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EXISTENCE AND UPPER SEMI-CONTINUITY OF UNIFORM ATTRACTORS FOR NON-AUTONOMOUS REACTION DIFFUSION EQUATIONS ON $\mathbb{R}^N$

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Abstract. We prove the existence of uniform attractors for the non-autonomous reaction diffusion equation

$$u_t - \Delta u + f(x,u) + \lambda u = g(t,x)$$

on $\mathbb{R}^N$, where the external force g is translation bounded and the nonlinearity f satisfies a polynomial growth condition. Also, we prove the upper semi-continuity of uniform attractors with respect to the nonlinearity.

1. Introduction

In this article, we study the following non-autonomous reaction diffusion equation

$$\begin{gathered} u_t - \Delta u + f(x,u) + \lambda u = g(t,x), \quad x \in \mathbb{R}^N, \\ u|_{t=\tau} = u_\tau, \end{gathered} \tag{1.1}$$

where $\lambda > 0$, the nonlinearity f and the external force g satisfy some specified conditions later.

Non-autonomous equation are of great importance and interest as they appear in many applications in natural sciences. One way to treat non-autonomous equations is that considering its uniform attractors, which are extended from global attractors for autonomous case. In the recent years, the existence of uniform attractors for non-autonomous reaction diffusion equations or its generalized forms is studied extensively by many authors (see e.g. [1, 2, 5, 7] for the case of bounded domains, and [10] for the case of unbounded domains). However, uniform attractors for (1.1) in the case of unbounded domains is not well understood. In this paper, we prove the existence and the upper semicontinuity of uniform attractors for (1.1) in unbounded domains with a large class of external force g.

To study problem (1.1), we assume the following hypotheses:

(H1) The nonlinearity f satisfies: there exists $p \geq 2$ such that

$$f(x,u)u \geq \alpha_1 |u|^p - \phi_1(x), \tag{1.2}$$

$$|f(x,u)| \leq \alpha_2 |u|^{p-1} + \phi_2(x), \tag{1.3}$$

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$$f'_u(x,u) \geq -\ell, \tag{1.4}$$

where $\phi_1 \in L^1(\mathbb{R}^N) \cap L^{p/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $\phi_2 \in L^q(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ with $\frac{1}{p} + \frac{1}{q} = 1$ and α_1, α_2, ℓ are positive constant. For the primitive $F(x,u) = \int_0^u f(x,\xi)d\xi$, we assume that there are positive constants α_3, α_4 and $\phi_3, \phi_4 \in L^1(\mathbb{R}^N)$ satisfy

$$\alpha_3|u|^p - \phi_3(x) \leq F(x,u) \leq \alpha_4|u|^p + \phi_4(x). \tag{1.5}$$

(H2) The external force $g \in L^2_{\rm loc}\big(\mathbb{R}; L^2(\mathbb{R}^N)\big)$ satisfies

$$\sup_{t\in\mathbb{R}} \int_t^{t+1} \Big(\|g(s)\|^2_{L^2(\mathbb{R}^N)} + \|\partial_t g(s)\|^2_{L^2(\mathbb{R}^N)}\Big)\, ds < +\infty. \tag{1.6}$$

We borrow from [10, Lemma 3.4] the following result:

$$\limsup_{k\to+\infty} \Big(\sup_{t\in\mathbb{R}^N} \int_t^{t+1} \int_{|x|\geq k} |g(s,x)|^2\, dx\, ds\Big) = 0. \tag{1.7}$$

Since $\mathbb{R}^N$ is unbounded, the embedding $H^1(\mathbb{R}^N) \subset L^2(\mathbb{R}^N)$ is no longer compact, that causes the main difficulty. By using "tail estimate" technique (see, e.g., [8, 9]), we overcome this difficulty and thus prove the existence of a uniform attractor $L^2(\mathbb{R}^N)$. For attractors in $L^p(\mathbb{R}^N)$, we use some *a priori* estimates (see, e.g., [7, 10]) to prove the uniform asymptotic compactness of the family of processes. Finally, the existence of a uniform attractor in $H^1(\mathbb{R}^N)$ is obtained by combining "tail estimate" method and useful estimates of nonlinearity. The first main theorem is as follows.

Theorem 1.1. *Suppose that f and g satisfy hypothesis* (H1)–(H2). *Moreover, we assume that g is normal (see Definition 3.7) and f satisfies*

$$|\frac{\partial f}{\partial x}(x,s)| \leq \psi_5(x), \quad \forall x \in \mathbb{R}^N, \forall s \in \mathbb{R}, \tag{1.8}$$

where $\psi_5 \in L^2(\mathbb{R}^N)$. Then, the family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$ has a unique uniform attractor in $H^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$.

Remark 1.2. To prove the existence of a uniform attractor in $L^2(\mathbb{R}^N)$ we only need f and g to satisfy (H1)-(H2). The addition conditions: g is normal is needed to obtain the uniform attractor in $L^p(\mathbb{R}^N)$; and (1.8) of f is to prove the asymptotic compactness of family of processes in $H^1(\mathbb{R}^N)$.

Remark 1.3. In the case external force g is bounded uniformly in $t \in \mathbb{R}$; that is,

$$\|g(t,\cdot)\|_{L^2(\mathbb{R}^N)} \leq M, \quad \forall t \in \mathbb{R},$$

where M is independent of t, we can use arguments similar to [1, 2] to obtain the existence of a uniform attractor in $H^1(\mathbb{R}^N)$ easily. In this paper, since g only belongs to $L^2_b(\mathbb{R}; L^2(\mathbb{R}^N))$ (see Definition 2.6), the required computations are more complicated and involved.

Remark 1.4. The positivity of λ is used for the dissipativity of the solution; that is, the solution of the equation should be bounded uniformly in all time $t > 0$ (See Proposition 3.3).

If we replace $\mathbb{R}^n$ by a domain Ω (bounded or unbounded) that satisfies Poincare's inequality

$$\int_\Omega |\nabla u|^2 dx \geq C \int_\Omega |u|^2 dx,$$

then we can let $\lambda = 0$ (or even $\lambda > -C$), and Proposition 3.3 still follows the same way.

If $\lambda < 0$ in general, solutions of (1.1) can be unbounded when $t \to +\infty$ even in bounded domains. For example, consider the one dimensional equation

$$\begin{gathered} u_t - u_{xx} + u - (2\pi^2 + 1)u = 0, \quad x \in (0,1), t > 0, \\ u(t,0) = u(t,1) = 0, \quad t > 0, \\ u(0,x) = \sin(\pi x), x \in (0,1). \end{gathered} \tag{1.9}$$

Here we have $f(u) \equiv u, g(t,x) \equiv 0$ and $\lambda = -(2\pi^2+1) < 0$. It is easy to check that $u(t,x) = e^{\pi^2 t}\sin(\pi x)$ is a solution to (1.9) and

$$\|u(t,\cdot)\|^2_{L^2(0,1)} = \int_0^1 e^{2\pi^2 t}|\sin(\pi x)|^2 dx \to +\infty \quad \text{as } t \to +\infty.$$

Another interesting feature of this paper is that we prove the upper semi-continuity of uniform attractors with respect to the nonlinearity. Uniform attractors are not invariant under the family of processes, this brings some difficulties in proving upper semi-continuous property. In this work, in order to prove this kind of continuity, we use the structure of uniform attractors, which says that each uniform attractor is a union of kernels (see Definition 2.4 and Theorem 2.5).

We consider a family of functions $f_\gamma, \gamma \in \Gamma$, such that for each $\gamma \in \Gamma$, f_γ satisfies (1.2)-(1.5) and (1.8) where the constants are independent of γ. The topology $\mathcal{T}$ in Γ can be defined as follows:

If $\gamma_m \to \gamma$ in $\mathcal{T}$ then $f_{\gamma_m}(x,s) \to f_\gamma(x,s)$ for all $x \in \mathbb{R}^N$ and $s \in \mathbb{R}$.

Let $\{U^\gamma_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$ be the family of processes corresponding to the problem

$$\begin{gathered} u_t - \Delta u + f_\gamma(x,u) + \lambda u = g(t,x), \quad x \in \mathbb{R}^N, t > \tau, \\ u(\tau) = u_\tau, \quad x \in \mathbb{R}^N. \end{gathered} \tag{1.10}$$

By Theorem 1.1, for each $\gamma \in \Gamma$, $\{U^\gamma_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$ has a compact uniform attractor $\mathcal{A}_\gamma$ in $H^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$. We have the second main result.

Theorem 1.5. *The family of uniform attractors $\{\mathcal{A}_\gamma\}_{\gamma\in\Gamma}$ is upper semi-continuous in $L^2(\mathbb{R}^N)$ with respect to the nonlinearity, that is,*

$$\lim_{\gamma_n\to\gamma} \operatorname{dist}_{L^2(\mathbb{R}^N)}(\mathcal{A}_{\gamma_n}, \mathcal{A}_\gamma) = 0,$$

whenever $\gamma_n \to \gamma$ in $\mathcal{T}$.

