Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 183, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR NONLINEAR ELLIPTIC DIRICHLET SYSTEMS

GABRIELE BONANNO, ELISABETTA TORNATORE

Abstract. The existence and multiplicity of solutions for systems of nonlinear elliptic equations with Dirichlet boundary conditions is investigated. Under suitable assumptions on the potential of the nonlinearity, the existence of one, two, or three solutions is established. Our approach is based on variational methods.

1. Introduction

The aim of this article is to establish the existence of solutions to the system

$$\begin{gathered} -\Delta u = \lambda \nabla_u F(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ (with $N \geq 3$) is a non-empty bounded open set with smooth boundary $\partial\Omega$, λ is a positive parameter. In the statement of problem (1.1), $u : \Omega \to \mathbb{R}^m$ (with $m \geq 1$) and $F : \Omega \times \mathbb{R}^m \to \mathbb{R}$ is a C^1-function, $F(x,0) = 0$ for every $x \in \Omega$ and $\nabla_u F = (F_{u_i})_{i=1,\dots,m}$ where F_{u_i} denotes the partial derivative of F respect on u_i $(i = 1,\dots,m)$.

Existence results for nonlinear elliptic systems of type (1.1) have received a great deal of interest in recent years. We refer the reader to [6] for a complete overview on this subject, and to [8] and the references therein for more recent developments.

In this article, at first, we prove the existence of a non-zero solution of problem (1.1), without assuming any asymptotic condition neither at zero nor at infinity (see Theorem 3.1) and, as a consequence, we obtain the existence of one solution, by assuming only that the potential F has a suitable behavior at zero (see Corollary 3.2). Next, we obtain the existence of two solutions, possibly both non-zero, assuming only the classical Ambrosetti-Rabinowitz condition; that is, without requiring that the potential F satisfies the usual condition at zero (see Theorem 3.3). Finally, we present a three solutions existence result under appropriate condition on the potential F (see Theorem 3.4).

It is worth noticing that in [8] the nonlinear elliptic Dirichlet system involves the (p,q)-Laplacian with $p, q > N$, since in a such result the compact embedding of the Sobolev space in $C^0(\bar{\Omega})$ is a crucial point in the proof; while in our results, the case $p = q = 2 < N$ is investigated. Some examples illustrate the obtained results

---

2000 *Mathematics Subject Classification.* 35J65, 35J20.
Key words and phrases. Elliptic systems; variational problems; multiple solutions.

Submitted August 28, 2012. Published October 28, 2012.

(see Examples 3.7, 3.8 and 3.9). Our approach is based on critical point theorems contained in [3] and [5]. The paper is arranged as follows. In Section 2 we recall our main tools, while Section 3 is devoted to our main results.

2. Preliminaries

In this section, we recall definitions and theorems to be used in this article. Let $(X, \|\cdot\|)$ be a real Banach space and Φ, $\Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals; put

$$I = \Phi - \Psi$$

and fix r_1, $r_2 \in [-\infty, +\infty]$, with $r_1 < r_2$. We say that functional I satisfies the *Palais-Smale condition cut off lower at r_1 and upper at r_2* (${}^{[r_1]}(PS)^{[r_2]}$-condition) if any sequence $\{u_n\} \in X$ such that

- $\{I(u_n)\}$ is bounded,
- $\lim_{n\to+\infty} \|I'(u_n)\|_{X^*} = 0$,
- $r_1 < \Phi(u_n) < r_2 \quad \forall n \in \mathbb{N}$,

has a convergent subsequence.

If $r_1 = -\infty$ and $r_2 = +\infty$ it coincides with the classical (PS)-condition, while if $r_1 = -\infty$ and $r_2 \in \mathbb{R}$ it is denoted by $(PS)^{[r_2]}$-condition.

Now we recall a result of local minimum obtained in [3], which is based on [2, Theorem 5.1].

Theorem 2.1 ([3, Theorem 2.2]). *Let X be a real Banach space, and let Φ, Ψ : $X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that* $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. *Assume that there exist $r \in \mathbb{R}$ and $\bar{u} \in X$, with $0 < \Phi(\bar{u}) < r$, such that*

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{r} < \frac{\Psi(\bar{u})}{\Phi(\bar{u})} \tag{2.1}$$

and, for each $\lambda \in \Lambda := \big]\frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}\big[$ the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the $(PS)^{[r]}$-condition. Then, for each $\lambda \in \Lambda := \big]\frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}\big[$, there is $u_\lambda \in \Phi^{-1}(]0, r[)$ such that $I_\lambda(u_\lambda) \le I_\lambda(u)$ for all $u \in \Phi^{-1}(]0, r[)$ and $I'_\lambda(u_\lambda) = 0$.

