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TRANSPORT EQUATION FOR GROWING BACTERIAL POPULATIONS (II)

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Abstract. This article studies the growing bacterial population. Each bacterium is distinguished by its degree of maturity and its maturation velocity. To complete the study in [3], we describe the bacterial profile of this population by proving that the generated semigroup possesses an asynchronous exponential growth property.

1. Introduction

This work studies a model for growing bacterial population, partially studied in [3], in which each bacteria is distinguished by its degree of maturity $0 \leq \mu \leq 1$ and its maturation velocity v. As each bacteria may not become less mature, then its maturation velocity must be positive ($0 \leq a < v < \infty$). If $f = f(t, \mu, v)$ denotes the bacterial density with respect to the degree of maturity μ and the maturation velocity v at time t, then

$$\frac{\partial f}{\partial t} = -v\frac{\partial f}{\partial \mu} - \sigma f, \tag{1.1}$$

where $\sigma = \sigma(\mu, v)$ denotes the rate of bacterial mortality or bacteria loss due to causes other than division.

At any time t, the density of mothers bacteria $f(t, 0, \cdot)$ is related to that of daughters bacteria $f(t, 1, \cdot)$ by biological laws, such as the *transition law* mathematically described by the following boundary condition

$$vf(t, 0, v) = p\int_a^\infty k(v, v')f(t, 1, v')v'\,dv', \tag{1.2}$$

where $k = k(v, v')$ denotes the correlation kernel between the maturation velocity of a mother bacteria v' and that of a daughter bacteria v, and, $p \geq 0$ denotes the average number of daughter bacteria viable per mitotic. However, for more generality, we are going to be concerned by a general biological law mathematically described by the following boundary condition

$$f(t, 0, v) = [Kf(t, 1, \cdot)](v), \tag{1.3}$$

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where K denotes a linear operator on suitable spaces (see Section 3).

Recently, we partially studied the model (1.1), (1.3) in the most interesting case $a = 0$ (see [3]). We proved that this model is governed by a strongly continuous semigroup $V_K = (V_K(t))_{t\geq 0}$. The purpose of this work is then to complete [3] by studying the asynchronous exponential growth of the generated semigroup $V_K = (V_K(t))_{t\geq 0}$.

According to the case $a > 0$, we have recently proved in [1] that the strongly continuous semigroup $V_K = (V_K(t))_{t\geq 0}$ is compact for a large time $t > \frac{2}{a}$ which led to an easy computation of the essential type ($\omega_{\rm ess}(V_K) = -\infty$). Biologically speaking, the case $a > 0$ means that after a transitory phase, all bacteria will be divided or dead.

In contrary to $a > 0$, the case $a = 0$ means that the maturation velocities can be arbitrary small and at any time there may be bacteria that are not yet divided. Consequently, the bacterial population never goes out of the transitory phase, which explains the non-compactness of the generated semigroup $V_K = (V_K(t))_{t\geq 0}$ and therefore the computation of its essential type, $\omega_{\rm ess}(V_K)$, presents a lot of difficulties.

In the sequel we organize the work as follows

(3) Generation Theorem
(4) Stability and Domination
(5) Asynchronous Exponential Growth

In third Section, we recall some properties of the generated semigroup $V_K = (V_K(t))_{t\geq 0}$ governing the model (1.1), (1.3). We also complete some claims already proved in [3]. In Fourth Section, we study the stability of the generated semigroup $V_K = (V_K(t))_{t\geq 0}$ with respect to the boundary operator K. Domination result is also given. In fifth Section, we prove that the generated semigroup possesses Asynchronous Exponential Growth property as follows