The rest of this article is organized as follows: In section 2, for convenience to readers, we recall some basic concepts related to uniform attractors and translation bounded functions. The proof of Theorems 1.1 and 1.5 is showed in Sections 3 and 4, respectively.

Throughout this article, we will denote by $\|\cdot\|$ and $(\cdot,\cdot)$ the norm and the inner product in $L^2(\mathbb{R}^N)$, respectively. For a Banach space X, $\|\cdot\|_X$ stands for its norm. The letter C denotes an arbitrary constant, which can be different from line to line and even in a same line.

2. Preliminaries

2.1. **Uniform attractors.** Let Σ be a parameter set, X, Y be two Banach spaces. $\{U_\sigma(t,\tau), t \geq \tau, \tau \in \mathbb{R}\}$, $\sigma \in \Sigma$, is said to be a family of processes in X, if for each $\sigma \in \Sigma, \{U_\sigma(t,\tau)\}$ is a process; that is, the two-parameter family of mappings $\{U_\sigma(t,\tau)\}$ from X to X satisfies

$$U_\sigma(t,s)U_\sigma(s,\tau) = U_\sigma(t,\tau), \quad \forall t \geq s \geq \tau,\ \tau \in \mathbb{R},$$
$$U_\sigma(\tau,\tau) = Id, \quad \tau \in \mathbb{R},$$

where Id is the identity operator, $\sigma \in \Sigma$ is the symbol, and Σ is called the symbol space. Denote by $\mathcal{B}(X), \mathcal{B}(Y)$ the set of all bounded subsets of X and Y respectively.

Definition 2.1. A set $B_0 \in \mathcal{B}(Y)$ is said to be a uniform absorbing set in Y for $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$, if for any $\tau \in \mathbb{R}$ and any $B \in \mathcal{B}(X)$, there exists $T_0 \geq \tau$ such that $\cup_{\sigma\in\Sigma} U_\sigma(t,\tau)B \subset B_0$ for all $t \geq T_0$.

Definition 2.2. A family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ is called uniform asymptotically compact in Y if for any $\tau \in \mathbb{R}$ and any $B \in \mathcal{B}(X)$, we have $\{U_{\sigma_n}(t_n,\tau)x_n\}$ is relatively compact in Y, where $\{x_n\} \subset B$, $\{t_n\} \subset [\tau, +\infty), t_n \to +\infty$ and $\{\sigma_n\} \subset \Sigma$ are arbitrary.

Definition 2.3. A subset $\mathcal{A}_\Sigma \subset Y$ is said to be the uniform attractor in Y of the family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ if

(i) $\mathcal{A}_\Sigma$ is compact in Y;
(ii) for an arbitrary fixed $\tau \in \mathbb{R}$ and $B \in \mathcal{B}(X)$ we have
$$\lim_{t\to\infty} (\sup_{\sigma\in\Sigma} (\operatorname{dist}_Y(U_\sigma(t,\tau)B, \mathcal{A}_\Sigma)) = 0,$$
where $\operatorname{dist}_Y(A,B) = \sup_{x\in A} \inf_{y\in B} \|x-y\|_Y$ for $A, B \subset Y$; and
(iii) if $\mathcal{A}'_\Sigma$ is a closed subset of Y satisfying (i), then $\mathcal{A}_\Sigma \subset \mathcal{A}'_\Sigma$.

Definition 2.4. The kernel $\mathcal{K}$ of a process $\{U(t,\tau)\}$ acting on X consists of all bounded complete trajectories of the process $\{U(t,\tau)\}$:

$$\mathcal{K} = \{u(\cdot) | U(t,\tau)u(\tau) = u(t), \operatorname{dist}(u(t), u(0)) \leq C_u, \forall t \geq \tau, \tau \in \mathbb{R}\}.$$

The set $\mathcal{K}(s) = \{u(s) | u(\cdot) \in \mathcal{K}\}$ is said to be kernel section at time $t = s, s \in \mathbb{R}$.

We have the following result about the existence and structure of uniform attractors.

Theorem 2.5 ([2]). *Assume that the family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ satisfies the following conditions:*

(i) *Σ is weakly compact, and $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ is $(X \times \Sigma, Y)$-weakly continuous, that is, for any fixed $t \geq \tau$, the mapping $(u,\sigma) \mapsto U_\sigma(t,\tau)u$ is weakly continuous in Y. Moreover, there is a weakly continuous semigroup $\{T(h)\}_{h\geq 0}$ acting on Σ satisfying*
$$T(h)\Sigma = \Sigma, U_\sigma(t+h, \tau+h) = U_{T(h)\sigma}(t,\tau), \quad \forall \sigma \in \Sigma, t \geq \tau, h \geq 0;$$
(ii) *$\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ has a uniform absorbing set B_0 in Y;*
(iii) *$\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ is uniform asymptotically compact in Y.*

Then it possesses a uniform attractor $\mathcal{A}_\Sigma$ in Y, and

$$\mathcal{A}_\Sigma = \cup_{\sigma\in\Sigma} \mathcal{K}_\sigma(s), \quad \forall s \in \mathbb{R},$$

where $\mathcal{K}_\sigma(s)$ is the section at time s of the process $\{U_\sigma(t,\tau)\}$.

2.2. **The translation bounded functions.**

Definition 2.6. Let $\mathcal{E}$ be a reflexive Banach space. A function $\varphi \in L^2_{loc}(\mathbb{R};\mathcal{E})$ is said to be translation bounded if

$$\|\varphi\|^2_{L^2_b} = \|\varphi\|^2_{L^2_b(\mathbb{R};\mathcal{E})} = \sup_{t\in\mathbb{R}} \int_t^{t+1} \|\varphi\|^2_{\mathcal{E}} ds < \infty.$$

Let $g \in L^2_b\left(\mathbb{R}, L^2(\mathbb{R}^N)\right)$, we denote by $\mathcal{H}_w(g)$ be the closure of the set $\{g(\cdot + h) : h \in \mathbb{R}\}$ in $L^2_b(\mathbb{R}; L^2(\mathbb{R}^N))$ with the weak topology. The following results are proved in [3].

Lemma 2.7 ([3, Proposition 4.2])**.** (1) *For all $\sigma \in \mathcal{H}_w(g)$, $\|\sigma\|^2_{L^2_b} \le \|g\|^2_{L^2_b}$;*
(2) *The translation group $\{T(h)\}$ is weakly continuous on $\mathcal{H}_w(g)$;*
(3) *$T(h)\mathcal{H}_w(g) = \mathcal{H}_w(g)$ for $h \ge 0$;*
(4) *$\mathcal{H}_w(g)$ is weakly compact.*

3. Existence of uniform attractors

In this section, we prove the existence of uniform attractors for the family of processes corresponding to problem (1.1). First, we state without proofs the results about the existence of a unique weak solution of (1.1) and then prove there exists a uniform absorbing set for $\{U_\sigma(t,\tau)u_\tau\}_{\sigma\in\mathcal{H}_w(g)}$. Next, by a technique so called "tail estimate" we obtain a uniform attractor in $L^2(\mathbb{R}^N)$. Then, using abstract result in [10], we prove the existence of a uniform attractor in $L^p(\mathbb{R}^N)$. Finally, the existence of the uniform attractor in $H^1(\mathbb{R}^N)$ is obtained by combining "tail estimate" and arguments in [5].

3.1. **Existence of uniform absorbing set.**

Definition 3.1. A function $u(t,x)$ is called a weak solution of (1.1) on (τ,T), $T > \tau$, if

$$\begin{gathered} u \in C\left([\tau,T]; L^2(\mathbb{R}^N)\right) \cap L^p\left(\tau,T; L^p(\mathbb{R}^N)\right) \cap L^2(\tau,T; H^1(\mathbb{R}^N)), \\ u_t \in L^2(\tau,T; L^2(\mathbb{R}^N)), \\ u(\tau,x) = u_\tau(x) \text{a.e. on } \mathbb{R}^N, \end{gathered}$$

and for any $v \in C^\infty([\tau,T]\times\mathbb{R}^N)$,

$$\int_\tau^T \int_{\mathbb{R}^N} (u_t v + \nabla u \nabla v + f(x,u)v + \lambda uv) = \int_\tau^T \int_{\mathbb{R}^N} gv.$$

By the standard Galerkin-Feado approximation, we can find the existence of unique weak solution for problem (1.1) in the case of bounded domains. To overcome the difficulties of unboundedness of the domains, following [6], one may take the domain to be a sequence of balls with radius approaching ∞ to deduce the existence of a weak solution to (1.1) in $\mathbb{R}^N$. Here we state results only, for the details of the proof, readers are referred to [6].

