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# CROSS-CONSTRAINED PROBLEMS FOR NONLINEAR SCHRÖDINGER EQUATION WITH HARMONIC POTENTIAL

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Abstract. This article studies a nonlinear Schödinger equation with harmonic potential by constructing different cross-constrained problems. By comparing the different cross-constrained problems, we derive different sharp criterion and different invariant manifolds that separate the global solutions and blowup solutions. Moreover, we conclude that some manifolds are empty due to the essence of the cross-constrained problems. Besides, we compare the three cross-constrained problems and the three depths of the potential wells. In this way, we explain the gaps in [J. Shu and J. Zhang, Nonlinear Shrödinger equation with harmonic potential, Journal of Mathematical Physics, 47, 063503 (2006)], which was pointed out in [R. Xu and Y. Liu, Remarks on nonlinear Schrödinger equation with harmonic potential, Journal of Mathematical Physics, 49, 043512 (2008)].

## 1. Introduction

In this paper, we study the following initial-value problem for the nonlinear Schödinger equation with harmonic potential:

$$\begin{gathered} i\varphi_t + \Delta\varphi - |x|^2\varphi + |\varphi|^{p-1}\varphi = 0, \quad t > 0,\ x \in \mathbb{R}^N, \\ \varphi(0,x) = \varphi_0(x). \end{gathered} \tag{1.1}$$

Hereafter we will use the following notation: $\varphi(x,t) : \mathbb{R}^N \times [0,T_a) \to \mathbb{C}$ is a complex valued wavefunction; $0 < T_a \leqslant +\infty$ is the maximal existence time; $N$ is the space dimension; $i = \sqrt{-1}$; $\Delta$ is the Laplace operator on $\mathbb{R}^N$; $p$ is the exponent of the nonlinear function, $\frac{4}{N} + 1 < p \leqslant \frac{N+2}{N-2}$; $\|\cdot\|_{H^1}$ is the norm of $H^1(\mathbb{R}^N)$; $\|\cdot\|_{L^p}$ is the norm of $L^p(\mathbb{R}^N)$; $\int \cdot dx = \int_{\mathbb{R}^N} \cdot dx$; $C$ is a positive constant that varies from expression to expression.

Note a more general form of (1.1) is

$$\begin{gathered} i\varphi_t + \Delta\varphi - V(x)\varphi + |\varphi|^{p}\varphi = 0, \quad t > 0,\ x \in \mathbb{R}^N, \\ \varphi(0,x) = \varphi_0(x). \end{gathered} \tag{1.2}$$

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It is well-known that

$$\begin{gathered} i\varphi_t + \Delta\varphi + |\varphi|^{p-1}\varphi = 0, \quad t > 0,\ x \in \mathbb{R}^N, \\ \varphi(0,x) = \varphi_0(x), \end{gathered} \tag{1.3}$$

is one of the basic evolution models for nonlinear waves in various branches of physics. Many papers have studied equation (1.3). In [8], Ginibre and Velo established the local existence of the Cauchy problems in the energy space $H^1(\mathbb{R}^N)$. Glassey [10], Tsutsumi [21], Ogawa and Tsutsumi [15, 16] proved that for some initial data, especially for a class of sufficiently large data, the solutions of the Cauchy problem for (1.3) blow up in finite time. Strauss and Cazenave also mentioned this topic in their monographs [20] and [4] respectively. There are also many mathematicians who addressed these problems with harmonic potential. It is found that for sufficiently small initial data, the solutions of the Cauchy problem for (1.3) globally exist (cf. [9, 12, 13, 7, 11], etc). Zhang [26] studied the global existence of (1.3) and the relationship between the Schrödinger equation and its ground state. For (1.2), Fujiwara [6] proved the smoothness of Schrödinger kernel for potentials of quadratic growth. It is shown that quadratic potentials are the highest order potentials for local well-posedness of the equation [17]. Yajima [25] showed that for super-quadratic potentials, the Schrödinger kernel is nowhere $C^1$.

When $p > 1 + 4/N$, Cazenave [4], Tsurumi and Wadati [22] and Carles [2, 3] showed that the solutions of the Cauchy problem of (1.1) blow up in finite time for some initial data, especially for a class of sufficiently large initial data; while the solutions of the Cauchy problem of (1.1) globally exist for other initial data, especially for a class of sufficiently small initial data, see [2], [3] and [22]. When $1 < p < 1 + 4/N$, Zhang [27] proved that global solutions of the Cauchy problem of (1.1) exist for any initial data in the energy space. When $p = 1 + 4/N$, Zhang [28] showed that there exists a sharp condition of the global existence. In [1], Chen and Zhang derived a global existence condition for the supercritical case for (1.1). Moreover, Shu and Zhang [19] also studied (1.1) for its global existence and blowup.