Now, we also recall a recent result obtained in [3] that ensures the existence of two critical points and which is based on [2, Theorem 3.1] and on the classical Ambrosetti-Rabinowitz Theorem (see [1]).

Theorem 2.2 ([3, Theorem 3.2]). *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is bounded from below and $\Phi(0) = \Psi(0) = 0$.*

Fix $r > 0$ and assume that, for each $\lambda \in \big]0, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}\big[$, the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies (PS)-condition and it is unbounded from below. Then, for each $\lambda \in]0, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}[$, the functional I_λ admits two distinct critical points in X.

Finally we point out an other result, which insures the existence of at least three critical points, that has been obtained in [5] and it is a more precise version of [4, Theorem 3.2].

Theorem 2.3 ([5, Theorem 3.6]). *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable, coercive and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, moreover*

$$\Phi(0) = \Psi(0) = 0.$$

Assume that there exist $r \in \mathbb{R}$ and $\bar{u} \in X$, with $0 < r < \Phi(\bar{u})$, such that

(i) $\frac{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{r} < \frac{\Psi(\bar{u})}{\Phi(\bar{u})}$

(ii) *for each* $\lambda \in \Lambda := \Big]\frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}\Big[$ *the functional $\Phi - \lambda\Psi$ is coercive.*

Then, for each $\lambda \in \Lambda$, the functional $I_\lambda = \Phi - \lambda\Psi$ has at least three distinct critical points in X.

Throughout in the article we assume the following conditions:

(H0) there exist two non negative constants a_1, a_2 and a constant $q \in]1, \frac{2N}{N-2}[$ such that
$$|F_{t_i}(x, t_1, \dots, t_m)| \le a_1 + a_2|t_i|^{q-1} \quad i = 1, \dots, m$$
for every $(x, t_1, \dots, t_m) \in \Omega \times \mathbb{R}^m$.

We consider the Sobolev space $H_0^1(\Omega)$ endowed with the norm
$$\|u\|_{H_0^1(\Omega)} := \Big(\int_\Omega |\nabla u(x)|^2 dx\Big)^{1/2}, \tag{2.2}$$
for all $u \in H_0^1(\Omega)$.

Now, let X be the Cartesian product of m Sobolev space $H_0^1(\Omega)$; i.e., $X = \prod_{i=1}^m H_0^1(\Omega)$ endowed with the norm
$$\|u\| := \sum_{i=1}^m \|u_i\|_{H_0^1(\Omega)}$$
for all $u = (u_1, \dots, u_m) \in X$.

A function $u = (u_1, \dots, u_m) \in X$ is said a weak solution to system (1.1) if
$$\int_\Omega \sum_{i=1}^m \nabla u_i(x) \cdot \nabla v_i(x) dx - \lambda \int_\Omega \sum_{i=1}^m F_{u_i}(x, u_1(x), \dots, u_m(x)) v_i(x) dx = 0$$
for every $v = (v_1, v_2, \dots, v_m) \in X$. Moreover, a weak solution $u \in X$ is called non negative if $u_i(x) \ge 0$ for every $i = 1, \dots, m$ and for each $x \in \Omega$.

Now, put $2^* = \frac{2N}{N-2}$ and denote by Γ the Gamma function defined by
$$\Gamma(s) = \int_0^{+\infty} z^{s-1} e^z dz, \ \forall s > 0.$$
From the Sobolev embedding theorem, for every $u \in H_0^1(\Omega)$ there exists a constant $c \in \mathbb{R}_+$ such that
$$\|u\|_{L^{2^*}(\Omega)} \le c\|u\|_{H_0^1(\Omega)} \tag{2.3}$$

the best (smallest) constant that appears in (2.3) is

$$c = \frac{1}{\sqrt{N(N-2)\pi}} \Big(\frac{N!}{2\Gamma(1+\frac{N}{2})} \Big)^{1/N} \tag{2.4}$$

(see [7]).

Fixing $q \in [1, 2^*[$ in virtue of Sobolev embedding theorem, for every $u \in H_0^1(\Omega)$, there exists a positive constant c_q such that

$$\|u\|_{L^q(\Omega)} \le c_q \|u\|_{H_0^1(\Omega)} \tag{2.5}$$

and, by the Rellich theorem the embedding is compact.