Theorem 3.2. *Assume that f and g satisfy* (H1)–(H2)*. For any $u_\tau \in L^2(\mathbb{R}^N)$ and any $T > \tau$, there exists a unique weak solution u for problem* (1.1)*, and*

$$u \in C\left([\tau,T]; L^2(\mathbb{R}^N)\right); \quad u_t \in L^2\left(\tau,T; L^2(\mathbb{R}^N)\right).$$

From Theorem 3.2, we can define a family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$ associated with (1.1) acting on $L^2(\mathbb{R}^N)$, where $U_\sigma(t,\tau)u_\tau$ is the solution of (1.1) at time t subject to initial condition $u(\tau)=u_\tau$ at time τ and with σ in place of g.

Proposition 3.3. *There exists a uniform absorbing set $\mathcal{B}$ in $H^1(\mathbb{R}^N)\cap L^p(\mathbb{R}^N)$ for the family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$ corresponding to* (1.1).

Proof. Consider the equation

$$u_t-\Delta u+f(x,u)+\lambda u=\sigma(t,x). \tag{3.1}$$

Taking the inner product of (3.1) with $2u$ in $L^2(\mathbb{R}^N)$, we have

$$\frac{d}{dt}\|u\|^2+2\|\nabla u\|^2+2(f(x,u),u)+2\lambda\|u\|^2=2(\sigma(t),u). \tag{3.2}$$

Using (1.2), applying the Cauchy and Young's inequalities,

$$\frac{d}{dt}\|u\|^2+\frac{3\lambda}{2}\|u\|^2+2\|\nabla u\|^2+2\alpha_1\|u\|^p_{L^p(\mathbb{R}^N)}\le\frac{2}{\lambda}\|\sigma(t)\|^2+2\|\phi_1\|_{L^1(\mathbb{R}^N)}. \tag{3.3}$$

By Gronwall's lemma, we find

$$\|u(t)\|^2\le e^{-\lambda(t-\tau)}\|u_\tau\|^2+\frac{2\|\phi_1\|_{L^1(\mathbb{R}^N)}}{\lambda}+\frac{2}{\lambda}\int_\tau^t e^{-\lambda(t-s)}\|\sigma(s)\|^2ds. \tag{3.4}$$

For the last term of the right hand side,

$$\begin{aligned}\int_\tau^t e^{-\lambda(t-s)}\|\sigma(s)\|^2ds&\le\Big(\int_{t-1}^t+\int_{t-2}^{t-1}+\int_{t-3}^{t-2}+\dots\Big)e^{-\lambda(t-s)}\|\sigma(s)\|^2ds\\&\le\int_{t-1}^t\|\sigma(s)\|^2ds+e^{-\lambda}\int_{t-2}^{t-1}\|\sigma(s)\|^2+\dots\\&\le\big(1+e^{-\lambda}+e^{-2\lambda}+\dots\big)\|\sigma\|^2_{L^2_b}\\&\le\frac{1}{1-e^{-\lambda}}\|g\|^2_{L^2_b}.\end{aligned} \tag{3.5}$$

Combining (3.4)-(3.5), and noting that u_τ belongs to a bounded set B, there exists a $T_0>0$ satisfies

$$\|u(t)\|^2\le\rho_0=1+\frac{2\|\phi_1\|_{L^1(\mathbb{R}^N)}}{\lambda}+\frac{2e^\lambda}{\lambda(e^\lambda-1)}\|g\|^2_{L^2_b}, \tag{3.6}$$

for all $t>T_0$, all $u_\tau\in B$ and all $\sigma\in\mathcal{H}_w(g)$. By integrating (3.3), we find that

$$\begin{aligned}&\int_t^{t+1}\Big(\frac{\lambda}{2}\|u(s)\|^2+2\|\nabla u(s)\|^2+2\alpha_1\|u(s)\|^p_{L^p(\mathbb{R}^N)}\Big)ds\\&\le\|u(t)\|^2+\frac{2}{\lambda}\int_t^{t+1}\|\sigma(s)\|^2ds+\frac{2\|\phi_1\|_{L^1(\mathbb{R}^N)}}{\lambda}\\&\le\rho_0+\frac{2}{\lambda}\|g\|^2_{L^2_b}+\frac{2\|\phi_1\|_{L^1(\mathbb{R}^N)}}{\lambda},\end{aligned} \tag{3.7}$$

for all $t\ge T_0$. From (1.5),

$$\|u\|^p_{L^p(\mathbb{R}^N)}\ge\frac{1}{\alpha_4}\Big(\int_{\mathbb{R}^N}F(x,u)dx-\|\phi_4\|_{L^1(\mathbb{R}^N)}\Big),$$

and (3.7), it leads to

$$\int_t^{t+1}\Big(\lambda\|u(s)\|^2+\|\nabla u(s)\|^2+2\int_{\mathbb{R}^N}F(x,u)dx\Big)ds\le C,\quad\text{for all } t\ge T_0. \tag{3.8}$$

On the other hand, by multiplying (3.1) by $2u_t$ then integrating over $\mathbb{R}^N$, after using Cauchy's inequality,

$$\|u_t\|^2 + \frac{d}{dt}\Big(\lambda\|u\|^2 + \|\nabla u\|^2 + 2\int_{\mathbb{R}^N} F(x,u)dx\Big) \le \|\sigma(t)\|^2. \tag{3.9}$$

From (3.8)-(3.9) and the uniform Gronwall inequality, we obtain

$$\lambda\|u\|^2 + \|\nabla u\|^2 + 2\int_{\mathbb{R}^N} F(x,u)dx \le C, \text{ for all } t \ge T_0. \tag{3.10}$$

Using (1.5) again, there exists $\rho_1 > 0$ such that, for all $t \ge T_0$,

$$\|u(t)\|^2 + \|\nabla u(t)\|^2 + \|u(t)\|^p_{L^p(\mathbb{R}^N)} \le \rho_1, \quad \forall u_\tau \in B, \forall \sigma \in \mathcal{H}_w(g). \tag{3.11}$$

This completes the proof. □

Lemma 3.4. *The family of processes associated with problem* (1.1) *is* $(L^2(\mathbb{R}^N) \times \mathcal{H}_w(g), H^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N))$ *weakly continuous, that is, for any* $x_n \rightharpoonup x_0$ *in* $L^2(\mathbb{R}^N)$ *and* $\sigma_n \rightharpoonup \sigma$ *in* $\mathcal{H}_w(g)$*, we have*

$$U_{\sigma_n}(t,\tau)x_n \rightharpoonup U_\sigma(t,\tau)x \quad \text{in } H^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N), \tag{3.12}$$

for all $t > \tau$.

Proof. Denote by $u_n(t) = U_{\sigma_n}(t,\tau)x_n$, then u_n solves

$$\partial_t u_n - \Delta u_n + f(x,u_n) + \lambda u_n = \sigma_n(t), \tag{3.13}$$

with initial condition $u_n(\tau) = x_n$. Using arguments in Proposition 3.3, we can deduce that there exists a function $w(t)$ such that

$$u_n \rightharpoonup w \text{ weak-star in } L^\infty(\tau,t;L^2(\mathbb{R}^N)), \tag{3.14}$$

$$u_n \rightharpoonup w \text{ in } L^p(\tau,t;L^p(\mathbb{R}^N)), \tag{3.15}$$

and the sequence

$$\{u_n(s)\}, \tau \le s \le t, \text{ is bounded in } H^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N). \tag{3.16}$$

By (1.3) and (3.15),

$$\{f(x,u_n)\} \text{ is bounded } L^q(\tau,t;L^q(\mathbb{R}^N)),$$

thus, by equation (3.13),

$$\{\partial_t u_n\} \text{ is bounded in } L^q(\tau,t;L^q(\mathbb{R}^N)) \cap L^2(\tau,t;H^{-1}(\mathbb{R}^N)).$$

Therefore, one can pass to the limit (in the weak sense) of equation (3.13) to have

$$w_t - \Delta w + f(x,w) + \lambda w = \sigma(t), \tag{3.17}$$

with $w(\tau) = x$. In fact, there are some difficulties to overcome when one wants to show $f(x,u_n) \rightharpoonup f(x,w)$, but it can be solved by taking the domain to be a sequence of balls with radius approaching ∞ as mentioned before Theorem 3.2. By the uniqueness of the weak solution, we obtain $U_\sigma(t,\tau)x = w(t)$ and thus complete the proof. □

3.2. **Existence of a uniform attractor in** $L^2(\mathbb{R}^N)$.

Lemma 3.5. *For any $\varepsilon > 0$, any $\tau \in \mathbb{R}$ and any $B \subset L^2(\mathbb{R}^N)$ is bounded, there exist $T_\varepsilon > \tau$ and $K_\varepsilon > 0$ such that*

$$\int_{|x|\geq K} |U_\sigma(t,\tau)u_\tau|^2 dx \leq \varepsilon, \tag{3.18}$$

for all $K \geq K_\varepsilon$, $t \geq T_\varepsilon$, all $u_\tau \in B$ and all $\sigma \in \mathcal{H}_w(g)$.

