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EXISTENCE OF SOLUTIONS FOR QUASILINEAR ELLIPTIC EQUATIONS WITH NONLINEAR BOUNDARY CONDITIONS AND INDEFINITE WEIGHT

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Abstract. In this article, we establish the existence and non-existence of solutions for quasilinear equations with nonlinear boundary conditions and indefinite weight. Our proofs are based on variational methods and their geometrical features. In addition, we prove that all the weak solutions are in $C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$.

1. Introduction

In this article, we consider the problem

$$\begin{gathered} \operatorname{div}(a(x)|Du|^{p-2}Du) = |u|^{p-2}u, \quad \text{in } \Omega, \\ a(x)|Du|^{p-2}\frac{\partial u}{\partial \nu} + |u|^{q-2}u + h(x) = \lambda V(x)|u|^{p-2}u, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, with a $C^{2,\alpha}$ boundary for some $\alpha \in (0,1)$, $1 < p < N$, $q < p^\star = \frac{(N-1)p}{N-p}$, $\frac{\partial}{\partial \nu}$ is the outer normal derivative, $0 < a_0 \le a(x) \in L^\infty(\overline{\Omega})$. The functions $V(x)$, $h(x)$ are defined on $\partial\Omega$ and satisfy the assumption

(H1) $V \in L^s(\partial\Omega)$, $V(x)$ is a indefinite weight, i.e.

$$V^+(x) = \max\{V(x), 0\} \neq 0, x \in \partial\Omega,$$

where $s > \frac{N-1}{p-1}$, and $h(x) \in L^s(\partial\Omega)$.

Elliptic problems with nonlinear boundary conditions arise in many and diverse contexts, such as differential geometry (e.g., in the scalar curvature problem and the Yamabe problem [6]), Non-Newtonian fluid mechanics [3], and mathematical biology problem (e.g., a prototype of pattern formation in biology and the steady-state problem for a chemotactic aggregation model [7]). In this paper, we consider the quasilinear problems with mixed nonlinear boundary condition and the indefinite character; i.e. $V(x)$ may change sign on $\partial\Omega$. Some existence and non-existence results are obtained.

On the other hand, the regularity for elliptic problems with nonlinear boundary conditions have been studied. For the semilinear elliptic problem, Ebmeyer [5]

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obtained that every weak solution belongs to $C^{\beta}(\Omega)(0<\beta<1)$. Using the result of Dibenedetto [4], Anane, Chakrone, Moradi [1] obtained that the eigenfunction of the first eigenvalue is in $C^{1,\beta}(\overline{\Omega})(0<\beta<1)$ for the linear eigenvalue problem of the p-Laplacian. In this paper, for problem (1.1) with nonlinear boundary conditions and indefinite weight, we obtain that all weak solutions are in $L^{\infty}(\partial\Omega)\cap L^{\infty}(\Omega)$ and $C^{1,\beta}(\overline{\Omega})$ for some $\beta\in(0,1)$.

This article is organized as follows: In Section 2, we state our main results. In section 3, we obtain some existence and non-existence results. Section 4 is devoted to proving the regularity of the solutions for the problem (1.1).

2. Main results

Let Ω be a bounded smooth domain in $\mathbb{R}^N$, and $V(x)$ satisfies (H1). We denote the Sobolev space

$$L^p(\partial\Omega;V)=\{u:\partial\Omega\to\mathbb{R};\int_{\partial\Omega}V(x)|u|^p d\sigma<+\infty\}, \tag{2.1}$$

and the norm $\|u\|_{L^p(\partial\Omega;V)}=(\int_{\partial\Omega}V(x)|u|^p d\sigma)^{1/p}$. Consider the Sobolev trace embedding $W^{1,p}(\Omega)\hookrightarrow L^p(\partial\Omega;V)$, we obtain that the embedding is compact when $V(x)$ satisfies (H1) (see [2]), where the norm in $W^{1,p}(\Omega)$ is defined as

$$\|u\|_{W^{1,p}(\Omega)}=\Big(\int_{\Omega}[|\nabla u|^p+|u|^p]dx\Big)^{1/p}.$$

As the function $a(x)$ satisfies $0<a_0\le a(x)\in L^{\infty}(\overline{\Omega})$, we define the space E is the reflexive Banach space under the norm

$$\|u\|_{a,\Omega}=\Big(\int_{\Omega}[a(x)|Du|^p+|u|^p]dx\Big)^{1/p}.$$

Of course, $E\sim W^{1,p}(\Omega)$, we obtain that the embedding $E\hookrightarrow L^p(\partial\Omega;V)$ is compact and there exists a $\widetilde{C}=\widetilde{C}(\overline{\Omega},V(x),p)>0$ such that

$$\widetilde{C}\|v\|^p_{L^p(\partial\Omega;V)}\le\|v\|^p_{a,\Omega}\quad\text{for any } v\in E. \tag{2.2}$$

Now, we state the main results in this article.