Shu and Zhang [19] studied (1.1) by constructing a cross-constrained problem, which originated from [29]. The main idea of the cross-constrained variational method introduced in [29] can be described as follows. In the energy functional, there are more than two terms, like $\int |\nabla\varphi|^2 dx$, $\int |x|^2|\varphi|^2 dx$ and $\int |\varphi|^{p+1} dx$ for problem (1.1). It is well-known that the "nonlinear source" is controlled by the "potential energy", using the variational method. If the "potential energy" is not as simple as being composed of just one term, then one can give some various combinations of the terms and consider different cases of these combinations. For instance, for problem (1.1), we can use various combinations of the terms $\int |\nabla\varphi|^2 dx$, $\int |x|^2|\varphi|^2 dx$ and $\int |\varphi|^2 dx$ to control the nonlinear term $\int |\varphi|^{p+1} dx$. Then we define the corresponding Nehari functional and potential energy functional to construct the variational problem, which is the so-called cross-constrained potential well method. This approach seems to work in the sense of finding the relationships between these different terms or functionals. But sometimes it may also arouse some confusion because of the complex structure. It may explain the occurrence of some self-contradiction criteria in [19]. Although Xu and Liu pointed out the self-contradiction in [24], they still never make a clear statement about the relationships among these different so-called cross-constrained problems. In other words, Xu and Liu just found the problem, but they did not clarify the essence behind it. So in this paper we mainly aim at a comprehensive study of the so-called cross-constrained problems, and finding the relations between the cross-constrained functionals and cross-constrained manifolds. Then occurrence of all of the gaps and problems mentioned above can be well explained. In the end, we simply illustrate the spatial structure by three concentric spheres.

In this paper, we study the Cauchy problem of (1.1) by constructing different cross-constrained problems and therefore derive different sharp criteria for both global existence and blowup. Moreover, we compare three different invariant manifolds defined in order to separate the global solutions and the blowup solutions of the Cauchy problem (1.1). We also dig the reason that some invariant manifolds are empty, which was previously pointed out in [24].

The organization of this paper is as follows. In Section 2, we give some concerned preliminaries. In Section 3, we construct three variational problems and invariant manifolds. In Section 4, we derive sharp criteria for both global existence and blowup. In Section 5, we compute the potential depth of one of the potential wells. In Section 6, we compare the three variational problems, point out some of the invariant manifolds are empty and explain why such phenomenon happens. Further we reveal the relationships among these different cross-constrained variational problems and the different manifolds.

## 2. Preliminaries

In this section, we like to introduce some functionals and a Hilbert space, which will be used to construct different cross-constrained problems. For (1.1), we first equip the following space

$$H = \big\{\psi \in H^1(\mathbb{R}^N) : \int |x|^2|\psi|^2 dx < \infty\big\} \tag{2.1}$$

with the inner product

$$\langle \psi, \phi\rangle := \int \nabla\psi\nabla\overline{\phi} + \psi\overline{\phi} + |x|^2\psi\overline{\phi} dx, \tag{2.2}$$

whose associated norm is $\|\cdot\|_H$.

Further we define the energy functional

$$E(\varphi) = \int \frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}|x|^2|\varphi|^2 - \frac{1}{p+1}|\varphi|^{p+1} dx, \tag{2.3}$$

and the following four auxiliary functionals

$$P(\varphi) = \int \frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}|\varphi|^2 + \frac{1}{2}|x|^2|\varphi|^2 - \frac{1}{p+1}|\varphi|^{p+1} dx, \tag{2.4}$$

$$I_1(\varphi) = \int |\nabla\varphi|^2 + |\varphi|^2 + |x|^2|\varphi|^2 - \frac{N(p-1)}{2(p+1)}|\varphi|^{p+1} dx, \tag{2.5}$$

$$I_2(\varphi) = \int |\nabla\varphi|^2 + |\varphi|^2 - \frac{N(p-1)}{2(p+1)}|\varphi|^{p+1} dx, \tag{2.6}$$

$$I_3(\varphi) = \int |\nabla\varphi|^2 + |x|^2|\varphi|^2 - \frac{N(p-1)}{2(p+1)}|\varphi|^{p+1} dx. \tag{2.7}$$