Proof. Let $\phi : [0,+\infty) \to [0,1]$ be a smooth function such that $\phi(s) = 0$ for all $0 \leq s \leq 1$ and $\phi(s) = 1$ for all $s \geq 2$. It is easy to see that $\phi'(s) \leq C$, for all s, and $\phi'(s) = 0$ for all $s \geq 2$. Denote $u(t) = U_\sigma(t,\tau)u_\tau$ and multiply (3.1) by $2\phi\big(\frac{|x|^2}{k^2}\big)u$, where $k > 0$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^N} \phi\Big(\frac{|x|^2}{k^2}\Big)|u|^2 dx + 2\int_{\mathbb{R}^N} \phi\Big(\frac{|x|^2}{k^2}\Big)|\nabla u|^2 dx + 2\int_{\mathbb{R}^N} \phi'\Big(\frac{|x|^2}{k^2}\Big)u\frac{2x}{k^2}\cdot\nabla u\, dx \\
&+ 2\int_{\mathbb{R}^N} \phi\Big(\frac{|x|^2}{k^2}\Big)f(x,u)u\, dx + 2\lambda\int_{\mathbb{R}^N} \phi\Big(\frac{|x|^2}{k^2}\Big)|u|^2 dx \\
&= 2\int_{\mathbb{R}^N} \phi\Big(\frac{|x|^2}{k^2}\Big)u\sigma(t,x) dx.
\end{aligned} \tag{3.19}$$

Now, we estimate terms in (3.19). First, using condition (1.2) of f, we find

$$\int_{\mathbb{R}^N} \phi\Big(\frac{|x|^2}{k^2}\Big)f(x,u)u\, dx \geq -\int_{\mathbb{R}^N} \phi\Big(\frac{|x|^2}{k^2}\Big)\phi_1(x) dx \geq -\int_{|x|\geq k} |\phi_1(x)| dx. \tag{3.20}$$

Next,

$$\begin{aligned}
\Big|\int_{\mathbb{R}^N} \phi'\Big(\frac{|x|^2}{k^2}\Big)u\frac{2x}{k^2}\cdot\nabla u\, dx\Big| &\leq \int_{|x|\leq k\sqrt{2}} \frac{C|x|}{k^2}|u||\nabla u| dx \\
&\leq \frac{C}{k}\int_{\mathbb{R}^N} |u||\nabla u| dx \\
&\leq \frac{C}{k}\Big(\|u\|^2 + \|\nabla u\|^2\Big) \leq \frac{C}{k},
\end{aligned} \tag{3.21}$$

for all $t \geq T_0$, since (3.11). Finally, for the right hand side of (3.19),

$$2\Big|\int_{\mathbb{R}^N} \phi\Big(\frac{|x|^2}{k^2}\Big)\sigma(t,x)u\, dx\Big| \leq \frac{1}{\lambda}\int_{\mathbb{R}^N} \phi\Big(\frac{|x|^2}{k^2}\Big)|\sigma(t,x)|^2 dx + \lambda\int_{\mathbb{R}^N} \phi\Big(\frac{|x|^2}{k^2}\Big)|u|^2 dx. \tag{3.22}$$

Combining (3.19)-(3.22), we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^N} \phi\Big(\frac{|x|^2}{k^2}\Big)|u|^2 dx + \lambda\int_{\mathbb{R}^N} \phi\Big(\frac{|x|^2}{k^2}\Big)|u|^2 dx + 2\int_{\mathbb{R}^N} \phi\Big(\frac{|x|^2}{k^2}\Big)|\nabla u|^2 dx \\
&\leq \frac{C}{k} + 2\int_{|x|\geq k} |\phi_1(x)| dx + \frac{1}{\lambda}\int_{|x|\geq k} |\sigma(t,x)|^2 dx.
\end{aligned} \tag{3.23}$$

By Gronwall's lemma, proceed as (3.5), we conclude that

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \phi\Big(\frac{|x|^2}{k^2}\Big)|u(t)|^2 dx + 2\int_\tau^t e^{-\lambda(t-\tau)} \int_{\mathbb{R}^N} \phi\Big(\frac{|x|^2}{k^2}\Big)|\nabla u|^2\, dx\, ds \\
&\le e^{-\lambda(t-\tau)} \int_{\mathbb{R}^N} \phi\Big(\frac{|x|^2}{k^2}\Big)|u_\tau|^2 dx + \frac{C}{\lambda k} \\
&\quad + \frac{2}{\lambda}\int_{|x|\ge k} |\phi_1(x)| dx + \frac{1}{\lambda}\int_\tau^t e^{-\lambda(t-s)} \int_{|x|\ge k} |\sigma(s,x)|^2\, dx\, ds \\
&\le e^{-\lambda(t-\tau)}\|u_\tau\|^2 + C\Big(\frac{1}{k} + \int_{|x|\ge k} |\phi_1(x)| dx\Big) \\
&\quad + \frac{1}{\lambda(1-e^{-\lambda})}\sup_{t\in\mathbb{R}} \int_t^{t+1}\int_{|x|\ge k} |g(s,x)|^2\, dx\, ds.
\end{aligned}
\tag{3.24}
$$

Using (1.7) and the fact that $\phi_1 \in L^1(\mathbb{R}^N)$, it can be followed from (3.24) that

$$
\limsup_{t\to+\infty}\limsup_{k\to+\infty}\int_{|x|\ge k\sqrt{2}} |u(t)|^2 dx = 0, \tag{3.25}
$$

which completes the proof of (3.18). □

Theorem 3.6. *Assume that assumptions* (H1)–(H2) *hold. Then the family of processes* $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$ *possesses a uniform attractor* $\mathcal{A}_2$ *in* $L^2(\mathbb{R}^N)$. *Moreover, we have*

$$
\mathcal{A}_2 = \cup_{\sigma\in\mathcal{H}_w(g)}\mathcal{K}_\sigma(s) \quad \textit{for all } s\in\mathbb{R}. \tag{3.26}
$$

Proof. By Proposition 3.3, the family $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$ has a uniform absorbing set in $L^2(\mathbb{R}^N)$. Thus, it is sufficient to prove the uniform asymptotic compactness of $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$. Let $\{x_n\}$ be a bounded set in $L^2(\mathbb{R}^N)$, $\{t_n\}$ be a sequence such that $t_n \to +\infty$ as $n\to\infty$, and $\{\sigma_n\}$ be an arbitrary sequence in $\mathcal{H}_w(g)$. We have to show that $\{U_{\sigma_n}(t_n,\tau)x_n\}$ is precompact in $L^2(\mathbb{R}^N)$. Let $\varepsilon>0$ arbitrary. For $K>0$, we denote $B_K = \{x\in\mathbb{R}^N : |x|\le K\}$. From Lemma 3.5 and $\lim_{n\to\infty} t_n = +\infty$, there exist $K>0$ and $N_0\in\mathbb{N}$ satisfy

$$
\|U_{\sigma_n}(t_n,\tau)x_n\|_{L^2(B_K^c)} \le \frac{\varepsilon}{3}, \quad \forall n\ge N_0, \tag{3.27}
$$

where $B_K^c = \mathbb{R}^N \backslash B_K$. On the other hand, from Proposition 3.3, $\{U_{\sigma_n}(t_n,\tau)x_n\}$ is bounded in $H^1(\mathbb{R}^N)$, and then $\{U_{\sigma_n}(t_n,\tau)x_n\}$ restrict on B_K is bounded in $H^1(B_K)$. Since, $H^1(B_K)\hookrightarrow L^2(B_K)$ compactly, $\{U_{\sigma_n}(t_n,\tau)x_n\}$ is precompact in $L^2(B_K)$, thus there exist a subsequence $\{n'\}\subset\{n\}$ and N_1 such that

$$
\|U_{\sigma_{m'}}(t_{m'},\tau)x_{m'} - U_{\sigma_{n'}}(t_{n'},\tau)x_{n'}\|_{L^2(B_K)} \le \frac{\varepsilon}{3}, \quad \text{for all } m',n'\ge N_1. \tag{3.28}
$$

Taking $N=\max\{N_0,N_1\}$, then for all $m',n'\ge N$,

$$
\begin{aligned}
&\|U_{\sigma_{m'}}(t_{m'},\tau)x_{m'} - U_{\sigma_{n'}}(t_{n'},\tau)x_{n'}\|_{L^2(\mathbb{R}^N)} \\
&\le \|U_{\sigma_{m'}}(t_{m'},\tau)x_{m'} - U_{\sigma_{n'}}(t_{n'},\tau)x_{n'}\|_{L^2(B_K)} \\
&\quad + \|U_{\sigma_{m'}}(t_{m'},\tau)x_{m'}\|_{L^2(B_K^c)} + \|U_{\sigma_{n'}}(t_{n'},\tau)x_{n'}\|_{L^2(B_K^c)} \le \varepsilon,
\end{aligned}
\tag{3.29}
$$

by (3.27) and (3.28). This prove that $\{U_{\sigma_n}(t_n,\tau)x_n\}$ is precompact in $L^2(\mathbb{R}^N)$. Relation (3.26) follows directly from Theorem 2.5 and Lemma 3.4. The proof is complete. □

3.3. **Existence of a uniform attractor in $L^p(\mathbb{R}^N)$.** To obtain the existence of a uniform attractor in $L^p(\mathbb{R}^N)$, we assume that the external force g belongs to L^2_n, the space of normal functions, which is defined as follows.