Theorem 2.1. *If $p<q<p^{\star}$ and $\int_{\partial\Omega}h\varphi\,d\sigma\ge 0$ for all $\varphi\in E$ with $\varphi|_{\partial\Omega}>0$, then there exists $\lambda_0>0$ such that*

1. *if $\lambda<\lambda_0$, then (1.1) does not have any weak solutions,*
2. *if $\lambda>\lambda_0$, then (1.1) has at least one weak solution.*

We remark that there are functions h such that $\int_{\partial\Omega}h\varphi\,d\sigma\ge 0$ for all $\varphi\in E$ with $\varphi|_{\partial\Omega}>0$: For $p=2$ and Ω is a unit circle, let $x=e^{i\alpha}$, $x\in\partial\Omega$, and

$$h=\begin{cases}1+\alpha^2, & 0<\alpha\le 2\pi,\\ -1, & \alpha=0.\end{cases}$$

Theorem 2.2. *If u is a weak solution of (1.1) and $q<\frac{p^2-2p+N}{N-p}$, then u has the following properties:*

1. *$u\in L^{\infty}(\Omega)\cap L^{\infty}(\partial\Omega)$,*

(2) $u \in C^{1,\beta}(\overline{\Omega})$ *for some* $\beta \in (0,1)$*, and* $\|u\|_{C^{1,\beta}(\overline{\Omega})} \le K$*, where*

$$K = K\big(p, N, G, \|u\|_{L^{s'q_0}(\partial\Omega)}, \|V\|_{L^s(\partial\Omega)}\big),$$
$$G = \Big(\int_{\partial\Omega} |(|u|^{q-2}u + h)|^s d\sigma\Big)^{1/s},$$

$s > \frac{N-1}{p-1}$*,* $s'q_0 \in [s'p, p^\star]$*, and* s' *is the conjugate of* s*.*

3. Proof of Theorem 2.1

For this proof we use direct methods in variational methods.

(1) We prove only that (1.1) does not have any weak solutions for λ small enough. Indeed, assume that $u \in E$ is a weak solution of (1.1); then we have

$$\begin{aligned} &\int_\Omega a(x)|Du|^{p-2}DuD\varphi\,dx + \int_\Omega |u|^{p-2}u\varphi\,dx + \int_{\partial\Omega} |u|^{q-2}u\varphi\,d\sigma + \int_{\partial\Omega} h\varphi\,d\sigma \\ &= \lambda \int_{\partial\Omega} V(x)|u|^{p-2}u\varphi\,d\sigma, \end{aligned} \tag{3.1}$$

for any $\varphi \in E$. Taking $\varphi = u$ in (3.1), we obtain

$$\|u\|^p_{a,\Omega} + \|u\|^q_{L^q(\partial\Omega)} + \int_{\partial\Omega} hu\,d\sigma = \lambda\|u\|^p_{L^p(\partial\Omega;V)}. \tag{3.2}$$

Clearly, for $p < q < p^\star$, problem (1.1) does not have non-trivial solution whenever $\lambda \le 0$.

Furthermore, by (2.2) and (3.2), we have

$$\lambda\|u\|^p_{L^p(\partial\Omega;V)} \ge \|u\|^p_{a,\Omega} \ge \widetilde{C}\|u\|^p_{L^p(\partial\Omega;V)}.$$

i.e., $\lambda \ge \widetilde{C}$, which implies that when $\lambda_0 \le \widetilde{C}$, problem (1.1) still does not have weak solution. This completes the proof of (1) of Theorem 2.1.

(2) Let the functional $J_\lambda : E \to \mathbb{R}$ be

$$J_\lambda(u) = \frac{1}{p}\|u\|^p_{a,\Omega} + \frac{1}{q}\|u\|^q_{L^q(\partial\Omega)} + \int_{\partial\Omega} hud\sigma - \frac{\lambda}{p}\|u\|^p_{L^p(\partial\Omega;V)}. \tag{3.3}$$

By (H1), we obtain the weak solution of the problem (1.1) is the critical point of the functional J_λ.