In the above four functionals, $P(\varphi)$ is composed of both energy and mass. And $I_i(\varphi)$ $(i = 1, 2, 3)$ can be considered as Nehari functionals. Throughout this paper, we assume

$$
\begin{aligned}
&1+\frac{4}{N}<p<\frac{N+2}{N-2}, \quad \text{for } N\geqslant 3;\\
&1+\frac{4}{N}<p<+\infty, \quad \text{for } N=1,2.
\end{aligned}
\tag{2.8}
$$

We now state the local well-posedness.

**Lemma 2.1** ([14])**.** *Let $\varphi_0\in H$. Then there exists a unique solution $\varphi$ of the Cauchy problem* (1.1) *in $C([0,T];H)$ for some $T\in(0,\infty]$ (maximal existence time), and either $T=\infty$ (global existence) or else $T<\infty$ and*

$$
\lim_{t\to T}\|\varphi\|_H=\infty \quad \text{(blowup)}.
$$

Now we have the conservation laws for both energy and mass.

**Lemma 2.2** ([5, 10, 23])**.** *Let $\varphi_0\in H$ and $\varphi$ be a solution of the Cauchy problem* (1.1) *in $C([0,T];H)$. Then one has*

$$
\int|\varphi|^2dx=\int|\varphi_0|^2dx, \tag{2.9}
$$

$$
E(\varphi)\equiv E(\varphi_0), \tag{2.10}
$$

$$
P(\varphi)\equiv P(\varphi_0). \tag{2.11}
$$

We introduce the following lemma, which will be used for proving the blowup phenomenon in Section 4.

**Lemma 2.3.** *Let $\varphi_0\in H$ and $\varphi$ be a solution of the Cauchy problem* (1.1) *in $C([0,T];H)$, Set $J(t)=\int|x|^2|\varphi|^2dx$. Then one has*

$$
J''(t)=8\int\Big(|\nabla\varphi|^2-|x|^2|\varphi|^2-\frac{N(p-1)}{2(p+1)}|\varphi|^{p+1}\Big)dx. \tag{2.12}
$$

## 3. Three variational problems and invariant manifolds

First we define the following three Nehari manifolds,

$$
\begin{aligned}
M_1&:=\{\psi\in H\backslash\{0\}:I_1(\psi)=0\},\\
M_2&:=\{\psi\in H\backslash\{0\}:I_2(\psi)=0\},\\
M_3&:=\{\psi\in H\backslash\{0\}:I_3(\psi)=0\}.
\end{aligned}
$$

Now we consider the following cross-constrained problems

$$
d_1=\inf_{\psi\in M_1}P(\psi), \tag{3.1}
$$

$$
d_2=\inf_{\psi\in M_2}P(\psi), \tag{3.2}
$$

$$
d_3=\inf_{\psi\in M_3}P(\psi), \tag{3.3}
$$

respectively. First we have the following lemma.

**Lemma 3.1.** *$d_i>0$ for $i=1,2,3$.*

*Proof.* (i) For any $\varphi\in M_1\cup M_2$, we have

$$
\int|\nabla\varphi|^2+|\varphi|^2dx\leqslant\frac{N(p-1)}{2(p+1)}\int|\varphi|^{p+1}dx.
$$

By Sobolev embedding inequality, this implies

$$\int |\nabla\varphi|^2 + |\varphi|^2 dx \geqslant C.$$

Note by assumption (2.8),

$$\frac{1}{2} - \frac{1}{p+1} \cdot \frac{2(p+1)}{N(p-1)} > 0.$$

Hence $P(\varphi) \geqslant C > 0$, which verifies $d_i > 0$ $(i = 1, 2)$.