Definition 3.7. *A function $\varphi \in L^2_{\rm loc}(\mathbb{R}; L^2(\mathbb{R}^N))$ is said to be normal if for any $\varepsilon > 0$ there exists $\eta > 0$ such that*

$$\sup_{t\in\mathbb{R}^N} \int_t^{t+\eta} \|\varphi(s)\|^2_{L^2(\mathbb{R}^N)} ds \le \varepsilon.$$

Lemma 3.8 ([4])**.** *If $g \in L^2_n(\mathbb{R}; L^2(\mathbb{R}^N))$ then $g \in L^2_b(\mathbb{R}; L^2(\mathbb{R}^N))$ and for any $\tau \in \mathbb{R}^N$,*

$$\lim_{\gamma\to\infty} \sup_{t\ge\tau} \int_\tau^t e^{-\gamma(t-s)} \|\sigma(s)\|^2_{L^2(\mathbb{R}^N)} ds = 0,$$

uniformly with respect to $\sigma \in \mathcal{H}_w(g)$.

We also need an additional result whose proof can be found in [10].

Lemma 3.9 ([10])**.** *Assume $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$ is a family of processes in $L^2(\mathbb{R}^N)$ and $L^p(\mathbb{R}^N)$, $p \ge 2$. If*

(i) *$\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$ possesses a uniform attractor in $L^2(\mathbb{R}^N)$;*
(ii) *$\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$ has a bounded uniform absorbing set in $L^p(\mathbb{R}^N)$;*
(iii) *for any $\varepsilon > 0$ and any bounded set $B \subset L^2(\mathbb{R}^N)$, there exist $T = T(\varepsilon, B)$ and $M = M(\varepsilon, B)$ such that*

$$\int_{\Omega(|U_\sigma(t,\tau)u_\tau|\ge M)} |U_\sigma(t,\tau)u_\tau|^p dx \le \varepsilon, \text{ for all } \sigma \in \mathcal{H}_w(g), t \ge T, u_\tau \in B, \tag{3.30}$$

where $\Omega(|U_\sigma(t,\tau)u_\tau| \ge M) = \{x \in \mathbb{R}^N : U_\sigma(t,\tau)u_\tau(x) \ge M\}$;

then $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$ has a uniform attractor in $L^p(\mathbb{R}^N)$.

Theorem 3.10. *Assume that f and g satisfy* (H1)–(H2)*. We also assume that g is a normal function. Then the family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$ has a uniform attractor $\mathcal{A}_p$ in $L^p(\mathbb{R}^N)$, moreover $\mathcal{A}_p$ coincides with $\mathcal{A}_2$.*

Proof. By Proposition 3.3, Theorem 3.6 and Lemma 3.9, we only have to prove that $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$ satisfies condition (iii) in Lemma 3.9. Let B be a bounded subset of $L^2(\mathbb{R}^N)$ and $\varepsilon > 0$ arbitrary. For $u(t) = U_\sigma(t,\tau)u_\tau$, we denote by $(u - M)_+$ the positive part of $u - M$; that is,

$$(u-M)_+ = \begin{cases} u - M & \text{if } u \ge M \\ 0 & \text{otherwise,} \end{cases} \tag{3.31}$$

Multiplying (1.1) by $p(u - M)_+^{p-1}$, we obtain

$$\begin{aligned} &\frac{d}{dt}\|(u-M)_+\|^p_{L^p(\mathbb{R}^N)} + p(p-1)\int_{\mathbb{R}^N} |\nabla u|^2 |(u-M)_+|^{p-2} dx \\ &+ p\int_{\mathbb{R}^N} f(u)(u-M)_+^{p-1} dx \\ &= \int_{\mathbb{R}^N} \sigma(t,x)(u-M)_+^{p-1} dx. \end{aligned} \tag{3.32}$$

By (1.2), we can take M large enough to get $f(x,u) \geq C|u|^{p-1}$ when $u \geq M$, and thus,

$$\begin{aligned}\int_{\mathbb{R}^N} f(u)(u-M)_+^{p-1}dx &\geq C\int_{\mathbb{R}^N} |u|^{p-1}(u-M)_+^p dx\\ &\geq C\int_{\mathbb{R}^N}(u-M)_+^{2p-2}dx + CM^{p-2}\|(u-M)_+\|^p_{L^p(\mathbb{R}^N)}.\end{aligned}$$

For the external force,

$$\int_{\mathbb{R}^N}\sigma(t,x)(u-M)_+^{p-1}dx \leq C\|\sigma(t)\|^2 + C\int_{\mathbb{R}^N}(u-M)_+^{2p-2}dx. \tag{3.33}$$

Combining (3.32)-(3.33), we obtain

$$\frac{d}{dt}\|(u-M)_+\|^p_{L^p(\mathbb{R}^N)} + CM^{p-2}\|(u-M)_+\|^p_{L^p(\mathbb{R}^N)} \leq C\|\sigma(t)\|^2. \tag{3.34}$$

By Gronwall's lemma,

$$\begin{aligned}\|(u(t)-M)_+\|^p_{L^p(\mathbb{R}^N)} &\leq e^{-CM^{p-2}(t-T_1)}\|(u(T_1)-M)_+\|^p_{L^p(\mathbb{R}^N)}\\ &\quad + C\int_{T_1}^t e^{-CM^{p-2}(t-s)}\|\sigma(s)\|^2ds,\end{aligned} \tag{3.35}$$

where T_1 is in (3.11). Applying (3.11) and Lemma 3.8, we obtain

$$\int_{\Omega_1}|(u(t)-M)_+|^p dx \leq \varepsilon, \quad \text{uniformly in } u_\tau \in B, \sigma \in \mathcal{H}_w(g), \tag{3.36}$$

when t and M are large enough. Repeat steps above, just replace $(u-M)_+$ by $(u+M)_-$, where

$$(u+M)_- = \begin{cases} u+M & \text{if } u \leq -M\\ 0 & \text{otherwise,}\end{cases} \tag{3.37}$$

we can find t and M large enough such that

$$\int_{\Omega(u\leq -M)}|(u+M)_-|^p dx \leq \varepsilon, \quad \forall u_\tau \in B, \forall \sigma \in \mathcal{H}_w(g). \tag{3.38}$$

From (3.36) and (3.38), we obtain (3.30) and hence complete the proof. $\square$

3.4. **Existence of a uniform attractor in $H^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$.** In this section, we prove the uniform attractor in $H^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$. For this purpose, we first assume an addition condition of the nonlinearity

$$\Big|\frac{\partial f}{\partial x}(x,u)\Big| \leq \phi_5(x), \tag{3.39}$$

where $\phi_5 \in L^2(\mathbb{R}^N)$. Next, we show that solutions of (1.1) is uniformly small when time and spatial variables are large enough. Finally, combining this and arguments similar to the ones used in [5], we can prove the uniform asymptotic compactness of $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$ in $H^1(\mathbb{R}^N)$.

Lemma 3.11. *For any $\tau \in \mathbb{R}$ and any bounded set $B \subset H^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$, there exist $\rho_2 > 0$ and $T_1 \geq \tau$ such that*

$$\|u_t(t)\|^2 \leq \rho_1, \forall t \geq T_1, \quad \forall u_\tau \in B,\ \forall \sigma \in \mathcal{H}_w(g). \tag{3.40}$$

Proof. Integrating (3.9) from t to $t+1$, where $t \geq T_0$, using (1.5) and (3.11), we have

$$\begin{aligned} &\int_t^{t+1} \|u_t(s)\|^2 ds + 2\|\phi_3\|_{L^1(\mathbb{R}^N)} \\ &\leq \lambda\|u(t)\|^2 + \|\nabla u(t)\|^2 + 2\int_{\mathbb{R}^N} F(x,u)dx + \int_t^{t+1} \|\sigma(s)\|^2 ds \\ &\leq (\lambda + 1 + 2\alpha_4)\rho_1 + 2\|\phi_4\|_{L^1(\mathbb{R}^N)} + \|g\|^2_{L_b^2}, \end{aligned} \tag{3.41}$$

thus

$$\int_t^{t+1} \|u_t(s)\|^2 ds \leq C, \quad \text{for all } t \geq T_0. \tag{3.42}$$

Now, differentiate (3.1) with respect to time, denote $v = u_t$, then multiply by $2v$ in $L^2(\mathbb{R}^N)$, we see that

$$\frac{d}{dt}\|v\|^2 + 2\|\nabla v\|^2 + (f'(x,u)v, 2v) + 2\lambda\|v\|^2 = (\sigma'(t), 2v). \tag{3.43}$$

By (1.4) and Cauchy's inequality,

$$\frac{d}{dt}\|v\|^2 \leq 2\ell\|v\|^2 + \frac{1}{2\lambda}\|\sigma'(t)\|^2. \tag{3.44}$$

Combining (3.42) and (3.44), then using the uniform Gronwall lemma, we obtain (3.40). ☐

Lemma 3.12. *For any $\tau \in \mathbb{R}$, and any bounded set $B \subset L^2(\mathbb{R}^N)$,*

$$\int_{\mathbb{R}} |f(x, U_\sigma(t,\tau)u_\tau)|^2 dx \leq C(1 + \|\sigma(t)\|^2_{L^2(\mathbb{R}^N)}), \tag{3.45}$$

for all $t \geq T_1$, all $u_\tau \in B$ and all $\sigma \in \mathcal{H}_w(g)$.