Firstly, we prove that the functional J_λ is coercive. Indeed, fix a $w \in E\backslash\{0\}$, by (2.2) and $p < q$, we have

$$\begin{aligned} J_\lambda(tw) &= \frac{t^p}{p}\|w\|^p_{a,\Omega} + \frac{t^q}{q}\|w\|^q_{L^q(\partial\Omega)} + t\int_{\partial\Omega} hwd\sigma - \frac{\lambda t^p}{p}\|w\|^p_{L^p(\partial\Omega;V)} \\ &\ge \frac{t^p}{p}(1 - \frac{\lambda}{p\widetilde{C}})\|w\|^p_{a,\Omega} + \frac{t^q}{q}\|w\|^q_{L^q(\partial\Omega)} + t\int_{\partial\Omega} hwd\sigma. \end{aligned} \tag{3.4}$$

Obviously we have $J_\lambda(tw) \to +\infty$ when $t \to +\infty$. So the coercivity of the functional J_λ is obtained.

Let $\{u_n\}_{n=1}^\infty$ be a minimizing sequence of J_λ in E, which is bounded in E by the coercivity of J_λ. By the non-negativity of the norm and $\int_{\partial\Omega} h\varphi\,d\sigma \ge 0$ for all $\varphi \in E$, we assume that $\{u_n\}_{n=1}^\infty$ is non-negative, converges weakly to some $u \in E$ and pointwise converges to u.

Secondly, we prove that the non-negative limit $u \in E$ is a weak solution of (1.1). Indeed, We already know that $\lim_{n\to\infty} J_\lambda(u_n) = \inf_{u'\in X} J_\lambda(u')$; i.e.,

$$\lim_{n\to\infty} J_\lambda(u_n) \le J_\lambda(u'), \quad \text{for all } u' \in E. \tag{3.5}$$

So we only need to prove

$$J_\lambda(u) \le \lim_{n\to\infty} \inf J_\lambda(u_n). \tag{3.6}$$

By (H1), we have

$$\int_{\partial\Omega} hu \, d\sigma = \lim_{n\to\infty} \int_{\partial\Omega} hu_n \, d\sigma,$$

and by the weak lower semicontinuity of the norm, we have

$$\frac{1}{p}\|u\|_{a,\Omega}^p + \frac{1}{q}\|u\|_{L^q(\partial\Omega)}^q \le \lim_{n\to\infty} \inf(\frac{1}{p}\|u_n\|_{a,\Omega}^p + \frac{1}{q}\|u_n\|_{L^q(\partial\Omega)}^q).$$

On the other hand, the boundedness of $\{u_n\}_{n=1}^\infty$ and the compact imbedding $E \hookrightarrow L^p(\partial\Omega; V)$ implies that

$$\|u\|_{L^p(\partial\Omega;V)} = \lim_{n\to\infty} \|u_n\|_{L^p(\partial\Omega;V)}.$$

So (3.6) is established. Then by (3.5) and (3.6) we have

$$J_\lambda(u) = \inf_{u'\in E} J_\lambda(u').$$

Thus, u is a global minimizer of J_λ in E.

Thirdly, we show that the weak limit u is a non-trivial weak solution of (1.1) if $\lambda > 0$ is large enough. Indeed, $J_\lambda(0) = 0$. Hence, we only need to prove that there exists $\lambda^0 > 0$, such that

$$\inf_{u'\in E} J_\lambda(u') < 0 \quad \text{for all } \lambda > \lambda^0.$$

Consider the minimization problem

$$\lambda^0 := \inf\{\frac{1}{p}\|\phi\|_{a,\Omega}^p + \frac{1}{q}\|\phi\|_{L^q(\partial\Omega)}^q + \int_{\partial\Omega} h\phi \, d\sigma : \phi \in E \text{ and } \|\phi\|_{L^p(\partial\Omega;V)}^p = p\}. \tag{3.7}$$

Let $\{\kappa_n\}_{n=1}^\infty \in E$ be a minimizing sequence of (3.7), which is obviously bounded in E. Hence, without loss of generality, we assume that it converges weakly to some $\kappa \in E$, with $\|\kappa\|_{L^p(\partial\Omega;V)}^p = p$. By the weak lower semicontinuity of $\|\cdot\|$, We can deduce that

$$\lambda^0 = \frac{1}{p}\|\kappa\|_{a,\Omega}^p + \frac{1}{q}\|\kappa\|_{L^q(\partial\Omega)}^q + \int_{\partial\Omega} h\kappa \, d\sigma.$$

So $J_\lambda(\kappa) = \lambda^0 - \lambda < 0$ for any $\lambda > \lambda^0$. Now we denote

$$\lambda_0 := \sup\{\lambda > 0 : \text{ problem (1.1) does not have weak solutions}\},$$
$$\lambda_1 := \inf\{\lambda > 0 : \text{ problem (1.1) admits a weak solution}\}.$$

Of course $\lambda_1 \ge \lambda_0 > 0$.