(ii) For $\varphi \in M_3$, we have

$$\|\nabla\varphi\|_2^2 \leqslant \frac{N(p-1)}{2(p+1)} \int |\varphi|^p dx,$$

which implies from Gagliardo-Nirenberg inequality and Cauchy-Schwartz inequality that there exists a constant $C(p, N) > 0$ such that

$$\begin{aligned} C(p,N) &\leqslant \|\nabla\varphi\|_2^{\frac{Np-(N+4)}{2}} \cdot \|\varphi\|_2^{\frac{(N+2)-(N-2)p}{2}} \\ &\leqslant \frac{1}{2}\Big(\|\nabla\varphi\|_2^{Np-(N+4)} + \|\varphi\|_2^{(N+2)-(N-2)p}\Big). \end{aligned}$$

This yields $\|\nabla\varphi\|_2 \geqslant C > 0$ or $\|\varphi\|_2 \geqslant C > 0$. Thus

$$P(\varphi) = \frac{1}{2}\|\varphi\|_2^2 + \frac{Np-(N+4)}{2N(p-1)}\Big[\|\nabla\varphi\|_2 + \int |x|^2|\varphi|^2 dx\Big] \geqslant C > 0,$$

which proves $d_3 > 0$. □

Next we give the invariance of some manifolds.

**Theorem 3.2.** *For $i = 1, 2, 3$, define*

$$\mathcal{G}_i := \{\psi \in H : P(\psi) < d_i,\ I_i(\psi) > 0\} \cup \{0\} \tag{3.4}$$

*Then $\mathcal{G}_i$ is an invariant manifold of* (1.1)*; that is, if $\varphi_0 \in \mathcal{G}_i$, then the solution $\varphi(x,t)$ of the Cauchy problem* (1.1) *also satisfies $\varphi(x,t) \in \mathcal{G}_i$ for any $t \in [0,T)$.*

*Proof.* If $\varphi_0 = 0$, from the mass conservation law; i.e., (2.9), we can find that $\varphi = 0$ for $t \in [0,T)$; i.e., $\varphi(x,t) \in \mathcal{G}_i$. If $\varphi_0 \neq 0$, we have $\varphi_0 \in \mathcal{G}_i\backslash\{0\}$; i.e., $P(\varphi_0) < d_i$ and $I_i(\varphi_0) > 0$. By Lemma 2.1, there exists a unique $\varphi(x,t) \in C([0,T); H)$ with $0 < T \leqslant \infty$ such that $\varphi(x,t)$ is a solution of problem (1.1). Now we shall show that $\varphi(x,t) \in \mathcal{G}_i$ for any $t \in [0,T)$. By (2.11), we have

$$P(\varphi(x,t)) = P(\varphi_0) \geqslant d_i. \tag{3.5}$$

Next we show $I_i(\varphi) > 0$ for $t \in [0,T)$. Note that $I_i(\varphi_0) > 0$. Arguing by contradiction, by the continuity of $I_i(\varphi)$, suppose that there were a $t_2 \in [0,T)$ such that $I_i(\varphi(x,t_2)) = 0$. If $\varphi(x,t_2) = 0$, then by (2.9), we have $0 = \int |\varphi(x,t_2)|^2 dx = \int |\varphi_0|^2 dx$, which indicates $\varphi_0 = 0$. Contradiction. So $\varphi(x,t_2) \neq 0$, by the definition of $d_i$, we have $P(\varphi(x,t_2)) \geqslant d_i$, which contradicts (3.5). Therefore $I_i(\varphi) > 0$ for all $t \in [0,T)$.

Combining all of the analysis above, we arrive at $\varphi(x,t) \in \mathcal{G}_i$ for any $t \in [0,T)$. The proof is complete. □

By a similar argument, we can obtain the following result.

**Theorem 3.3.** *For $i = 1, 2, 3$, define*

$$\mathcal{B}_i := \{\psi \in H : P(\psi) < d_i,\, I_i(\psi) < 0\}$$

*Then $\mathcal{B}_i$ is an invariant manifold of* (1.1).

## 4. Sharp conditions for global existence

**Theorem 4.1.** *If $\varphi_0 \in \mathcal{G}_i$ $(i = 1, 2, 3)$, then the solution $\varphi(x, t)$ of the Cauchy problem* (1.1) *globally exists on $t \in [0, \infty)$.*

*Proof.* Here we prove only the case $\varphi_0 \neq 0$, for $\varphi_0 = 0$ is a trivial case. For any nontrivial $\varphi_0 \in \mathcal{G}_i$ $(i = 1, 2, 3)$, let $\varphi_i(x, t)$ be the solution of the Cauchy problem (1.1) with initial condition $\varphi_i(x, 0) = \varphi_0$, and $0 < T \leqslant \infty$ be the maximal existence time. It follows from Theorem 3.2 that $\varphi_i(x, t) \in \mathcal{G}_i$ $(i = 1, 2, 3)$ for all $t \in [0, T)$. Fix $t \in [0, T)$, and simply denote $\varphi_i(x, t)$ by $\varphi_i$, then the definition of $\mathcal{G}_i$ implies that

$$d_i > P(\varphi_i),\ I_i(\varphi_i) > 0\ (i = 1, 2, 3).$$

For $i = 1, 2, 3$, it always follows from $I_i(\varphi_i) > 0$ that

$$\frac{1}{p+1}|\varphi_i|^{p+1}dx < \frac{2}{N(p-1)}\int |\nabla\varphi_i|^2 + |\varphi_i|^2 + |x|^2|\varphi_i|^2 dx$$