By using (2.4), we have

$$c_q \le \frac{\mu(\Omega)^{\frac{2^*-q}{2^* q}}}{\sqrt{N(N-2)\pi}} \Big(\frac{N!}{2\Gamma(1+\frac{N}{2})} \Big)^{1/N} \tag{2.6}$$

where $\mu(\Omega)$ denotes the Lebesgue measure of the set Ω. Moreover, let

$$D := \sup_{x \in \Omega} \operatorname{dist}(x, \partial\Omega). \tag{2.7}$$

Simple calculations show that there is $x_0 \in \Omega$ such that $B(x_0, D) \subseteq \Omega$.

Finally, we set

$$\kappa = \frac{D}{\sqrt{2}\pi^{\frac{N}{4}}} \Big(\frac{\Gamma(1+\frac{N}{2})}{D^N - (D/2)^N} \Big)^{1/2}, \tag{2.8}$$

and

$$K_1 = \frac{2\sqrt{2} m c_1 (2^N - 1)}{D^2} \quad K_2 = \frac{2^{\frac{q+2}{2}} m^q c_q^q (2^N - 1)}{q D^2}. \tag{2.9}$$

To study system (1.1), we will use the functionals $\Phi, \Psi : X \to \mathbb{R}$ defined by putting

$$\Phi(u) := \frac{1}{2} \sum_{i=1}^m \|u_i\|^2_{H_0^1(\Omega)}, \quad \Psi(u) := \int_\Omega F(x, u_1(x), \dots, u_m(x)) dx \tag{2.10}$$

for every $u = (u_1, u_2, \dots, u_m) \in X$.

Clearly, Φ is a coercive, continuously Gâteaux differentiable and weakly sequentially lower semicontinuous, whose Gâteaux derivative admits a continuous inverse on X^*. On the other hand Ψ is well defined, continuously Gâteaux differentiable with compact derivative. One has

$$\Phi'(u)(v) = \int_\Omega \sum_{i=1}^m \nabla u_i(x) \cdot \nabla v_i(x) dx$$

$$\Psi'(u)(v) = \int_\Omega \sum_{i=1}^m F_{u_i}(x, u_1(x), \dots, u_m(x)) v_i(x) dx,$$

for every $v = (v_1, v_2, \dots, v_m)$, $u = (u_1, u_2, \dots, u_m) \in X$.

A critical point for the functional $I_\lambda := \Phi - \lambda\Psi$ is any $u \in X$ such that

$$\Phi'(u)(v) - \lambda \Psi'(u)(v) = 0 \quad \forall v \in X,$$

Hence, the critical points for functional $I_\lambda := \Phi - \lambda\Psi$ are exactly the weak solutions to system (1.1).

3. Main results

In this Section, we present our main results. First, we establish the existence of one non-trivial solution.

Theorem 3.1. *We suppose that* (H0) *holds and assume that*

(J1) $F(x,t) \geq 0$ *for every* $(x,t) \in \Omega \times \mathbb{R}^m_+$ *where* $\mathbb{R}^m_+ = \{t = (t_1, \dots, t_m) \in \mathbb{R}^m : t_i \geq 0 \quad i = 1, \dots, m\}$;

(J2) *there exist a positive constant* γ *and a vector* $\delta \in \mathbb{R}^m_+$ *with* $|\delta| < \gamma\kappa$, *such that*
$$\frac{\inf_{x\in\Omega} F(x,\delta)}{|\delta|^2} > a_1\frac{K_1}{\gamma} + a_2K_2\gamma^{q-2},$$
where a_1, a_2, q *are given by* (H0) *and* κ, K_1, K_2 *are given by* (2.8) *and* (2.9).

Then, for each $\lambda \in \left]\frac{2(2^N-1)}{D^2}\frac{|\delta|^2}{\inf_{x\in\Omega} F(x,\delta)}, \frac{2(2^N-1)}{D^2}\frac{1}{a_1\frac{K_1}{\gamma}+a_2K_2\gamma^{q-2}}\right[$, *the system* (1.1) *has at least one non-zero weak solution.*