Lastly, we prove two facts: (i) problem (1.1) has a weak solution for any $\lambda > \lambda_1$; (ii) $\lambda_0 = \lambda_1$.

Now, we fix $\lambda > \lambda_1$, by the definition of λ_1, there exists $\mu \in (\lambda_1, \lambda)$, such that J_μ has a non-trivial critical point $u_\mu \in E$; i.e.,

$$\|u_\mu\|_{a,\Omega}^p + \|u_\mu\|_{L^q(\partial\Omega)}^q + \int_{\partial\Omega} hu_\mu d\sigma = \mu\|u_\mu\|_{L^p(\partial\Omega;V)}^p,$$

Clearly, u_μ is a sub-solution of problem (1.1). So next we need to find a super-solution of problem (1.1) which is greater than u_μ.

Consider the minimization problem

$$\inf\{\frac{1}{p}\|\phi\|^p_{a,\Omega}+\frac{1}{q}\|\phi\|^q_{L^q(\partial\Omega)}+\int_{\partial\Omega}h\phi\,d\sigma-\frac{\lambda}{p}\|\phi\|^p_{L^p(\partial\Omega;V)}:\phi\in E\text{ and }\phi\geq u_\mu\}.$$

From above argument, we can know that the minimization problem has a solution $u_\lambda\geq u_\mu$, which is also a weak solution of (1.1) provided $\lambda>\lambda_1$. So for the fixed λ, we have a sub-solution u_μ and a super-solution u_λ with $u_\lambda\geq u_\mu$, using [8, Theorem 2.4], we obtain a weak solution. Let us recall the definition of λ_1, we obtain that (1.1) does not have solutions for any $\lambda<\lambda_1$. Then by the define of λ_0, immediately we have $\lambda_1\leq\lambda_0$, so $\lambda_1=\lambda_0$.

4. Proof of Theorem 2.2

This is an adaptation of the proof in [1], and is presented here, for the reader's convenience. Let $g=-|u|^{q-2}u-h$, then by $q<\frac{p^2-2p+N}{N-p}$, we have $g\in L^s(\partial\Omega)$.

Lemma 4.1. *If $u\in E$ is a weak solution of* (1.1)*, then there exists a constant $C>0$, such that*

$$(\|u\|^{q_n}_{L^{q_n}(\Omega)}+\|u\|^{s'q_n}_{L^{s'q_n}(\partial\Omega)})^{1/q_n}\leq C,\quad\textit{for all } n>n_0,$$

where the sequence $\{q_n\}_{n=0}^\infty$ is defined as

$$s'q_0\in[s'p,p^\star],\quad p^\star=\frac{(N-1)p}{N-p},\quad q_{n+1}=\frac{q_0}{p}q_n.$$

Furthermore, $u\in L^{q_n}(\Omega)$ and $u\in L^{s'q_n}(\partial\Omega)$ for all $n\geq0$, where $s'=s/(s-1)$.

Proof. Assume that $u\in E$ is a weak solution of (1.1). By $E\sim W^{1,p}(\Omega)$, u is also in $W^{1,p}(\Omega)$. Since $s>\frac{N-1}{p-1}$, we have $1<s'=\frac{s}{s-1}<\frac{N-1}{N-p}$, and $[p,p^\star]\cap[s'p,s'p^\star]=[s'p,p^\star]\neq\emptyset$.

Let $q_0\in[p,p^\star/s']$. Then

$$W^{1,p}(\Omega)\hookrightarrow L^{q_0}(\Omega)\quad\text{and}\quad W^{1,p}(\Omega)\hookrightarrow L^{q_0s'}(\partial\Omega).$$

Obviously, $u\in L^{q_0}(\Omega)$ and $u\in L^{s'q_0}(\partial\Omega)$. Of course, u is also in $L^{q_0}(\partial\Omega)$. Suppose that $\|u\|_{L^{s'q_0}(\partial\Omega)}\geq1$, if not we consider $u_0=u/\|u\|_{L^{s'q_0}(\partial\Omega)}$, which is a solution of

$$\begin{gathered}\operatorname{div}(a(x)|Du|^{p-2}Du)=|u|^{p-2}u\quad\text{in }\Omega,\\ a(x)|Du|^{p-2}\frac{\partial u}{\partial\nu}=\lambda V(x)|u|^{p-2}u+g'\quad\text{on }\partial\Omega,\end{gathered}$$

with $g'=(\|u\|_{L^{s'q_0}(\partial\Omega)})^{p-1}g\in L^s(\partial\Omega)$.