Thus we obtain

$$\begin{aligned} d_i &> P(\varphi_i) \\ &= \int \frac{1}{2}|\nabla\varphi_i|^2 + \frac{1}{2}|\varphi_i|^2 + \frac{1}{2}|x|^2|\varphi_i|^2 - \frac{1}{p+1}|\varphi_i|^{p+1}dx \\ &> \Big(\frac{1}{2} - \frac{2}{N(p-1)}\Big)\int |\nabla\varphi_i|^2 + |\varphi_i|^2 + |x|^2|\varphi_i|^2 dx, \end{aligned} \tag{4.1}$$

which yields

$$\int |\nabla\varphi_i|^2 + |\varphi_i|^2 + |x|^2|\varphi_i|^2 dx < \frac{2N(p-1)d_i}{N(p-1)-4}.$$

Therefore, it follows from Lemma 2.1 that $\varphi$ globally exists on $t \in [0, \infty)$. At this point, we proved this theorem. □

**Theorem 4.2.** *If $\varphi_0 \in \mathcal{B}_i$ $(i = 1, 2, 3)$, then the solution $\varphi(x, t)$ of the Cauchy problem* (1.3) *blows up in finite time.*

*Proof.* We prove this theorem case by case.
**Case I**: $\varphi_0 \in \mathcal{B}_1 \cup \mathcal{B}_2$. In this case, Theorem 3.2 implies that the solution $\varphi(x, t)$ of the Cauchy problem (1.3) satisfies that $\varphi(x, t) \in \mathcal{B}_1 \cup \mathcal{B}_2$ for $t \in [0, T)$. For $J(t) = \int |x|^2|\varphi|^2 dx$, the definitions of $P(\varphi)$ and $I_i(\varphi)$ $(i = 1, 2)$ imply that

$$J''(t) < -8\int |\varphi|^2 dx. \tag{4.2}$$

Then

$$J'(t) < J'(0) - 8\Big(\int |\varphi|^2 dx\Big)\, t.$$

Further we have

$$J(t) < J(0) + J'(0)t - 4\Big(\int |\varphi|^2 dx\Big)t^2.$$

Note that $I_1(\varphi) < 0$ yields $\int |\varphi|^2 dx > 0$. Therefore there exists a $T_1 \in (0, \infty)$ such that $J(t) > 0$ for $t \in [0, T_1)$ and $J(T_1) = 0$. By the inequality (see [23])

$$\|\varphi\|^2 \leqslant \frac{2}{N} \|\nabla \varphi\| \cdot \|x\varphi\|$$

we obtain $\lim_{t \to T_1} \|\nabla \varphi\| = \infty$, which indicates

$$\lim_{t \to T_1} \|\varphi\|_H = \infty.$$

**Case ii**: $\varphi_0 \in \mathcal{B}_3$. In this case, Theorem 3.2 implies that the solution $\varphi(x,t)$ of the Cauchy problem (1.3) satisfies that $\varphi(x,t) \in \mathcal{B}_3$ for $t \in [0,T)$. For $J(t) = \int |x|^2 |\varphi|^2 dx$, (2.4) and (2.7) imply that

$$J''(t) < -16 \int |x|^2 |\varphi|^2 dx. \tag{4.3}$$

Now we show that there exists a $T_1 \in (0, \infty)$ such that $J(t) > 0$ for $t \in [0, T_1)$ and $J(T_1) = 0$. Arguing by contradiction, suppose $\forall t \in [0, \infty)$, $J(t) > 0$. Set

$$g(t) = \frac{J'(t)}{J(t)}.$$

It is easy to show that

$$g'(t) = \frac{J''(t)}{J(t)} - \Big(\frac{J'(t)}{J(t)}\Big)^2 < -16 - g^2(t). \tag{4.4}$$