Proof. Our goal is to apply Theorem 2.1. Consider the Sobolev space X and the operators defined in (2.10). By using (H0) one has
$$|F(x,t_1,\dots,t_m)| \leq a_1\sum_{i=1}^m |t_i| + \frac{a_2}{q}\sum_{i=1}^m |t_i|^q, \tag{3.1}$$
for every $(x,t) \in \Omega \times \mathbb{R}^m$. Taking into account (3.1) it follows that
$$\Psi(u) = \int_\Omega F(x,u)dx \leq a_1\sum_{i=1}^m \|u_i\|_{L^1(\Omega)} + \frac{a_2}{q}\sum_{i=1}^m \|u_i\|^q_{L^q(\Omega)}. \tag{3.2}$$
Let $r \in]0,+\infty[$, then for every $u = (u_1,\dots,u_m) \in X$ such that $\Phi(u) < r$, by using (2.5) from (3.2) we obtain
$$\Psi(u) \leq a_1c_1m\sqrt{2r} + \frac{a_2}{q}m^qc_q^q2^{q/2}r^{q/2}. \tag{3.3}$$
Hence, from (3.3), the following relation holds
$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}{r} \leq \sqrt{\frac{2}{r}}mc_1a_1 + \frac{2^{q/2}m^qc_q^qa_2}{q}r^{\frac{q}{2}-1}, \tag{3.4}$$
for every $r > 0$. Now, we choose the function $\bar{u} = (\bar{u}_1,\dots,\bar{u}_m) \in X$ defined by
$$\bar{u}_i(x) = \begin{cases} 0 & \text{if } x \in \Omega\setminus B(x_0,D) \\ \frac{2\delta_i}{D}(D - \sqrt{\sum_{j=1}^N (x_j - x_{j0})^2} & \text{if } x \in B(x_0,D)\setminus B(x_0,\frac{D}{2}) \\ \delta_i & \text{if} x \in B(x_0,\frac{D}{2}) \end{cases} \tag{3.5}$$

for $i = 1, \dots, m$. Clearly $\bar{u} \in X$ and we have

$$
\begin{aligned}
\Phi(\bar{u}) &= \frac{1}{2} \sum_{i=1}^{m} \int_{\Omega} |\nabla u_i(x)|^2 dx \\
&= \frac{1}{2} \sum_{i=1}^{m} \int_{B(x_0,D)\setminus B(x_0,\frac{D}{2})} \frac{4\delta_i^2}{D^2} dx \\
&= \frac{2|\delta|^2}{D^2} (\mu(B(x_0,D)) - \mu(B(x_0,\frac{D}{2}))) \\
&= \frac{2|\delta|^2}{D^2} \frac{\pi^{\frac{N}{2}}}{\Gamma(1+\frac{N}{2})} (D^N - (D/2)^N).
\end{aligned}
\tag{3.6}
$$

Put $r = \gamma^2$, bearing in mind that $|\delta| < \gamma\kappa$, we obtain

$$
0 < \Phi(\bar{u}) < r
$$

and by using (J1) we have

$$
\Psi(\bar{u}) = \int_{\Omega} F(x, \bar{u}(x))dx \geq \int_{B(x_0,\frac{D}{2})} F(x,\delta)dx \geq \inf_{x\in\Omega} F(x,\delta) \frac{\pi^{\frac{N}{2}}}{\Gamma(1+\frac{N}{2})} \frac{D^N}{2^N}. \tag{3.7}
$$

Hence, by (3.6) and (3.7), one has

$$
\frac{\Psi(\bar{u})}{\Phi(\bar{u})} \geq \frac{D^2 \inf_{x\in\Omega} F(x,\delta)}{2(2^N-1)|\delta|^2}. \tag{3.8}
$$

By using (2.9), (3.4), (3.8) and taking into account (J2), we obtain

$$
\begin{aligned}
\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{r} &\leq \frac{\sqrt{2}}{\gamma} m c_1 a_1 + \frac{2^{q/2} m^q c_q^q a_2}{q} \gamma^{q-2} \\
&= \frac{D^2}{2(2^N-1)} \Big(a_1 \frac{K_1}{\gamma} + a_2 K_2 \gamma^{q-2} \Big) \\
&< \frac{D^2 \inf_{x\in\Omega} F(x,\delta)}{2(2^N-1)|\delta|^2} \leq \frac{\Psi(\bar{u})}{\Phi(\bar{u})}.
\end{aligned}
$$

Moreover, by using [2, Proposition 2.1], it is easy to prove that the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies $(PS)^{[r]}$-condition.

Therefore, all the assumptions of Theorem 2.1 are satisfied. So, for each $\lambda \in \Big] \frac{2(2^N-1)}{D^2} \frac{|\delta|^2}{\inf_{x\in\Omega} F(x,\delta)}, \frac{2(2^N-1)}{D^2} \frac{1}{a_1 \frac{K_1}{\gamma} + a_2 K_2 \gamma^{q-2}} \Big[\subseteq \Big] \frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \frac{\gamma^2}{\sup_{u\in\Phi^{-1}(]-\infty,\gamma^2[)} \Psi(u)} \Big[$, the functional I_λ has at least one non-zero critical point that is weak solution of system (1.1). $\square$

We now point out the case when F does not depend on $x \in \Omega$, we consider problem

$$
\begin{gathered}
-\Delta u = \lambda \nabla_u F(u) \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial\Omega
\end{gathered}
\tag{3.9}
$$

we have the following result.