Using mathematical induction, suppose that $u\in L^{q_n}(\Omega)$, $u\in L^{s'q_n}(\partial\Omega)$ and $\|u\|_{L^{s'q_n}(\partial\Omega)}\geq1$, we show that

$$u\in L^{q_{n+1}}(\Omega),\quad u\in L^{q_{n+1}}(\partial\Omega),\quad u\in L^{s'q_{n+1}}(\partial\Omega),\quad\|u\|_{L^{s'q_{n+1}}(\partial\Omega)}\geq1.$$

Define a sequence $\{\omega_k\}_{k=0}^\infty$ in E by

$$\omega_k(x)=\begin{cases}k,&\text{if }u(x)\geq k;\\ u(x),&\text{if }-k\leq u(x)\leq k,\ \forall x\in\overline{\Omega};\\ -k,&\text{if }u(x)\leq-k;\end{cases}$$

Obviously, $\{\omega_k\}_{k=0}^{\infty}$ is in $W^{1,p}(\Omega)$. Set $\delta = q_n - p > 0$, then take the test function $|\omega_k|^{\delta}\omega_k$ in (3.1), we obtain

$$\begin{aligned} \langle \operatorname{div}(a(x)|Du|^{p-2}Du), |\omega_k|^{\delta}\omega_k\rangle &= \int_\Omega |u|^{p-2}u|\omega_k|^{\delta}\omega_k dx \\ &\geq \int_\Omega |\omega_k|^{\delta+p}dx = \int_\Omega |\omega_k|^{q_n}dx, \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} &\langle \operatorname{div}(a(x)|Du|^{p-2}Du), |\omega_k|^{\delta}\omega_k\rangle \\ &= -\int_\Omega a(x)|Du|^{p-2}DuD(|\omega_k|^{\delta}\omega_k)dx + \lambda\int_{\partial\Omega}(V(x)|u|^{p-2}u+g)|\omega_k|^{\delta}\omega_k d\sigma \\ &\leq \lambda\int_{\partial\Omega}|u|^{q_n}|V(x)|d\sigma + G\|\omega_k^{\delta+1}\|_{L^{s'}(\partial\Omega)} - B_n\|D(|\omega_k|^{\frac{\delta}{p}}\omega_k)\|^p_{L^p(\Omega)} \\ &\leq \lambda\|u\|^{q_n}_{L^{s'q_n}(\partial\Omega)}\|V\|_{L^s(\partial\Omega)} + G\|\omega_k\|^{\delta+1}_{L^{(\delta+1)s'}(\partial\Omega)} - B_n\|D(|\omega_k|^{\frac{\delta}{p}}\omega_k)\|^p_{L^p(\Omega)}, \end{aligned} \tag{4.2}$$

where

$$G = \Big(\int_{\partial\Omega}||u|^{q-2}u+h|^s d\sigma\Big)^{1/s}, \quad B_n = a_0(\delta+1)(\frac{p}{q_n})^p.$$

Then by (4.1) and (4.2), we have

$$\begin{aligned} &\int_\Omega |\omega_k|^{q_n}dx \\ &\leq \lambda\|u\|^{q_n}_{L^{s'q_n}(\partial\Omega)}\|V\|_{L^s(\partial\Omega)} + G\|\omega_k\|^{\delta+1}_{L^{(\delta+1)s'}(\partial\Omega)} - B_n\|D(|\omega_k|^{\frac{\delta}{p}}\omega_k)\|^p_{L^p(\Omega)}. \end{aligned} \tag{4.3}$$

Since $W^{1,p}(\Omega) \hookrightarrow L^{q_0}(\Omega)$, there exists $C_1 = C_1(\Omega, p, q_0) > 0$, such that

$$\begin{aligned} \|D(|\omega_k|^{\frac{\delta}{p}}\omega_k)\|^p_{L^p(\Omega)} &\geq C_1\||\omega_k|^{\frac{\delta+p}{p}}\|^p_{L^{q_0}(\Omega)} - \||\omega_k|^{\frac{\delta+p}{p}}\|^p_{L^p(\Omega)} \\ &\geq C_1\|\omega_k\|^{q_n}_{L^{q_{n+1}}(\Omega)} - \|\omega_k\|^{\delta+p}_{L^{\delta+p}(\Omega)}. \end{aligned} \tag{4.4}$$