Next we like to show $g(t) \neq 0$ for any $t \in [0, \infty)$. Arguing by contradiction again, suppose there is a $t_0$ such that $g(t_0) = 0$. By (4.4), we have $g(t) < 0$ for $t \in (t_0, \infty)$. For any fixed $t_1 > t_0$, dividing (4.4) by $g^2(t)$, we have

$$\frac{g'(t)}{g^2(t)} < -\frac{16}{g^2(t)} - 1 < -1.$$

Further we derive

$$\int_{t_1}^t \frac{g'(\tau)}{g^2(\tau)} d\tau < \int_{t_1}^t -1 d\tau,$$

namely,

$$\frac{1}{g(t)} > \frac{1}{g(t_1)} + (t - t_1), \tag{4.5}$$

which indicates that there exists a $t_2 > t_1$ such that

$$g(t) > 0 \quad \text{for any } t \in (t_2, \infty) \tag{4.6}$$

This contradicts $g(t) < 0$ for $t \in (t_0, \infty)$. Hence we have $g(t) \neq 0$ for any $t \in [0, \infty)$. By (4.5), for $t \in (0, \infty)$ , we have

$$\frac{1}{g(t)} > \frac{1}{g(0)} + t.$$

Hence, $J'(t) > 0$ for $t \in (|\frac{1}{g(0)}|, \infty)$. Therefore $J(t)$ is increasing in $(|\frac{1}{g(0)}|, \infty)$. Let $t_0 = |\frac{1}{g(0)}|$. By

$$J''(t) < -16J(t) < 0,$$

we have for $t > t_0$,

$$J'(t) < J'(t_0) + 16J(t_0)t_0 - 16J(t_0)t.$$

Further

$$J(t) - J(0) < (J'(t_0) + 16J(t_0)t_0)t - 8J(t_0)t^2;$$

i.e.,

$$J(t) < J(0) + (J'(t_0) + 16J(t_0)t_0)t - 8J(t_0)t^2,$$

which implies that there exists $0 < T_1 \leqslant t_0$ such that $J(t) > 0$ for $t \in [0, T_1)$ and $J(T_1) = 0$. Again by the inequality (see [23])

$$\|\varphi\|^2 \leqslant \frac{2}{N}\|\nabla\varphi\| \cdot \|x\varphi\|$$

we obtain

$$\lim_{t\to T_1} \|\nabla\varphi\| = \infty.$$

So far we have shown that for the initial data $\varphi \in \mathcal{B}_i$, the solution of the Cauchy problem (1.1) blows up in finite time. This completes the proof of the theorem. □

**Remark 4.3.** It is clear that

$$\{\psi \in H, P(\psi) < d_i\} = \mathcal{G}_i \cup \mathcal{B}_i,$$

which indicates Theorem 4.1 and Theorem 4.2 are sharp.

By Theorem 3.3, we obtain another condition for global existence of the solution of (1.1).

**Corollary 4.4.** *If $\varphi_0$ satisfies $\|\varphi_0\|_H^2 < 2d_i$, then the solution $\varphi$ of the Cauchy problem* (1.1) *globally exists on $t \in [0, \infty)$.*

*Proof.* We consider only the nontrivial case. Suppose $\varphi_0 \neq 0$, from $\|\varphi_0\|_H^2 < 2d_i$, we have $P(\varphi_0) < d_i$. Moreover, we claim that $I_i(\varphi_0) > 0$. Otherwise, there is a $0 < \mu \leqslant 1$ such that $I_i(\mu\varphi_0) = 0$. Thus $P(\mu\varphi_0) \geqslant d_i$. On the other hand,

$$\|\mu\varphi_0\|_H^2 = \mu^2\|\varphi_0\|_H^2 < 2\mu^2 d_i < 2d_i.$$

It follows that $P(\mu\varphi_0) < d_i$. This is a contradiction. Therefore we have $\varphi_0 \in \mathcal{G}_i$. Thus Theorem 3.3 implies this corollary. □

