one has $0 < \omega_{\alpha,d} < \frac{|\cos(\pi\alpha)|}{\Omega^2}c^2$; that is, $0 < \Phi(u_0) < r$. Moreover, for all $u \in E_0^\alpha$ such that $u \in \Phi^{-1}(]-\infty, r])$, by (2.7) we have

$$|\cos(\pi\alpha)|\|u\|_\alpha^2 \le \Phi(u) \le r,$$

which implies

$$\|u\|_\alpha^2 \le \frac{1}{|\cos(\pi\alpha)|}r. \tag{3.8}$$

Thus, by (3.3), (3.8) and (3.7) we obtain

$$|u(t)| < \Omega\|u\|_\alpha \le \Omega\sqrt{\frac{r}{|\cos(\pi\alpha)|}} = c, \quad \forall t \in [0,T].$$

Hence,

$$\Psi(u) = \int_0^T F(u(t))dt \le \int_0^T \max_{|\eta|\le c} F(\eta)dt = T\max_{|\eta|\le c} F(\eta),$$

for all $u \in E_0^\alpha$ such that $u \in \Phi^{-1}(]-\infty, r])$. Hence,

$$\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u) \le T \max_{|\eta|\le c} F(\eta).$$

Hence, one has

$$\begin{aligned}
\beta(0,r) &\le \frac{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)-\Psi(u_0)}{r-\Phi(u_0)} \\
&\le \Omega^2 T\frac{\max_{|\eta|\le c}F(\eta)-\frac{1}{\Gamma(2-\alpha)|d|}\int_0^{\Gamma(2-\alpha)|d|}F(x)dx}{|\cos(\pi\alpha)|c^2-\omega_{\alpha,d}\Omega^2} \\
&< \Omega^2 T\frac{\max_{|\eta|\le c}F(\eta)-\frac{\omega_{\alpha,d}\Omega^2}{|\cos(\pi\alpha)|c^2}\max_{|\eta|\le c}F(\eta)}{|\cos(\pi\alpha)|c^2-\omega_{\alpha,d}\Omega^2} \\
&= \Omega^2 T\frac{\max_{|\eta|\le c}F(\eta)}{c^2|\cos(\pi\alpha)|},
\end{aligned} \tag{3.9}$$

by condition (3.6). On the other hand, if $u \in \Phi^{-1}(]-\infty, 0])$, then $\Phi(u) \le 0$. Thus, by (2.4) and (2.7) we have $\|u\|_{L^2} = 0$; that is, $u(t) = 0$, a.e. $t \in [0, T]$. Hence,

$$\begin{aligned}
\rho_2(0,r) &\ge \frac{\Psi(u_0)-\sup_{u\in\Phi^{-1}(]-\infty,0])}\Psi(u)}{\Phi(u_0)} = \frac{\Psi(u_0)}{\Phi(u_0)} \\
&= T\,\frac{\frac{1}{\Gamma(2-\alpha)|d|}\int_0^{\Gamma(2-\alpha)|d|}F(x)dx}{\omega_{\alpha,d}}.
\end{aligned} \tag{3.10}$$

Thus, by (3.9), (3.10) and (3.6) it follows that $\beta(0, r) < \rho_2(0, r)$. So, from Theorem 2.11 for each

$$\lambda \in \Big(\frac{\omega_{\alpha,d}\Gamma(2-\alpha)|d|}{T\int_0^{\Gamma(2-\alpha)|d|}F(x)dx}, \frac{c^2|\cos(\pi\alpha)|}{T\Omega^2\max_{|\eta|\le c}F(\eta)}\Big) \subset \Big(\frac{1}{\rho_2(0,r)},\ \frac{1}{\beta(0,r)}\Big),$$

the function $\Phi - \lambda\Psi$ admits at least one critical point $\bar{u}$ such that $0 < \Phi(\bar{u}) < r$; that is, $\|\bar{u}\|_\alpha < \frac{c}{\Omega}$, and the conclusion is achieved. □

We conclude with an example that illustrates the results obtained here. Let $\alpha = 0.8$, $T = 1$, and $f(u) = \cos(\pi u/3)$. Then (1.1) reduces to the boundary-value problem

$$\begin{gathered}
\frac{d}{dt}\Big({}_0D_t^{-0.2}({}_0^cD_t^{0.8}u(t)) - {}_tD_1^{-0.2}({}_t^cD_1^{0.8}u(t))\Big) + \lambda\cos(\frac{\pi}{3}u(t)) = 0, \quad \text{a.e. } t\in[0,1], \\
u(0) = u(1) = 0.
\end{gathered} \tag{3.11}$$

Owing to Theorem 3.1, for each $\lambda \in (3.2964, 4.30512)$, boundary-value problem (3.11) admits at least one solution. In fact, put $c = 2.5$ and $d = 1$, it is easy to calculate that $\Omega = 1.1089$, $\omega_{0.8,1} = 1.4$ and

$$\sqrt{\frac{\omega_{0.8,1}}{|\cos(0.8\pi)|}}\,\Omega = 1.4588 < 2.5 = c.$$

Moreover, we have

$$\frac{\frac{1}{\Gamma(2-\alpha)|d|}\int_0^{\Gamma(2-\alpha)|d|}F(x)dx}{\omega_{\alpha,d}\Omega^2} = \frac{\frac{1}{\Gamma(1.2)}\int_0^{\Gamma(1.2)}\frac{3}{\pi}\sin(\pi x/3)dx}{\omega_{0.8,1}\cdot\Omega^2} = 0.2467, \tag{3.12}$$

and

$$\frac{\max_{|\eta|\le c} F(\eta)}{c^2|\cos(\pi\alpha)|} = \frac{3/\pi}{2.5^2 \cdot |\cos(0.8\pi)|} = 0.1889, \tag{3.13}$$

which implies that condition (3.6) holds. Thus, by Theorem 3.1, for each $\lambda \in (3.2964, 4.3051)$, problem (3.11) admits at least one solution $\bar{u}$ such that $\|\bar{u}\|_{0.8} < 2.2545$.

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Chuanzhi Bai
Department of Mathematics, Huaiyin Normal University, Huaian, Jiangsu 223300, China
E-mail address: czbai8@sohu.com