Lemma 1.1 ([4, Theorems 9.10 and 9.11])**.** *Let $U = (U(t))_{t\geq 0}$ be a positive and irreducible strongly continuous semigroup, on the Banach lattice space X, satisfying the inequality $\omega_{\rm ess}(U) < \omega_0(U)$. Then, there exist a rank one projector $\mathbb{P}$ into X and an $\varepsilon > 0$ such that: for any $\eta \in (0, \varepsilon)$, there exists $M(\eta) \geq 1$ satisfying*

$$\|e^{-\omega_0(U)t}U(t) - \mathbb{P}\|_{\mathcal{L}(X)} \leq M(\eta)e^{-\eta t} \quad t \geq 0.$$

Thanks to [4, Thereom 8.7], the rank one projection $\mathbb{P}$ can be written as follows: $\mathbb{P}\varphi =< \varphi, \varphi_0^* > \varphi_0^*$, where $\varphi_0^* \in (X^*)_+$ is a strictly positive functional. A strongly continuous semigroup $U = (U(t))_{t\geq 0}$ satisfying Lemma 1.1 possesses the *asynchronous exponential growth* with *intrinsic growth density* φ_0^*.

Lemma 1.1 describes the bacterial profile whose privileged direction is mathematically explained by the vector φ_0^*. This is what the biologist observes in his laboratory. Finally, note the novelty of this work. For all used theoretical results, we refer the reader to [4] or [7].

2. Mathematical preliminaries

This section deals with some useful mathematical tools that we will need in the sequel. These tools concern strongly continuous semigroups of linear operators in Banach spaces and Banach lattice spaces.

Let X be a Banach space and let $U = (U(t))_{t\geq 0}$ be a strongly continuous semigroup of linear operators, on X. Following [7, Chapter IV], the *type* $\omega_0(U)$ and the *essential type* $\omega_{\text{ess}}(U)$ of the semigroup $U = (U(t))_{t\geq 0}$ are given by

$$\omega_0(U) = \lim_{t\to\infty} \frac{\ln \|U(t)\|_{\mathcal{L}(X)}}{t} \tag{2.1}$$

$$\omega_{\text{ess}}(U) = \lim_{t\to\infty} \frac{\ln \|U(t)\|_{\text{ess}}}{t}, \tag{2.2}$$

where $\|\cdot\|_{\text{ess}}$ denotes the norm of Calkin algebra $\mathcal{L}(X)/\mathcal{K}(X)$ with $\mathcal{K}(X)$ stands for the two-sided closed ideal in $\mathcal{L}(X)$ of all compact operators. The types $\omega_{\text{ess}}(U)$ and $\omega_{\text{ess}}(U)$ are always ordered as follows

$$\omega_{\text{ess}}(U) \leq \omega_0(U). \tag{2.3}$$

Lemma 2.1. *Let $U = (U(t))_{t\geq 0}$ and $V = (V(t))_{t\geq 0}$ be two strongly continuous semigroups, on X, and let $\lambda \in \rho(U(t_0))$ for some $t_0 > 0$. If*

$$\Big[\Big(V(t_0) - U(t_0)\Big)\Big(\lambda - U(t_0)\Big)^{-1}\Big]^n$$

is a compact operator for some integer $n > 0$, then $\omega_{\text{ess}}(U) = \omega_{\text{ess}}(V)$

Proof. It suffices to apply [8] for the operators $V(t_0) - U(t_0)$ and $U(t_0)$. □

Lemma 2.2 ([6])**.** *Let (Ω, Σ, μ) be a positive measure space and let S and T be linear and bounded operators on $L^1(\Omega, \mu)$. Then*

(1) *The set of all weakly compact operators is norm-closed subset.*
(2) *If T is weakly compact and $0 \leq S \leq T$ then S is weakly compact too.*
(3) *If S and T are weakly compact, then ST is compact.*