## 5. Computing $d_1$

In this section, we compute $d_1$ using the method definded by Payne and Sattinger [18]. Since $M_1$ is a closed nonempty set, there exists an $\omega_1 \in M_1$ such that

$$P(\omega_1) = \inf_{\psi\in M_1} P(\psi) = d_1,$$

where $\omega_1$ is a solution of the following Euler equation

$$\Delta\omega - |x|^2\omega + |\omega|^{p-1}\omega - \omega = 0.$$

We define

$$\begin{aligned} C_{p,1} &= \frac{\left(\int |\nabla\omega_1|^2 + |\omega_1|^2 + |x|^2|\omega_1|^2 dx\right)^{1/2}}{\left(\int |\omega_1|^{p+1} dx\right)^{\frac{1}{p+1}}} \\ &= \inf_{\psi\in M_1} \frac{\left(\int |\nabla\psi|^2 + |\psi|^2 + |x|^2|\psi|^2 dx\right)^{1/2}}{\left(\int |\psi|^{p+1} dx\right)^{\frac{1}{p+1}}} \end{aligned}$$

be the Sobolev constant from $H$ to $L^{p+1}(\mathbb{R}^N)$. By $I_1(\omega_1) = 0$, we derive

$$d_1 = P(\omega_1) = \frac{N(p-1)-4}{4(p+1)} \int |\omega_1|^{p+1} dx. \tag{5.1}$$

Using the definition of $C_{p,1}$, we obtain

$$\Big(\int|\omega_1|^{p+1}dx\Big)^{\frac{1}{p+1}}=\Big(\frac{2(p+1)C_{p,1}^2}{N(p-1)}\Big)^{\frac{1}{p-1}}. \tag{5.2}$$

Combining (5.1) and (5.2), we have

$$d_1=\frac{N(p-1)-4}{4(p+1)}\Big(\frac{2(p+1)C_{p,1}^2}{N(p-1)}\Big)^{\frac{p+1}{p-1}}. \tag{5.3}$$

## 6. Cross-constrained problems and some trivial manifolds

In this section, we like to address the relations of some cross-constrained various manifolds for problem (1.1).

Next we define three manifolds as follows. First, we define some new potential well depths. For $i<j$, $i,j=1,2,3$,

$$d_{i,j}=\min\{d_i,d_j\} \tag{6.1}$$

and

$$d_{1,2,3}=\min\{d_1,d_2,d_3\}. \tag{6.2}$$

Second, we define the following manifolds. Note similar as the proof of Theorem 3.2, it is trivial to show that these manifolds are also invariant. For $i<j$, $i,j=1,2,3$,

$$\mathcal{G}_{i,j}:=\{\psi\in H:P(\psi)<d_{i,j},I_i(\psi)>0,I_j(\psi)>0\}\cup\{0\}, \tag{6.3}$$

$$\mathcal{V}_{+i,-j}:=\{\psi\in H:P(\psi)<d_{i,j},I_i(\psi)>0,I_j(\psi)<0\}, \tag{6.4}$$

$$\mathcal{V}_{-i,+j}:=\{\psi\in H:P(\psi)<d_{i,j},I_i(\psi)<0,I_j(\psi)>0\}, \tag{6.5}$$

$$\mathcal{B}_{i,j}:=\{\psi\in H:P(\psi)<d_{i,j},I_i(\psi)<0,I_j(\psi)<0\}. \tag{6.6}$$

We have the following theorems to clarify the relations among all the invariant manifolds.

**Theorem 6.1.** *$\mathcal{G}_{i,j}=\mathcal{G}_i\cap\mathcal{G}_j$; $\mathcal{V}_{+i,-j}=\emptyset$; $\mathcal{V}_{-i,+j}=\emptyset$; $\mathcal{B}_{i,j}=\mathcal{B}_i\cap\mathcal{B}_j$, where $i<j$ and $i,j=1,2,3$.*

**Theorem 6.2.** *For $i<j$ and $i,j=1,2,3$, define*

$$\mathcal{G}_{ij}:=\{\psi\in H:P(\psi)<d_{i,j},I_j(\psi)>0\},$$
$$\mathcal{B}_{ij}:=\{\psi\in H:P(\psi)<d_{i,j},I_j(\psi)<0\}.$$

*Then*

$$\mathcal{G}_{1i}\subset\mathcal{G}_{i1},\quad \mathcal{B}_{i1}\subset\mathcal{B}_{1i}.$$