Corollary 3.2. *Let $F : \mathbb{R}^m \to \mathbb{R}$ be a non-negative and C^1-function satisfying (H0) and assume that*

$$
\limsup_{|t|\to 0^+} \frac{F(t)}{|t|^2} = +\infty.
$$

Then, there is $\lambda^* > 0$ such that, for each $\lambda \in]0, \lambda^*[$, problem (3.9) admits at least one non-zero weak solution.

Proof. Taking into account condition (H0), fix

$$\lambda^* = \frac{1}{\sqrt{2}a_1 c_1 m + 2^{q/2}\frac{a_2}{q}c_q^q m^q}.$$

From

$$\limsup_{|t|\to 0^+} \frac{F(t)}{|t|^2} = +\infty$$

for all $\lambda \in]0, \lambda^*[$, there is a vector $\delta^* \in \mathbb{R}^m_+$ with $|\delta^*| < k$ such that

$$\frac{D^2}{2(2^N-1)}\frac{F(\delta^*)}{|\delta^*|^2} > \frac{1}{\lambda}$$

Put $\bar{u} \in X$ as in (3.5), and by choosing $\gamma = 1$ we obtain

$$\frac{F(\delta^*)}{|\delta^*|^2} > \frac{2(2^N-1)}{\lambda D^2} > \frac{2(2^N-1)}{\lambda^* D^2} = a_1 K_1 + a_2 K_2$$

All the assumptions of Theorem 3.1 are satisfied and the proof is complete. □

The following result, in which the global Ambrosetti-Rabinowitz condition is also used, ensures the existence at least two weak solutions.

Theorem 3.3. *We suppose that* (H0) *holds and $\nabla_u F(x, 0) \neq 0$ for every $x \in \Omega$. Assume that there are two positive constants $\mu > 2$ and R such that*

$$0 < \mu F(x, t) \leq t \cdot \nabla_t F(x, t) \tag{3.10}$$

for all $x \in \Omega$ and $|t| \geq R$. Then, there exists $\lambda^* > 0$ such that for each $\lambda \in]0, \lambda^*[$, problem (1.1) *has at least two non trivial weak solutions.*

Proof. Put

$$\lambda^* = \frac{1}{\sqrt{2}a_1 c_1 m + 2^{q/2}\frac{a_2}{q}c_q^q m^q},$$

and fix $\lambda < \lambda^*$. From (3.10), by standard computations, there is a positive constant C such that

$$F(x, t) \geq C|t|^\mu \tag{3.11}$$

for all $x \in \Omega$, $|t| > R$. In fact, setting $a(x) = \min_{|\xi|=R} F(x, \xi)$ and

$$\varphi_t(s) = F(x, st) \quad \forall s > 0, \tag{3.12}$$

by (3.10), for every $x \in \Omega$ and $|t| > R$ one has

$$0 < \mu\varphi_t(s) = \mu F(x, st) \leq st \cdot \nabla F(x, st) = s\varphi_t'(s) \quad \forall s > 0.$$

Therefore,

$$\int_{R/|t|}^1 \frac{\varphi_t'(s)}{\varphi_t(s)} ds \geq \int_{R/|t|}^1 \frac{\mu}{s} ds.$$

Then

$$\varphi_t(1) \geq \varphi_t\Big(\frac{R}{|t|}\Big)|t|^\mu.$$

Taking into account of (3.12), we obtain

$$F(x, t) \geq F\Big(x, \frac{R}{|t|}t\Big)|t|^\mu \geq a(x)|t|^\mu \geq C|t|^\mu$$

and (3.11) is proved. From (3.11) it follows that I_λ is unbounded from below.

Now, to verify the (PS)-condition it is sufficient to prove that any sequence of Palais-Smale is bounded. To this end, taking into account (3.10) one has

$$\begin{aligned}
&\mu I_\lambda(u_n) - \|I'_\lambda(u_n)\|_{X'}\|u_n\| \geq \mu I_\lambda(u_n) - I'_\lambda(u_n)(u_n)\\
&= \mu\Phi(u_n) - \lambda\mu\Psi(u_n) - \Phi'(u_n)(u_n) + \lambda\mu\Psi'(u_n)(u_n)\\
&= (\frac{\mu}{2}-1)\sum_{i=1}^m \|u_{in}\|^2 - \lambda\int_\Omega(\mu F(x,u_n(x)) - \sum_{i=1}^m F_{u_i}(x,u_1(x),\dots,u_m(x))u_i(x))\\
&\geq (\frac{\mu}{2}-1)\sum_{i=1}^m \|u_{in}\|^2 \geq \frac{1}{m}(\frac{\mu}{2}-1)\|u_n\|^2.
\end{aligned} \tag{3.13}$$