3. Generation Theorem

In this section, we recall briefly some properties of the model (1.1), (1.3) and its associated semigroup $V_K = (V_K(t))_{t\geq 0}$ (see [3]). Also, other needed properties will be proved. So, let us consider the functional framework $L^1(\Omega)$ whose norm is

$$\|\varphi\|_1 = \int_\Omega |\varphi(\mu, v)|\, d\mu\, dv, \tag{3.1}$$

where $\Omega = (0,1) \times (0,\infty) := I \times J$. We also consider our regularity space

$$W_1 = \big\{\varphi \in L^1(\Omega)\ \ v\frac{\partial \varphi}{\partial \mu} \in L^1(\Omega) \quad \text{and} \quad v\varphi \in L^1(\Omega)\big\}$$

and the trace space $Y_1 := L^1(J, v\, dv)$ whose norms are

$$\|\varphi\|_{W_1} = \|v\frac{\partial\varphi}{\partial\mu}\|_1 + \|v\varphi\|_1 \quad \text{and} \quad \|\psi\|_{Y_1} = \int_0^\infty |\psi(v)| v\, dv.$$

In this context, our recent result [2, Theorem 2.2] leads to

Lemma 3.1. *The trace mappings $\gamma_0\varphi = \varphi(0,\cdot)$ and $\gamma_1\varphi = \varphi(1,\cdot)$ are linear bounded from W_1 into Y_1.*

Let T_0 be the following unbounded operator

$$\begin{gathered} T_0\varphi = -v\frac{\partial\varphi}{\partial\mu} \quad \text{on the domain,} \\ D(T_0) = \{\varphi \in W_1\ \ \gamma_0\varphi = 0\}, \end{gathered} \tag{3.2}$$

corresponding to the model (1.1), (1.3) without bacterial division and without bacterial mortality. Some of its properties can be summarized as follows.

Lemma 3.2. *The operator T_0 generates, on $L^1(\Omega)$, a strongly continuous positif semigroup $U_0 = (U_0(t))_{t\geq 0}$ of contraction given by*
$$U_0(t)\varphi(\mu, v) := \chi(\mu, v, t)\varphi(\mu - tv, v), \tag{3.3}$$
where
$$\chi(\mu, v, t) = \begin{cases} 1 & \text{if } \mu \geq tv; \\ 0 & \text{if } \mu < tv. \end{cases} \tag{3.4}$$

Next, let us impose on the bacterial mortality rate σ the hypothesis
$$\sigma \in (L^\infty(\Omega))_+ \tag{3.5}$$
which obviously leads to the boundedness of the perturbation operator
$$S\varphi := -\sigma\varphi \tag{3.6}$$
from $L^1(\Omega)$ into itself. In this context, the model (1.1), (1.3), without bacterial division, can be modeled by the following unbounded operator $L_0 := T_0 + S$ on the domain $D(L_0) = D(T_0)$, for which we have

Lemma 3.3. *Suppose that* (3.5) *holds. Then, the operator L_0 generates, on $L^1(\Omega)$, a strongly continuous positif semigroup $V_0 = (V_0(t))_{t\geq 0}$ satisfying*
$$\|V_0(t)\|_{\mathcal{L}(L^1(\Omega))} \leq e^{-t\underline{\sigma}} \quad t \geq 0, \tag{3.7}$$
where
$$\underline{\sigma} := \operatorname{ess\,inf}_{(\mu,v)\in\Omega} \sigma(\mu, v). \tag{3.8}$$
Furthermore,
$$V_0(t) = U_0(t) + \int_0^t U_0(t-s)SV_0(s)ds \quad t \geq 0. \tag{3.9}$$

Proof. L_0 is clearly a perturbation of the generator T_0. □

Let us consider now the model (1.1), (1.3), without cell mortality, corresponding to the following unbounded operator
$$\begin{gathered} T_K\varphi = -v\frac{\partial\varphi}{\partial\mu} \quad \text{on the domain,} \\ D(T_K) = \{\varphi \in W_1 : \gamma_0\varphi = K\gamma_1\varphi\}, \end{gathered} \tag{3.10}$$
where the boundary operator K can fulfil the following definition.