By (4.3) and (4.4), we have

$$\begin{aligned} &\|\omega_k\|^{q_n}_{L^{q_{n+1}}(\Omega)} \\ &\leq A_n(\lambda\|u\|^{q_n}_{L^{s'q_n}(\partial\Omega)}\|V\|_{L^s(\partial\Omega)} + G\|\omega_k\|^{\delta+1}_{L^{(\delta+1)s'}(\partial\Omega)} + D_n\|\omega_k\|^{q_n}_{L^{q_n}(\Omega)}), \end{aligned} \tag{4.5}$$

where $A_n = \frac{1}{B_nC_1}$ and $D_n = B_n - 1$. By $\delta + 1 < q_n$, we have

$$\|\omega_k\|^{\delta+1}_{L^{(\delta+1)s'}(\partial\Omega)} \leq \|u\|^{\delta+1}_{L^{(\delta+1)s'}(\partial\Omega)} \leq \|u\|^{\delta+1}_{L^{s'q_n}(\partial\Omega)}(\text{meas}_\sigma(\partial\Omega)^{\frac{p-1}{s'q_n}}).$$

Suppose that $\text{meas}_\sigma(\partial\Omega) \leq 1$ and with the assumption $\|u\|_{L^{s'q_n}(\partial\Omega)} \geq 1$, we obtain

$$\|\omega_k\|^{\delta+1}_{L^{(\delta+1)s'}(\partial\Omega)} \leq \|u\|^{\delta+1}_{L^{s'q_n}(\partial\Omega)} \leq \|u\|^{q_n}_{L^{s'q_n}(\partial\Omega)}. \tag{4.6}$$

So by (4.5) and (4.6), we obtain

$$\begin{aligned} \|\omega_k\|^{q_n}_{L^{q_{n+1}}(\Omega)} &\leq A_n[(\lambda\|V\|_{L^s(\partial\Omega)} + G)\|u\|^{q_n}_{L^{s'q_n}(\partial\Omega)} + |D_n|\|u\|^{q_n}_{L^{q_n}(\Omega)}] \\ &\leq A_n\max(R,\ |D_n|)(\|u\|^{q_n}_{L^{s'q_n}(\partial\Omega)} + \|u\|^{q_n}_{L^{q_n}(\Omega)}), \end{aligned}$$

where $R = \lambda\|V\|_{L^s(\Omega)} + G$. Then we deduce that

$$\begin{aligned} \|u\|^{q_n}_{L^{q_{n+1}}(\Omega)} &\leq \lim_{|k|\to+\infty}\inf(\|\omega_k\|^{q_n}_{L^{q_{n+1}}(\Omega)}) \\ &\leq A_n\max(R,\ |D_n|)(\|u\|^{q_n}_{L^{s'q_n}(\partial\Omega)} + \|u\|^{q_n}_{L^{q_n}(\Omega)}) \end{aligned} \tag{4.7}$$

Thus $u \in L^{q_{n+1}}(\Omega)$.

Next we prove $u \in L^{s'q_{n+1}}(\partial\Omega)$ (so $u \in L^{q_{n+1}}(\partial\Omega)$), and $\|u\|_{L^{s'q_{n+1}}(\partial\Omega)} \geq 1$. By (4.3) and (4.6), we have

$$\int_\Omega |\omega_k|^{q_n}\,dx + B_n\|D(|\omega_k|^{\frac{\delta}{p}}\omega_k)\|^p_{L^p(\Omega)} \leq R\|u\|^{q_n}_{L^{s'q_n}(\partial\Omega)}. \tag{4.8}$$

The embedding $W^{1,p}(\Omega) \hookrightarrow L^{s'q_0}(\partial\Omega)$ implies the existence of $C_2 = C_2(\overline{\Omega}, p, s'q_0) > 0$ such that

$$\begin{aligned} \|D(|\omega_k|^{\frac{\delta}{p}}\omega_k)\|^p_{L^p(\Omega)} &\geq C_2\||\omega_k|^{\frac{\delta+p}{\delta}}\|^p_{L^{s'}q_0(\partial\Omega)} - \||\omega_k|^{\frac{\delta+p}{p}}\|^p_{L^p(\Omega)} \\ &\geq C_2\|\omega_k\|^{q_n}_{L^{s'q_{n+1}}(\partial\Omega)} - \|\omega_k\|^{\delta+p}_{L^{\delta+p}(\Omega)} \end{aligned} \tag{4.9}$$