**Remark 6.3.** Theorem 6.1 can well explain the gaps in [19], which was pointed out in [24].

**Remark 6.4.** It is easy to see that the reason for $\mathcal{V}_{-1,+i}=\emptyset$ $(i=2,3)$ is due to the fact $I_1(\varphi)>I_i$ $(i=2,3)$ for $\varphi\neq 0$. Now we like to analyze why $\mathcal{V}_{+2,-3}=\emptyset$, $\mathcal{V}_{-2,+3}=\emptyset$, $\mathcal{V}_{+1,-i}=\emptyset$ for $i=2,3$. In fact, we know that $\mathcal{V}_{+1,-2}$ is a subset of both $\mathcal{G}_1$ and $\mathcal{B}_2$. We have proved that $\mathcal{G}_1$ is a manifold of all the global solutions of the Cauchy problem (1.1) for $P(\varphi)<d_1$ while $\mathcal{B}_2$ is a manifold of all the blowup solutions of the Cauchy problem (1.1) for $P(\varphi)<d_2$. Hence by Lemma 2.1, there should be no intersection of the two manifolds, which indicates $\mathcal{V}_{+1,-2}=\emptyset$. Hence it is natural to deduce that the two surfaces

$$S_{12,1}:=\{\psi\in H:P(\psi)<d_{1,2},I_1(\psi)=0\}$$

and

$$S_{12,2} := \{\psi \in H : P(\psi) < d_{1,2}, I_2(\psi) = 0\}$$

coincide. Similarly,

$$S_{13,1} := \{\psi \in H : P(\psi) < d_{1,3}, I_1(\psi) = 0\}$$

and

$$S_{13,3} := \{\psi \in H : P(\psi) < d_{1,3}, I_3(\psi) = 0\}$$

coincide. Further we define the following three Nehari manifolds

$$S_i := \{\psi \in H : P(\psi) < d_{1,2,3}, I_i(\psi) = 0\},$$

for $i = 1, 2, 3$. It is easy to see that $S_1 = S_2 = S_3$.

To have an intuitive feeling of the relations among some of the above manifolds, we draw the Figures 1 and 2.

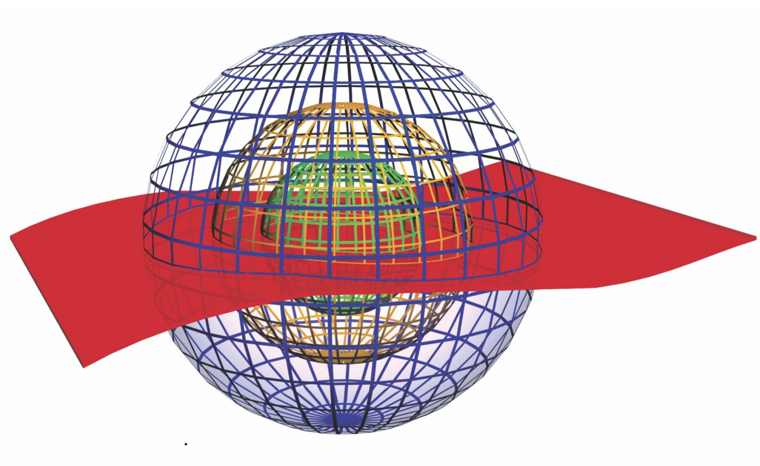

FIGURE 1. The relations among $I_1(\varphi)$, $I_2(\varphi)$ and $I_3(\varphi)$, where $d_1 > d_2 > d_3$; The intersections of the three spheres and red surface represent the three manifolds $I_i(\varphi) = 0$ $(i = 1, 2, 3)$, respectively.

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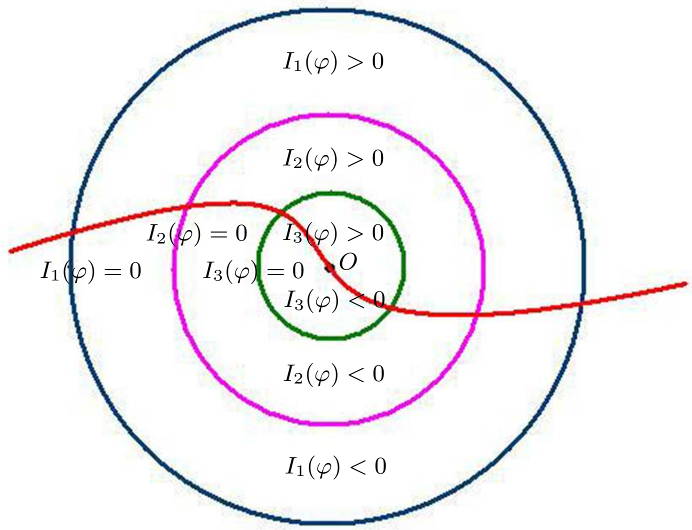


Figure 2. Cross section for Figure 1

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