Definition 3.4. Let K be a linear operator from Y_1 into itself. Then, K is said to be an *admissible* if one of the following hypotheses holds

- (Kb) K is bounded and $\|K\|_{\mathcal{L}(Y_1)} < 1$;
- (Kc) K is compact and $\|K\|_{\mathcal{L}(Y_1)} \geq 1$.

Thanks to Definition above, we can state the following.

Lemma 3.5. *Let K be an admissible operator. Then, for all $\lambda \geq 0$, the following linear operators*
$$K_\lambda\psi := K(\theta_\lambda\psi) \quad \textit{and} \quad \overline{K}_\lambda\psi := \theta_\lambda K\psi, \quad \textit{where } \theta_\lambda(v) = e^{-\lambda/v} \tag{3.11}$$
are bounded from Y_1 into itself satisfying
$$\|K_\lambda\|_{\mathcal{L}(Y_1)} < 1 \quad \textit{and} \quad \|\overline{K}_\lambda\|_{\mathcal{L}(Y_1)} < 1 \quad \textit{for all large } \lambda. \tag{3.12}$$

Furthermore, if K is positive then the spectral radius of K_λ and $\overline{K}_\lambda$ are the same; i.e.,

$$r(\overline{K}_\lambda) = r(K_\lambda) \quad \text{for all } \lambda \geq 0. \tag{3.13}$$

Proof. All announced properties of the operator K_λ are proved in [3, Lemma 3.3]. So, let us proved those of the operator $\overline{K}_\lambda$. Let $\lambda \geq 0$. Firstly, it is clear that $\overline{K}_\lambda$ is bounded because of

$$\|\overline{K}_\lambda \psi\| \leq \|K\| \|\psi\|$$

for all $\psi \in Y_1$. Furthermore, if (Kb) holds then we infer that

$$\lambda > 0 \Longrightarrow \|\overline{K}_\lambda\| < 1. \tag{3.14}$$

Next, if (Kc) holds, then

$$\begin{aligned}
\|\overline{K}_\lambda\|_{\mathcal{L}(Y_1)} &= \sup_{\psi \in B} \|\theta_\lambda K \psi\|_{Y_1} \\
&= \sup_{\varphi \in K(B)} \|\theta_\lambda \varphi\|_{Y_1} \\
&\leq \sup_{\varphi \in \overline{K(B)}} \|\theta_\lambda \varphi\|_{Y_1},
\end{aligned}$$

where B_0 be the unit ball into Y_1. By virtue of the compactness of $\overline{K(B)}$, there exists $\varphi_0 \in \overline{K(B)}$ such that

$$\|\overline{K}_\lambda\|_{\mathcal{L}(Y_1)} \leq \|\theta_\lambda \varphi_0\|_{Y_1} = \int_0^\infty e^{-\lambda/v} |\varphi_0(v)| v \, dv$$

which leads to

$$\lim_{\lambda \to \infty} \|\overline{K}_\lambda\|_{\mathcal{L}(Y_1)} \leq \lim_{\lambda \to \infty} \int_0^\infty e^{-\lambda/v} |\varphi_0(v)| v \, dv = 0$$

and therefore

$$\|\overline{K}_\lambda\| < 1 \quad \text{for all large } \lambda. \tag{3.15}$$

Now, by (3.14) and (3.15) we can infer that both hypotheses (Kb) and (Kc) imply that $\|\overline{K}_\lambda\| < 1$ for large λ.

Suppose now that K is positive. So, for all $\lambda \geq 0$ we clearly get that

$$K_\lambda \leq K \quad \text{and} \quad \overline{K}_\lambda \leq K \tag{3.16}$$