If $\{u_n\}$ is not bounded from (3.13) we have a contradiction. Moreover, from (3.4) by choosing $r=1$ one has

$$\sup_{u\in\Phi^{-1}(]-\infty,1[)}\Psi(u) \leq \sqrt{2}a_1c_1m + 2^{q/2}\frac{a_2}{q}c_q^q m^q = \frac{1}{\lambda^*}.$$

Hence, Theorem 2.2 ensures that problem (1.1), for each $\lambda \in]0,\lambda^*[$, admits at least two weak solutions. □

Now, we point out the following result of three weak solutions.

Theorem 3.4. *We suppose that* (H0) *holds and assume that*

- (H1) *$F(x,t) \geq 0$ for every $(x,t) \in \Omega\times\mathbb{R}^m_+$ where $\mathbb{R}^m_+ = \{t=(t_1,\dots,t_m)\in\mathbb{R}^m : t_i \geq 0 \quad i=1,\dots,m\}$;*
- (H2) *there exist two positive constants b and $s<2$ such that*
$$F(x,t) \leq b(1+\sum_{i=1}^m |t_i|^s)$$
for almost every $x\in\Omega$ and for every $t\in\mathbb{R}^m$;
- (H3) *there exist a positive constant γ and a vector $\delta\in\mathbb{R}^m_+$ such that $|\delta| > \gamma\kappa$, such that*
$$\frac{\inf_{x\in\Omega}F(x,\delta)}{|\delta|^2} > a_1\frac{K_1}{\gamma} + a_2K_2\gamma^{q-2},$$
where a_1, a_2, q are given by (H0) and κ, K_1, K_2 are given by (2.8) and (2.9).

Then, for each $\lambda \in \Big]\frac{2(2^N-1)}{D^2}\frac{\delta^2}{\inf_{x\in\Omega}F(x,\delta)}, \frac{2(2^N-1)}{D^2}\frac{1}{a_1\frac{K_1}{\gamma}+a_2K_2\gamma^{q-2}}\Big[$, system (1.1) *has at least three weak solutions.*

Proof. Our goal is to apply Theorem 2.3. Consider the Sobolev space X and the operators defined in (2.10) taking into account that the regularity assumptions on Φ and Ψ are satisfied, our aim is to verify (i) and (ii). Arguing as in the proof of Theorem 3.1, put $\bar{u}$ as in (3.5) and $r=\gamma^2$, bearing in mind that $|\delta|>\gamma\kappa$, we obtain

$$\Phi(\bar{u}) > r > 0.$$

Therefore, the assumption (i) of Theorem 2.3 is satisfied.

We prove that the functional $I_\lambda = \Phi - \lambda\Psi$ is coercive for all positive parameter, in fact by using condition (H2) we have

$$\begin{aligned}
I_\lambda(u) = \Phi(u) - \lambda\Psi(u) &\geq \frac{1}{2m}\|u\|^2 - \lambda\int_\Omega F(x,u(x))dx \\
&\geq \frac{1}{2m}\|u\|^2 - \lambda\int_\Omega b(1+\sum_{i=1}^m |u_i(x)|^s)dx \\
&\geq \frac{1}{2m}\|u\|^2 - \lambda b\mu(\Omega) - \lambda b c_2^s \mu(\Omega)^{\frac{2-s}{2}}\|u\|^s.
\end{aligned}$$

Then also condition (ii) holds, hence all the assumptions of Theorem 2.3 are satisfied. So, for each λ in $\big]\frac{2(2^N-1)}{D^2}\frac{|\delta|^2}{\inf_{x\in\Omega}F(x,\delta)}, \frac{2(2^N-1)}{D^2}\frac{1}{a_1\frac{K_1}{\gamma}+a_2K_2\gamma^{q-2}}\big[$, which is a subsect of $\big]\frac{\Phi(\bar u)}{\Psi(\bar u)}, \frac{\gamma^2}{\sup_{u\in\Phi^{-1}(]-\infty,\gamma^2[)}\Psi(u)}\big[$, the functional I_λ has at least three distinct critical points that are weak solutions of system (1.1). □

An immediate consequence of Theorem 3.4 is the following result.