Then by (4.8) and (4.9), we obtain

$$B_n(C_2\|\omega_k\|^{q_n}_{L^{s'q_{n+1}}(\partial\Omega)} - \|\omega_k\|^{\delta+p}_{L^{\delta+p}(\Omega)}) \leq R\|u\|^{q_n}_{L^{s'q_n}(\partial\Omega)} - \int_\Omega |\omega_k|^{q_n}\,dx.$$

Then

$$\begin{aligned} \|\omega_k\|^{q_n}_{L^{s'q_{n+1}}(\partial\Omega)} &\leq B'_n(R\|u\|^{q_n}_{L^{s'q_n}(\partial\Omega)} + |D_n|\|\omega_k\|^{q_n}_{L^{q_n}(\Omega)}) \\ &\leq B'_n \max(R,\ |D_n|)(\|u\|^{q_n}_{L^{s'q_n}(\partial\Omega)} + \|u\|^{q_n}_{L^{q_n}(\Omega)}), \end{aligned}$$

where $B'_n = 1/(C_2B_n)$. Then

$$\begin{aligned} \|u\|^{q_n}_{L^{s'q_{n+1}}(\partial\Omega)} &\leq \lim_{|k|\to+\infty}\inf(\|\omega_k\|^{q_n}_{L^{s'q_{n+1}}(\partial\Omega)}) \\ &\leq B'_n \max(R,\ |D_n|)(\|u\|^{q_n}_{L^{s'q_n}(\partial\Omega)} + \|u\|^{q_n}_{L^{q_n}(\Omega)}). \end{aligned} \tag{4.10}$$

Consequently, $u \in L^{s'q_{n+1}}(\partial\Omega)$ and $\|u\|_{L^{s'q_{n+1}}(\partial\Omega)} > \|u\|_{L^{s'q_n}(\partial\Omega)} \geq 1$. Thus

$$u \in L^{q_n}(\Omega), \quad u \in L^{s'q_n}(\partial\Omega), \quad \|u\|_{L^{s'q_n}(\partial\Omega)} \geq 1, \quad \text{for all } n \geq 0$$

Lastly, we have to show that there exists $C > 0$ such that

$$(\|u\|^{q_n}_{L^{q_n}(\Omega)} + \|u\|^{s'q_n}_{L^{s'q_n}(\partial\Omega)})^{1/q_n} \leq C, \quad \text{for all } n > n_0,$$

By (4.7) and (4.10), we have

$$\|u\|^{q_{n+1}}_{L^{s'q_{n+1}}(\partial\Omega)} + \|u\|^{q_{n+1}}_{L^{q_{n+1}}(\Omega)} \leq T_n(\max(R,\ |D_n|)(\|u\|^{q_n}_{L^{s'q_n}(\partial\Omega)} + \|u\|^{q_n}_{L^{q_n}(\Omega)}))^{q_0/p},$$

where

$$T_n = \Big((\frac{1}{C_1} + \frac{1}{C_2})\frac{1}{B_n}\Big)^{q_0/p}.$$

Obviously, $\lim_{n\to+\infty} B_n = 0$, so we have $\lim_{n\to+\infty}|D_n| = 1$; so there exists $n_0 \in \mathbf{N}^+$, such that $|D_n| \leq 2$ when $n > n_0$. Consequently,

$$\|u\|^{q_{n+1}}_{L^{s'q_{n+1}}(\partial\Omega)} + \|u\|^{q_{n+1}}_{L^{q_{n+1}}(\Omega)} \leq \overline{C}(q_n)^{q_0}(\|u\|^{q_n}_{L^{s'q_n}(\partial\Omega)} + \|u\|^{q_n}_{L^{q_n}(\Omega)})^{\frac{q_0}{p}},$$

where

$$\overline{C} = \frac{1}{p^{q_0}}\Big((\frac{1}{C_1} + \frac{1}{C_2})\max(R,\ 2)\Big)^{q_0/p}.$$