On the other hand, due to the obvious relation $K_\lambda K = K \overline{K}_\lambda$, it follows by induction that

$$K_\lambda^n K = K \overline{K}_\lambda^n \quad \text{for all integers } n \geq 1 \tag{3.17}$$

which leads, by (3.16), to

$$K_\lambda^{n+1} \leq K \overline{K}_\lambda^n \quad \text{and} \quad \overline{K}_\lambda^{n+1} \leq K_\lambda^n K$$

for all integers $n \geq 1$ and therefore

$$\begin{aligned}
\|K_\lambda^{(n+1)}\|^{\frac{1}{(n+1)}} &\leq \|K\|^{\frac{1}{(n+1)}} (\|\overline{K}_\lambda^n\|^{\frac{1}{n}})^{\frac{n}{(n+1)}}, \\
\|\overline{K}_\lambda^{(n+1)}\|^{\frac{1}{(n+1)}} &\leq (\|K_\lambda^n\|^{\frac{1}{n}})^{\frac{n}{(n+1)}} \|K\|^{\frac{1}{(n+1)}}.
\end{aligned}$$

This easily leads to (3.13) and completes the proof. $\square$

Some useful properties of the unbounded operator (3.10) are given next.

Lemma 3.6. *Let K be an admissible operator.*

(1) *The operator T_K generates, on $L^1(\Omega)$, a strongly continuous semigroup $U_K = (U_K(t))_{t\geq 0}$ satisfying*

$$U_K(t)\varphi(\mu,v) = U_0(t)\varphi(\mu,v) + \xi(\mu,v,t)K(\gamma_1 U_K(t-\tfrac{\mu}{v})\varphi)(v) \tag{3.18}$$

for almost all $(\mu,v)\in\Omega$, where

$$\xi(\mu,v,t) = \begin{cases} 0 & \text{if } \mu \geq tv;\\ 1 & \text{if } \mu < tv. \end{cases} \tag{3.19}$$

(2) *For all large λ, we have*

$$(\lambda - T_K)^{-1}\varphi = \varepsilon_\lambda K(I-\overline{K}_\lambda)^{-1}\gamma_1(\lambda-T_0)^{-1}\varphi + (\lambda-T_0)^{-1}\varphi, \tag{3.20}$$

where $\varepsilon_\lambda(\mu,v) = e^{-\lambda\frac{\mu}{v}}$.

(3) *If K is positive, then $U_K = (U_K(t))_{t\geq 0}$ is a positive semigroup and*

$$U_K(t) \geq U_0(t) \quad t\geq 0. \tag{3.21}$$

Proof. All the announced properties, but (3.20) and (3.21), follow from [3, Theorems 3.2 and 4.1 and Proposition 6.1]. So, let us prove (3.20) and (3.21).

Let λ be large and let $\varphi\in L^1(\Omega)$. Thanks to [3, Proposition 3.1] we infer that

$$(\lambda-T_K)^{-1} = \varepsilon_\lambda(I-K_\lambda)^{-1}K\gamma_1(\lambda-T_0)^{-1} + (\lambda-T_0)^{-1}$$

which leads, by (3.12) and (3.17), to

$$\begin{aligned}(\lambda-T_K)^{-1}\varphi - (\lambda-T_0)^{-1}\varphi &= \varepsilon_\lambda\Big(\sum_{n\geq 0}K_\lambda^n\Big)K\gamma_1(\lambda-T_0)^{-1}\varphi\\ &= \varepsilon_\lambda K\Big(\sum_{n\geq 0}\overline{K}_\lambda^n\Big)\gamma_1(\lambda-T_0)^{-1}\varphi\\ &= \varepsilon_\lambda K(I-\overline{K}_\lambda)^{-1}\gamma_1(\lambda-T_0)^{-1}\varphi\end{aligned}$$

and therefore (3.20) follows.