Proof. Multiply (1.1) by $|u|^{p-2}u$ in $L^2(\mathbb{R}^N)$, we obtain

$$\begin{aligned} &(u_t, |u|^{p-2}u) + (p-1)\int_{\mathbb{R}^N} |\nabla u|^2 |u|^{p-2} dx \\ &+ \int_{\mathbb{R}^N} f(x,u)u|u|^{p-2}dx + \lambda\|u\|^p_{L^p(\mathbb{R}^N)} = (\sigma(t,x), |u|^{p-2}u). \end{aligned} \tag{3.46}$$

By the Cauchy and Young's inequalities,

$$(u_t, |u|^{p-2}u) \leq C\|u_t\|^2 + \frac{\alpha_1}{4}\int_{\mathbb{R}^N} |u|^{2p-2}dx, \tag{3.47}$$

$$(\sigma(t,x), |u|^{p-2}u) \leq C\|\sigma(t)\|^2 + \frac{\alpha_1}{4}\int_{\mathbb{R}^N} |u|^{2p-2}dx. \tag{3.48}$$

Using (1.2), we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} f(x,u)u|u|^{p-2}dx &\geq \alpha_1 \int_{\mathbb{R}^N} |u|^{2p-2}dx - \int_{\mathbb{R}^N} \phi_1(x)|u|^{p-2}dx \\ &\geq \alpha_1 \int_{\mathbb{R}^N} |u|^{2p-2}dx - C\|\phi_1\|^{p/2}_{L^{p/2}(\mathbb{R}^N)} - C\|u\|^p_{L^p(\mathbb{R}^N)}. \end{aligned} \tag{3.49}$$

From (3.46)–(3.49), we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} |u(t)|^{2p-2}dx &\leq C(1 + \|u_t(t)\|^2 + \|u(t)\|^2_{L^p(\mathbb{R}^N)} + \|\sigma(t)\|^2) \\ &\leq C(1 + \|\sigma(t)\|^2), \end{aligned} \tag{3.50}$$

for all $t \geq \max\{T_0, T_1\}$, since (3.11) and (3.40). On the other hand, by (1.3),

$$\int_{\mathbb{R}^N} |f(x,u)|^2 dx \leq 2\alpha_2^2 \int_{\mathbb{R}^N} |u|^{2p-2} dx + 2\|\phi_2\|^2. \tag{3.51}$$

This, combining with (3.50), completes the proof. □

Lemma 3.13. *For any $\varepsilon > 0$, any $\tau \in \mathbb{R}$ and any $B \subset L^2(\mathbb{R}^N)$ is bounded, there exist $T_\varepsilon > \tau$ and $K_\varepsilon > 0$ such that*

$$\int_{|x|\geq K} |\nabla U_\sigma(t,\tau)u_\tau|^2 dx \leq \varepsilon, \tag{3.52}$$

for all $K \geq K_\varepsilon$, $t \geq T_\varepsilon$, all $u_\tau \in B$ and all $\sigma \in \mathcal{H}_w(g)$.

Proof. By multiplying (1.1) by $-2\phi(|x|^2/k^2)\Delta u$, where ϕ is in Lemma 3.5, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^N} \phi\Big(\frac{|x|^2}{k^2}\Big)|\nabla u|^2 dx + 2\int_{\mathbb{R}^N} \phi'\Big(\frac{|x|^2}{k^2}\Big) u_t \frac{2x}{k^2}\cdot \nabla u\, dx \\
&+ 2\int_{\mathbb{R}^N} \phi\Big(\frac{|x|^2}{k^2}\Big)|\Delta u|^2 dx + 2\int_{\mathbb{R}^N} \phi\Big(\frac{|x|^2}{k^2}\Big) f'_u(x,u)|\nabla u|^2 dx \\
&+ 2\int_{\mathbb{R}^N} \phi'\Big(\frac{|x|^2}{k^2}\Big) f(u)\frac{2x}{k^2}\cdot \nabla u\, dx + 2\int_{\mathbb{R}^N} \phi\Big(\frac{|x|^2}{k^2}\Big) f'_x(x,u)\nabla u \\
&+ 2\lambda\int_{\mathbb{R}^N} \phi\Big(\frac{|x|^2}{k^2}\Big)|\nabla u|^2 + 2\lambda\int_{\mathbb{R}^N} \phi'\Big(\frac{|x|^2}{k^2}\Big) u\frac{2x}{k^2}\cdot\nabla u\, dx \\
&= -\int_{\mathbb{R}^N} \sigma(t,x)\phi\Big(\frac{|x|^2}{k^2}\Big)\Delta u\, dx.
\end{aligned} \tag{3.53}$$

Using arguments similar to Lemma 3.5, taking into account (1.8), we find that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^N} \phi\Big(\frac{|x|^2}{k^2}\Big)|\nabla u|^2 dx + \lambda\int_{\mathbb{R}^N} \phi\Big(\frac{|x|^2}{k^2}\Big)|\nabla u|^2 dx \\
&\leq C\int_{\mathbb{R}^N} \phi\Big(\frac{|x|^2}{k^2}\Big)|\nabla u|^2 dx + \frac{C}{k}\Big(\|u_t\|^2 + \|u\|^2 + \|\nabla u\|^2 + \int_{\mathbb{R}^N} |f(x,u)|^2 dx\Big) \\
&\quad + \int_{\mathbb{R}^N} \phi\Big(\frac{|x|^2}{k^2}\Big)|\phi_5(x)|^2 dx + C\int_{|x|\geq k} |\sigma(t,x)|^2 dx.
\end{aligned} \tag{3.54}$$

By Gronwall's lemma, Lemma 3.5 and Lemma 3.12,

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \phi\Big(\frac{|x|^2}{k^2}\Big)|\nabla u(t)|^2 dx \\
&\leq e^{-\lambda(t-T)}\|\nabla u(T)\|^2 + C\int_T^t e^{-\lambda(t-s)} \int_{\mathbb{R}^N} \phi\Big(\frac{|x|^2}{k^2}\Big)|\nabla u(s)|^2\,dx\,ds \\
&\quad + \frac{C}{k}\int_T^t e^{-\lambda(t-s)}(1+\|u_t(s)\|^2+\|\nabla u(s)\|^2+\|\sigma(s)\|^2)ds \\
&\quad + C\int_{|x|\geq k}|\phi_5(x)|^2 dx + C\int_T^t e^{-\lambda(t-s)}\int_{|x|\geq k}|\sigma(t,x)|^2\,dx\,ds \qquad (3.55)\\
&\leq e^{-\lambda(t-T)}\|\nabla u(T)\|^2 + C\int_T^t e^{-\lambda(t-s)} \int_{\mathbb{R}^N} \phi\Big(\frac{|x|^2}{k^2}\Big)|\nabla u(s)|^2\,dx\,ds \\
&\quad + \frac{C}{k}\int_T^t e^{-\lambda(t-s)}(1+\rho_0+\rho_1+\|\sigma(s)\|^2)ds \\
&\quad + C\int_{|x|\geq k}|\phi_5(x)|^2 dx + C\sup_{t\in\mathbb{R}^N}\int_t^{t+1}\int_{|x|\geq k}|g(t,x)|^2\,dx\,ds.
\end{aligned}
$$

From (3.11), (3.24) and the fact that $\phi_5 \in L^2(\mathbb{R}^N)$, after detailed computations, we obtain from (3.55) the desired result. □

Now, we define a smooth function $\psi = 1-\phi$, and for a given positive number k, define $v^k(t,x) = \psi(|x|^2/k^2)u(t,x)$. Then, v^k is a unique solution to the initial Cauchy problem

$$
\begin{gathered}
v_t^k - \Delta v^k + \psi\Big(\frac{|x|^2}{k^2}\Big)f(x,u) + \lambda v^k \\
= u\Delta\psi + \frac{4}{k^2}\psi'\Big(\frac{|x|^2}{k^2}\Big)x\cdot\nabla u + \psi\Big(\frac{|x|^2}{k^2}\Big)g(t), \\
v^k|_{\partial B_{2k}} = 0 \\
v^k(\tau) = \psi\Big(\frac{|x|^2}{k^2}\Big)u_\tau.
\end{gathered} \tag{3.56}
$$

Consider the eigenvalue problem

$$
-\Delta w = \lambda w \text{ in } B_{2k}, \quad \text{with } w|_{\partial B_{2k}} = 0.
$$

Then the problem has a family of eigenfunctions $\{e_j\}_{j\geq 1}$ with corresponding eigenvalues $\{\lambda_j\}_{j\geq 1}$ such that $\{e_j\}_{j\geq 1}$ form an orthogonal basis of $H_0^1(B_{2k})$ and $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \to \infty$. For given integer m, any $u \in H_0^1(B_{2k})$ has a unique decomposition $u = u_1 + u_2 = P_m u + (Id - P_m)u$, where P_m is the canonical projector from $H_0^1(B_{2k})$ onto the subspace span$\{e_1, e_2, \ldots, e_m\}$.