Corollary 3.5. *We suppose that* (H0) *holds and assume that*

- (H1') $F(t) \geq 0$ *for every* $t \in \mathbb{R}^m_+$ *where* $\mathbb{R}^m_+ = \{t = (t_1,\dots,t_m) \in \mathbb{R}^m : t_i \geq 0 \quad i = 1,\dots,m\}$*;*
- (H2') *there exist two positive constants* b *and* $s < 2$ *such that*
$$F(t) \leq b(1+\sum_{i=1}^m |t_i|^s)$$
for every $t \in \mathbb{R}^m$*;*
- (H3') *there exist a positive constant* γ *and a vector* $\delta \in \mathbb{R}^m_+$ *with* $|\delta| > \gamma\kappa$*, such that*
$$\frac{F(\delta)}{|\delta|^2} > a_1\frac{K_1}{\gamma} + a_2K_2\gamma^{q-2},$$
where a_1*,* a_2 *are given by (H1) and* κ*,* K_1*,* K_2 *are given by* (2.8) *and* (2.9)*.*

Then, for each $\lambda \in \big]\frac{2(2^N-1)}{D^2}\frac{\delta^2}{F(\delta)}, \frac{2(2^N-1)}{D^2}\frac{1}{a_1\frac{K_1}{\gamma}+a_2K_2\gamma^{q-2}}\big[$*, system* (3.9) *has at least three weak solutions.*

Remark 3.6. If we assume that $F_{u_i} : \Omega\times\mathbb{R}^m \to \mathbb{R}$ $(i = 1,\dots,m)$ are non negative, continuous functions then the previous theorems guarantee the existence of non negative weak solutions. In fact, let $\bar u = (\bar{u_1},\dots,\bar{u_m})$ be a weak solution of system (1.1). Fixed i, we consider the problem

$$\begin{gathered}
-\Delta u_i = \lambda F_{u_i}(x,\bar{u_1},\dots,u_i,\dots,\bar{u_m}) \quad \text{in } \Omega, \\
u_i\big|_{\partial\Omega} = 0 \quad i = 1,\dots,m.
\end{gathered} \tag{3.14}$$

Clearly, one has $\bar{u_i} \in H^1_0(\Omega)$ and it is a weak solution of (3.14). Hence, the Strong Maximum Principle ensures that either $\bar{u_i}(x) = 0$ or $\bar{u_i}(x) > 0$ on Ω.

Now, we present some examples that illustrate our results.

Example 3.7. Let Ω be an open ball of radius one in $\mathbb{R}^3$. Consider the function $F : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$F(t_1,t_2) = |t_1|^{3/2} + |t_2|^{3/2}$$

for every $(t_1, t_2) \in \mathbb{R}^2$. We observe that

$$F_{t_1}(t_1, t_2) = \frac{3}{2}|t_1|^{\frac{3}{2}-2}t_1,$$
$$F_{t_2}(t_1, t_2) = \frac{3}{2}|t_2|^{\frac{3}{2}-2}t_2\,.$$

Then, choosing $q = 3/2$, $a_1 = 0$ and $a_2 = 3/2$ the condition (H0) holds. Then by using Corollary 3.2, put

$$\lambda^* = \frac{3^{3/2}\pi^{1/4}}{2^{19/4}}$$

for all $\lambda \in]0, \lambda^*[$, the system

$$\begin{gathered} -\Delta u = \lambda F_u(u, v) \quad \text{in } \Omega, \\ -\Delta v = \lambda F_v(u, v) \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{3.15}$$

admits at least one non-zero weak solution in $X = H_0^1(\Omega) \times H_0^1(\Omega)$.

Example 3.8. Let Ω be an open ball of radius one in $\mathbb{R}^3$. Consider the function $F : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$F(x, t_1, t_2) = \frac{1}{6}t_1 + \frac{1}{6}t_2 + \frac{1}{4}\big(|t_1|^4 + |t_2|^4\big)$$

for every $x \in \Omega$ and for every $(t_1, t_2) \in \mathbb{R}^2$. We observe that

$$F_{t_1}(x, t_1, t_2) = \frac{1}{6} + |t_1|^2 t_1,$$
$$F_{t_2}(x, t_1, t_2) = \frac{1}{6} + |t_2|^2 t_2,$$

therefore, $\nabla_u F(x, 0) \neq 0$ for every $x \in \Omega$, choosing $q = 4$, $a_1 = 1/6$ and $a_2 = 1$ the condition (H0) holds. Moreover, choose $\mu = 3$ we have

$$0 < 3F(x, t_1, t_2) \leq t_1 F_{t_1}(x, t_1, t_2) + t_2 F_{t_2}(x, t_1, t_2)$$

for every $x \in \Omega$ and for every $t \in \mathbb{R}^2$. Then, by using Theorem 3.3, put

$$\lambda^* = \frac{\pi^{7/12}3^{7/3}}{2^{17/6}(\pi^{3/4} + 2^2 3^{7/4})}$$

for all $\lambda \in]0, \lambda^*[$ the system

$$\begin{gathered} -\Delta u = \lambda F_u(x, u, v) \quad \text{in } \Omega, \\ -\Delta v = \lambda F_v(x, u, v) \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{3.16}$$

admits at least two non-zero weak solutions in $X = H_0^1(\Omega) \times H_0^1(\Omega)$.