Setting

$$v_n = \big(\|u\|^{q_n}_{L^{s'q_n}(\partial\Omega)} + \|u\|^{q_n}_{L^{q_n}(\Omega)}\big)^{1/q_n},$$

we have $v_{n+1}^{q_{n+1}} \le \overline{C}(q_n)^{q_0}(v_n^{q_n})^{q_0/p}$ for all $n \ge n_0$, and

$$\ln(v_{n+1}) \le \frac{B}{q_{n+1}} + p\frac{\ln(q_n)}{q_n} + \ln(v_n) \le B \sum_{n_0+1 \le k \le n+1} (\frac{1}{q_k}) + p \sum_{n_0 \le k \le n} (\frac{\ln(q_k)}{q_k}) + \ln(v_{n_0}),$$

for all $n \ge n_0$, where $B = \ln(\overline{C})$. By $0 < \frac{p}{q_0} < 1$, we have

$$\sum_{n_0+1 \le k \le n+1} (\frac{1}{q_k}) \le \frac{q_0}{q_0 - p}.$$

Since

$$\begin{aligned} \sum_{n_0 \le k \le n} \frac{\ln(q_k)}{q_k} &= \sum_{n_0 \le k \le n} (\frac{\ln(q_0)}{q_0} + \frac{\ln(q_0) - \ln(p)}{q_0} k)(\frac{p}{q_0})^k := \sum_{n_0 \le k \le n} (\theta + \eta k)(\frac{p}{q_0})^k \\ &\le \sum_{k \ge 0} (\theta + \eta k)(\frac{p}{q_0})^k = \frac{\theta q_0}{q_0 - p} + \frac{\eta p q_0}{(q_0 - p)^2}, \end{aligned}$$

we have

$$\ln(v_n) \le \frac{q}{(q_0 - p)}(B + \theta p) + \frac{\eta p^2 q_0}{(q_0 - p)^2} + \ln(v_{n_0}) := A, \quad \forall n \ge n_0.$$

Thus

$$v_n = (\|u\|_{L^{s' q_n}(\partial\Omega)}^{q_n} + \|u\|_{L^{q_n}(\Omega)}^{q_n}) \le \exp^A := C, \quad \forall n \ge n_0.$$

$\square$

Lemma 4.2. *Let $\partial\Omega$ be $C^{2,\alpha}(\partial\Omega)$ with $\alpha \in (0,1)$ and u be in $E \cap L^\infty(\Omega)$ such that $\operatorname{div}(a(x)|Du|^{p-2}Du) \in L^\infty(\Omega)$, then $u \in C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$ and*

$$\|u\|_{C^{1,\beta}(\overline{\Omega})} \le K\big(N,\ p,\ \|u\|_{L^\infty(\Omega)}, \|\operatorname{div}(a(x)|Du|^{p-2}Du)\|_{L^\infty(\Omega)}\big).$$

The above lemma is similar to [5, Lemma 2.2], and is also a result in [4].

Proof of Theorem 2.2. (1) By Lemma 4.1 we know that

$$\|u\|_{L^{q_n}(\Omega)} \le C, \quad \|u\|_{L^{s' q_n}(\partial\Omega)} \le C, \quad \forall n \ge n_0.$$

then we obtain

$$\|u\|_{L^\infty(\Omega)} \le \lim_{n \to +\infty} \sup \|u\|_{L^{q_n}(\Omega)} \le C,$$
$$\|u\|_{L^\infty(\partial\Omega)} \le \lim_{n \to +\infty} \sup \|u\|_{L^{s' q_n}(\partial\Omega)} \le C.$$

Hence, (1) of Theorem 2.2 is proved.

(2) By (1) of Theorem 2.2, we obtain that the solution u is in $E \cap L^\infty(\Omega)$. Using $\|\operatorname{div}(a(x)|Du|^{p-2}Du)\|_{L^\infty(\Omega)} = \|u\|_{L^\infty(\Omega)}^{p-1}$, we have $\operatorname{div}(a(x)|Du|^{p-2}Du) = |u|^{p-2}u \in L^\infty(\Omega)$. So u is in $C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$ and $\|u\|_{C^{1,\beta}(\overline{\Omega})} \le K(N,\ p,\ \|u\|_{L^\infty(\Omega)})$. Indeed, we have $\|u\|_{L^\infty(\Omega)} \le C$ for $1 < p < N$, where C depends on G, $\|u\|_{L^{s' q_0}(\partial\Omega)}$, and $\|V\|_{L^s(\partial\Omega)}$, then we have

$$K = K\big(p, N, G, \|u\|_{L^{s' q_0}(\partial\Omega)}, \|V\|_{L^s(\partial\Omega)}\big).$$

$\square$

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