We have the following lemma about the precompactness of v^k.

Lemma 3.14. *Let $k > 0$ is fixed. Then, for any $\tau \in \mathbb{R}$ and any $\varepsilon > 0$, there exist $T > \tau$, $m_0 \in \mathbb{N}$ such that*

$$
\|(Id-P_m)v^k(t)\|^2_{H_0^1(B_{2k})} \leq \varepsilon, \quad \forall t \geq T, m \geq m_0 \text{ and } \forall \sigma \in \mathcal{H}_w(g). \tag{3.57}
$$

Proof. Let $v^k = P_m v^k + (Id - P_m)v^k = v_1 + v_2$, and then multiply (3.56) by $-\Delta v_2$ in $L^2(B_{2k})$, we find that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v_2\|^2_{H_0^1(B_{2k})} + \|\Delta v_2\|^2_{L^2(B_{2k})} \\
&- \int_{B_{2k}} \psi\Big(\frac{|x|^2}{k^2}\Big)\Delta v_2 f(x,u)dx + \lambda\|v_2\|^2_{H_0^1(B_{2k})} \\
&\le -\int_{B_{2k}} u\Delta v_2\Delta\psi dx - \frac{4}{k^2}\int_{B_{2k}} \psi'\Big(\frac{|x|^2}{k^2}\Big)\Delta v_2 x\cdot\nabla u\,dx \\
&\quad - \int_{B_{2k}} \psi\Big(\frac{|x|^2}{k^2}\Big)g(t)\Delta v_2 dx.
\end{aligned} \tag{3.58}$$

From definition of ψ, we obtain

$$\Big|\int_{B_{2k}} \psi\Big(\frac{|x|^2}{k^2}\Big)\Delta v_2 f(x,u)dx\Big| \le \frac{1}{8}\|\Delta v_2\|^2_{L^2(B_{2k})} + C\int_{\mathbb{R}^N} |f(x,u)|^2 dx, \tag{3.59}$$

$$\int_{B_{2k}} u\Delta v_2\Delta\psi dx \le \frac{1}{8}\|\Delta v_2\|^2_{L^2(B_{2k})} + C\|u\|^2, \tag{3.60}$$

$$\int_{B_{2k}} \psi'\Big(\frac{|x|^2}{k^2}\Big)\Delta v_2 x\cdot\nabla u\,dx \le \frac{1}{8}\|\Delta v_2\|^2_{L^2(B_{2k})} + C\|\nabla u\|^2, \tag{3.61}$$

$$\int_{B_{2k}} \psi\Big(\frac{|x|^2}{k^2}\Big)g(t)\Delta v_2 dx \le \frac{1}{8}\|\Delta v_2\|^2_{L^2(B_{2k})} + C\|g(t)\|^2. \tag{3.62}$$

From (3.58)-(3.62) and noting that $\|\Delta v_2\|^2_{L^2(B_{2k})} \ge \lambda_m\|v_2\|^2_{H_0^1(B_{2k})}$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\|v_2\|^2_{H_0^1(B_{2k})} + \lambda_m\|v_2\|^2_{H_0^1(B_{2k})} \\
&\le C\Big(\|u\|^2 + \|\nabla u\|^2 + \int_{\mathbb{R}^N} |f(x,u)|^2 dx + \|\sigma(t)\|^2\Big).
\end{aligned} \tag{3.63}$$

Take T large enough such that (3.11) and (3.45) hold for all $t \ge T$. Integrating (3.63) from T to $t \ge T$, and using (3.11) and (3.45), we find that

$$\begin{aligned}
&\|v_2(t)\|^2_{H_0^1(B_{2k})} \\
&\le e^{-\lambda_m(t-T)}\|v_2(T)\|^2_{H_0^1(B_{2k})} \\
&+ C\int_T^t e^{-\lambda_m(t-s)}\Big(\|u(s)\|^2 + \|\nabla u(s)\|^2 + \int_{\mathbb{R}} |f(x,u(s))|^2 dx + \|\sigma(s)\|^2\Big)ds \\
&\le e^{-\lambda_m(t-T)}\|v_2(T)\|^2_{H_0^1(B_{2k})} + C\int_T^t e^{-\lambda_m(t-s)}\left(1 + \rho_1 + \|\sigma(s)\|^2\right)ds.
\end{aligned} \tag{3.64}$$

Noting that

$$\|v_2(T)\|^2_{H_0^1(B_{2k})} \le \|v(T)\|^2_{H_0^1(B_{2k})} \le \|u(T)\|^2_{H^1(\mathbb{R}^N)} \le \rho_1$$

and taking into account Lemma 3.8, we obtain (3.57) by letting t and m tend to infinity. □

Proof of Theorem 1.1. From Proposition 3.3, there is a bounded absorbing set in $H^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ for $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$. Thus, by Theorem 3.10, it is sufficient to prove the uniform asymptotic compactness of $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$ in $H^1(\mathbb{R}^N)$.

For $\tau \in \mathbb{R}$, let $\{x_n\}$ be a bounded sequence in $L^2(\mathbb{R}^N)$, $\{t_n\}$ such that $t_n \to +\infty$ and $\{\sigma_n\} \subset \mathcal{H}_w(g)$, we have to prove that $\{U_{\sigma_n}(t_n,\tau)x_n\}_{n\geq 1}$ is precompact in $H^1(\mathbb{R}^N)$. Given $\varepsilon > 0$, from Lemmas 3.5 and 3.13, there exist $k_1 > 0$ and N_1 such that

$$\int_{|x|\geq 2k} \left(|U_{\sigma_n}(t_n,\tau)x_n|^2 + |\nabla U_{\sigma_n}(t_n,\tau)x_n|^2\right) dx \leq \varepsilon, \tag{3.65}$$

as $n \geq N_1$ and $k \geq k_1$. Denote

$$v^k(t_n) = \psi\Big(\frac{|x|^2}{k^2}\Big) U_{\sigma_n}(t_n,\tau)x_n. \tag{3.66}$$

From Lemma 3.14, we obtain N_2 and $m \in \mathbb{N}$ satisfying

$$\|\,(Id - P_m)\, v^k(t_n)\|^2_{H^1_0(B_{2k})} \leq \varepsilon, \tag{3.67}$$

whenever $n \geq N_2$. By Proposition 3.3, we find that $\{P_m(v^k(t_n))\}_{n\geq 1}$ is bounded in a finite dimensional space, which along with (3.67) shows that $\{v^k(t_n)\}_{n\geq 1}$ is precompact in $H^1_0(B_{2k})$. Thus, we obtain by (3.66) that $\{U_{\sigma_n}(t_n,\tau)x_n\}$ is precompact in $H^1(B_{2k})$ since $\psi(|x|^2/k^2) = 1$ as $|x| \leq k$. Combining this with inequality (3.65) implies the uniform asymptotic compactness of $\{U_{\sigma_n}(t_n,\tau)x_n\}$ in $H^1(\mathbb{R}^N)$. This completes the proof. $\square$

4. Continuous dependence of the attractor on the nonlinearity

Recall that in this section, we consider a family of function $f_\gamma, \gamma \in \Gamma$, such that for each $\gamma \in \Gamma$, f_γ satisfies (1.2)-(1.5) and (1.8) where the constants are independent of γ. The topology $\mathcal{T}$ in Γ can be defined as follows:

If $\gamma_m \to \gamma$ in $\mathcal{T}$ then $f_{\gamma_m}(x,s) \to f_\gamma(x,s)$ for all $x \in \mathbb{R}^N$ and $s \in \mathbb{R}$.

Let $\{U^\gamma_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$ be the family of processes corresponding to the problem

$$\begin{gathered} u_t - \Delta u + f_\gamma(x,u) + \lambda u = g(t,x), \quad x \in \mathbb{R}^N, t > \tau, \\ u(\tau) = u_\tau, \quad x \in \mathbb{R}^N. \end{gathered} \tag{4.1}$$

From the previous section, for each $\gamma \in \Gamma$, the family of processes $\{U^\gamma_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$ has a compact uniform attractor $\mathcal{A}_\gamma$ in $H^1(\mathbb{R}^N)$. Our aim in this section is proving the upper semicontinuity of a family uniform attractors $\{\mathcal{A}_\gamma\}_{\gamma\in\Gamma}$; that is, if $\gamma_m \to \gamma$ in $\mathcal{T}$ as $m \to \infty$, then $\mathcal{A}_{\gamma_m}$ tends to $\mathcal{A}_\gamma$ in the sense that

$$\lim_{m\to\infty} \mathrm{dist}_{L^2(\mathbb{R}^N)}(\mathcal{A}_{\gamma_m}, \mathcal{A}_\gamma) = 0. \tag{4.2}$$

The following lemma is the key.