Example 3.9. Let Ω be an open ball of radius one in $\mathbb{R}^3$. Set $q = 5 \in]2, 6[$, $s = 3/2 < 2$, choose $a_1 = 1$, $a_2 = 10/3$ and

$$r = 9 > \Big(\frac{K_1 + a_2 K_2}{5}\Big)^{1/3}$$

where K_1 and K_2 are given by (2.9). Consider the function $F:\mathbb{R}^2\to\mathbb{R}$ defined by

$$F(t_1,t_2)=\begin{cases} t_1+t_2+\frac{1}{5}(t_1^5+t_2^5) & \text{if } t_1\le 9,\ t_2\le 9\\ t_1+t_2-\frac{7}{5}3^9+\frac{1}{5}t_1^5+2\cdot 3^6 t_2^{3/2} & \text{if } t_1\le 9,\ t_2>9\\ t_1+t_2-\frac{7}{5}3^9+\frac{1}{5}t_2^5+2\cdot 3^6 t_1^{3/2} & \text{if } t_1>9,\ t_2\le 9\\ t_1+t_2-\frac{14}{5}3^9+2\cdot 3^6(t_1^{3/2}+t_2^{3/2}) & \text{if } t_1>9,\ t_2>9. \end{cases}$$

Clearly (H0) holds. Moreover, for each $(t_1,t_2)\in\mathbb{R}^2$, one has

$$F(t_1,t_2)\le 2(9+2\cdot 3^6)(1+|t_1|^{3/2}+|t_2|^{3/2}),$$

therefore, if we choose $\gamma=1$, $b=2(9+2\cdot 3^6)$ and $\delta=(9,9)$ the hypotheses of Corollary 3.5 are satisfied. Then, for each $\lambda\in\big]\frac{630}{6566},\ \frac{\pi^{\frac{19}{6}}\cdot 3^{\frac{10}{3}}}{(3^2\cdot\pi^{\frac{10}{3}}+2^{\frac{25}{6}})2^{\frac{23}{6}}}\big[$, the system

$$\begin{gathered} -\Delta u=\lambda F_u(u,v)\quad\text{in }\Omega,\\ -\Delta v=\lambda F_v(u,v)\quad\text{in }\Omega,\\ u=v=0\quad\text{on }\partial\Omega \end{gathered}\tag{3.17}$$

admits at least three non negative weak solutions in $X=H_0^1(\Omega)\times H_0^1(\Omega)$.

References

[1] A. Ambrosetti, P. H. Rabinowitz; *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349–381.
[2] G. Bonanno; *A critical point theorem via Ekeland variational principle*, Nonlinear Analysis **75** (2012), 2992–3007.
[3] G. Bonanno; *Relations between the mountain pass theorem and local minima*, Adv. Nonlinear Anal. **1** (2012), 205–220.
[4] G. Bonanno, P. Candito; *Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities*, J. Differential Equations **244** (2008), 3031–3059.
[5] G. Bonanno, S. A. Marano; *On the structure of the critical set of non-differentiable functions with a weak compactness condition*, Applicable Analysis **89** (2010), 1–10.
[6] D. G. Defigueiredo; *Nonlinear Elliptic Systems*, An. Acad. Bras. Ci. **72** (2000) 453–469.
[7] G. Talenti; *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. **110** (1976) 353–372.
[8] C. Li, C.-L. Tang; *Three solutions for a class of quasilinear elliptic systems involving the (p,q)−Laplacian*, Nonlinear Anal. **69** (2008) 3322–3329.
[9] E. Zeidler; Nonlinear functional analysis and its applications, vol. III, Springer, Berlin, 1990.

Gabriele Bonanno
Department of Civil, Information Technology, Construction, Environmental Engineering and Applied Mathematics, University of Messina, 98166 - Messina, Italy
E-mail address: bonanno@unime.it

Elisabetta Tornatore
Dipartimento di Ingegneria Elettrica, Elettronica e delle Telecomunicazioni di Tecnologie Chimiche, Automatica e Modelli Matematici (DIEETCAM), Università degli studi di Palermo, Viale delle Scienze, 90128 Palermo, Italy
E-mail address: elisa.tornatore@unipa.it