Lemma 4.1. *Let $\{x_n\} \subset L^2(\mathbb{R}^N), \{\sigma_n\} \in \mathcal{H}_w(g)$ and $\{\gamma_n\} \subset \Gamma$ such that*

$$x_n \rightharpoonup x_0 \text{ weakly in } L^2(\mathbb{R}^N), \tag{4.3}$$

$$\sigma_n \rightharpoonup \sigma \text{ weakly in } \mathcal{H}_w(g), \tag{4.4}$$

$$\gamma_n \to \gamma \text{ in } \Gamma \tag{4.5}$$

as $n \to \infty$. Then, for any $t \geq \tau$, there exists a subsequence $\{j\}$ of $\{n\}$ such that

$$U^{\gamma_j}_{\sigma_j}(t,\tau)x_j \to U^\gamma_\sigma(t,\tau)x_0 \quad \text{strongly in } L^2(\mathbb{R}^N). \tag{4.6}$$

Proof. Denote by $u_n(t) = U^{\gamma_n}_{\sigma_n}(t,\tau)x_n$, we find that u_n solves the problem

$$\begin{gathered}\partial_t u_n - \Delta u_n + f_{\gamma_n}(x,u_n) + \lambda u_n = \sigma_n(t),\\ u_n(\tau) = x_n.\end{gathered} \tag{4.7}$$

Using Proposition 3.3 and noting that all constants are independent of n, we obtain

$$\{u_n(t)\} \text{ is bounded in } H^1(\mathbb{R}^N) \text{ uniformly in } n. \tag{4.8}$$

Thus, there exists a function $v_0 \in L^2(\mathbb{R}^N)$ such that $u_n(t) \rightharpoonup v_0$ weakly in $L^2(\mathbb{R}^N)$ (up to a subsequence). For each $m > 0$, take any $\psi \in L^2(B_m)$, we set $\bar\psi(x) = \psi(x)$ for all $x \in B_m$ and $\bar\psi(x) = 0$ for all $x > m$. It is obviously that $\bar\psi \in L^2(\mathbb{R}^N)$ and

$$(u_n(t),\psi)_{L^2(B_m)} = (u_n(t),\bar\psi)_{L^2(\mathbb{R}^N)} \to (v_0,\bar\psi)_{L^2(\mathbb{R}^N)} = (v_0,\psi)_{L^2(B_m)}. \tag{4.9}$$

It implies that $u_n(t) \rightharpoonup v_0$ in $L^2(B_m)$ for all $m > 0$. On the other hand, by (4.8), for $m > 0$, $\{u_n(t)\}$ is bounded in $H^1(B_m)$, then we find that $\{u_n(t)\}$ is precompact in $L^2(B_m)$ since $H^1(B_m) \hookrightarrow L^2(B_m)$ compactly. By a diagonalization process, we can choose a subsequence $\{j\}$ of $\{n\}$ and $v_m \in L^2(B_m)$ such that $u_j(t) \to v_m$ strongly in $L^2(B_m)$ for all $m > 0$. Taking into account that $u_n(t) \rightharpoonup v_0$ weakly in $L^2(B_m)$ for all $m > 0$, we obtain, by the uniqueness of weak limit,

$$u_j(t) \to v_0 \quad \text{strongly in } L^2(B_m) \text{ for all } m > 0. \tag{4.10}$$

We will prove that $u_j(t) \to v_0$ in $L^2(\mathbb{R}^N)$. Indeed, we have

$$\int_{\mathbb{R}^N} |u_j(t) - v_0|^2 \le \int_{B_m} |u_j(t) - v_0|^2 + 2\int_{B_m^c} |u_j(t)|^2 + 2\int_{B_m^c} |v_0|^2. \tag{4.11}$$

We now control terms of the right hand side of (4.11). First, by (4.10) we obtain

$$\int_{B_m} |u_j(t) - v_0|^2 \to 0 \text{ as } n \to +\infty. \tag{4.12}$$

Next, using arguments in Lemma 3.5, we easily deduce that

$$\begin{aligned}\int_{B_m^c} |u_j(t)|^2 dx \le e^{-\lambda(t-\tau)} \int_{B_m^c} |x_j|^2 dx + C \sup_{t\in\mathbb{R}} \int_t^{t+1} \int_{|x|\ge m} |g(s,x)|^2\,dx\,ds\\ + C\int_{B_m^c} |\phi_1(x)| dx + \frac{C}{m}\int_\tau^t \big(\|u_j(s)\|^2 + \|\nabla u_j(s)\|^2\big)\,ds.\end{aligned} \tag{4.13}$$

Applying (1.7), (4.3), $\phi_1 \in L^1(\mathbb{R}^N)$ and Proposition 3.3 in (4.13) gives us

$$\int_{B_m^c} |u_j(t)|^2 dx \to 0 \text{ as } j, m \to +\infty. \tag{4.14}$$

Because $v_0 \in L^2(\mathbb{R}^N)$,

$$\int_{B_m^c} |v_0|^2 dx \to 0 \text{ as } m \to +\infty. \tag{4.15}$$

Combining (4.11)-(4.15), we claim that

$$u_j(t) \to v_0 \text{ in } L^2(\mathbb{R}^N) \text{ as } n \to +\infty. \tag{4.16}$$

On the other hand, doing similarly to Lemma 3.4, we have

$$U^{\gamma_j}_{\sigma_j}(t,\tau)x_j \rightharpoonup U^{\gamma}_{\sigma}(t,\tau)x_0 \quad \text{in } L^2(\mathbb{R}^N). \tag{4.17}$$

From (4.16) and (4.17) we obtain the desired result. $\square$

Proof of Theorem 1.5. Assume that $\mathrm{dist}_{L^2(\mathbb{R}^N)}(\mathcal{A}_{\gamma_n}, \mathcal{A}_\gamma) \not\to 0$. Hence, by the compactness of $\mathcal{A}_\gamma$, we can choose a positive constant $\delta > 0$, a subsequence $\{m\}$ of $\{n\}$ and $\psi_m \in \mathcal{A}_{\gamma_m}$ satisfying

$$\mathrm{dist}_{L^2(\mathbb{R}^N)}(\psi_m, \mathcal{A}_\gamma) \geq \delta \quad \text{for all } m \geq 1. \tag{4.18}$$

Since $\{U_\sigma^{\gamma_m}(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$ has a uniform absorbing set, which is independent of m, we see that the set $\mathfrak{A} = \cup_{m\geq 1}\mathcal{A}_{\gamma_m}$ is bounded in $L^2(\mathbb{R}^N)$, and then by the uniform attracting property of $\mathcal{A}_\gamma$, we can choose t large enough such that

$$\mathrm{dist}_{L^2(\mathbb{R}^N)}\left(U_\sigma^\gamma(t,0)\mathfrak{A}, \mathcal{A}_\gamma\right) \leq \frac{\delta}{2}, \quad \text{for all } \sigma \in \mathcal{H}_w(g). \tag{4.19}$$

On the other hand,

$$\mathcal{A}_{\gamma_m} = \cup_{\sigma\in\mathcal{H}_w(g)}\mathcal{K}_\sigma^{\gamma_m}(t), \tag{4.20}$$

thus there exists a $\sigma_m \in \mathcal{H}_w(g)$ such that $\psi_m \in \mathcal{K}_{\sigma_m}^{\gamma_m}(t)$. By definition of $\mathcal{K}_{\sigma_m}^{\gamma_m}$, we obtain an $x_m \in \mathcal{K}_{\sigma_m}^{\gamma_m}(0)$ that satisfies $\psi_m = U_{\sigma_m}^{\gamma_m}(t,0)x_m$. Since $\{x_n\} \subset \cup_{m\geq 1}\mathcal{K}_{\sigma_m}^{\gamma_m}(0)$ is bounded in $L^2(\mathbb{R}^N)$, $\mathcal{H}_w(g)$ is weakly compact, we can assume without loss of generality that

$$x_m \rightharpoonup x_0 \text{ in } L^2(\mathbb{R}^N), \tag{4.21}$$

$$\sigma_m \rightharpoonup \sigma_0 \text{ in } \mathcal{H}_w(g). \tag{4.22}$$

Now, applying Lemma 4.1, we deduce that

$$\psi_m = U_{\sigma_m}^{\gamma_m}(t,0)x_m \to U_{\sigma_0}^\gamma(t,0)x_0 \in U_{\sigma_0}^\gamma(t,0)\mathfrak{A}, \tag{4.23}$$

which contradicts with (4.18) and (4.19). This completes the proof. □

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