Next, let $\varphi\in(L^1(\Omega))_+$. As $U_0=(U_0(t))_{t\geq 0}$ is a positive semigroup (Lemma 3.2), it follows that $(\lambda-T_0)^{-1}\varphi$ is a positive function and therefore $\gamma_1(\lambda-T_0)^{-1}\varphi$ is a positive function too. So, the computation above clearly leads to

$$(\lambda-T_K)^{-1}\varphi \geq (\lambda-T_0)^{-1}\varphi$$

because of the positivity of K and therefore

$$\big[\lambda(\lambda-T_K)^{-1}\big]^n\varphi \geq \big[\lambda(\lambda-T_0)^{-1}\big]^n\varphi \tag{3.22}$$

for all integers $n\geq 1$. Putting now $\lambda=\frac{n}{t}$ $(t>0)$ and passing at the limit $n\to\infty$ we infer that

$$\lim_{n\to\infty}\Big[\frac{n}{t}(\frac{n}{t}-T_K)^{-1}\Big]^n\varphi \geq \lim_{n\to\infty}\Big[\frac{n}{t}(\frac{n}{t}-T_0)^{-1}\Big]^n\varphi \tag{3.23}$$

which leads to (3.21) because of the exponential formula. $\square$

Finally, let us consider the general model (1.1), (1.3) corresponding to the unbounded operator $L_K := T_K + S$ on the domain $D(L_K) = D(T_K)$ and for which we have

Lemma 3.7. *Let K be an admissible operator. If the hypothesis* (3.5) *holds, then we have*

(1) *The operator L_K generates, on $L^1(\Omega)$, a strongly continuous semigroup $V_K = (V_K(t))_{t\geq 0}$. Furthermore*

$$V_K(t) = U_K(t) + \int_0^t U_K(t-s)SV_K(s)ds \quad t \geq 0. \tag{3.24}$$

(2) *Suppose that K is positive. Then the semigroup $V_K = (V_K(t))_{t\geq 0}$ is positive too and*

$$V_K(t) \geq V_0(t) \quad t \geq 0. \tag{3.25}$$

Moreover, if K is irreducible then $V_K = (V_K(t))_{t\geq 0}$ is also irreducible.

(3) *If K is a positive, irreducible and compact operator such that*

$$r(K_{\overline{\sigma}-\underline{\sigma}}) > 1$$

then the type $\omega_0(V_K)$ of the semigroup $V_K = (V_K(t))_{t\geq 0}$ satisfies to

$$\omega_0(V_K) > -\underline{\sigma}, \tag{3.26}$$

where $\underline{\sigma}$ is given by (3.8) *and*

$$\overline{\sigma} := \operatorname{ess\,sup}_{(\mu,v)\in\Omega} \sigma(\mu, v). \tag{3.27}$$

Proof. Almost all the announced properties follow from [3, Theorem 5.1] and [3, Theorem 6.1]. So, it only remains to prove (3.25) and (3.26).

Let $t > 0$ and let $\varphi \in (L^1(\Omega))_+$. Then (3.21) clearly leads to

$$\Big[e^{-\frac{t}{n}\sigma}U_K(\tfrac{t}{n})\Big]^n\varphi \geq \Big[e^{-\frac{t}{n}\sigma}U_0(\tfrac{t}{n})\Big]^n\varphi \quad \text{for all integers } n \in \mathbb{N}.$$

Passing at the limit $n \to \infty$, then (3.25) follows because of Trotter Formula. Finally, note that (3.26) follows from [3, Th.7.1] together with (3.13). $\square$

We end this section by the following particular case

Corollary 3.8. *Let K be a linear bounded operator from Y_1 into itself such that $\|K\| < 1$. If the hypothesis* (H_σ) *holds, then the semigroup $V_K = (V_K(t))_{t\geq 0}$ satisfies*

$$\|V_K(t)\|_{\mathcal{L}(L^1(\Omega))} \leq e^{-t\underline{\sigma}} \quad t \geq 0. \tag{3.28}$$

The above corollary forllows from Lemma 3.7 above together with [3, Corollary. 5.1].

4. Stability and domination

In this section we are concerned with stability and domination results of the unperturbed semigroup $U_K = (U_K(t))_{t\geq 0}$. That is one of the most useful results which will be used to insure Asynchronous Exponential Growth property for the semigroup $V_K = (V_K(t))_{t\geq 0}$. Before we start, let us give the following useful result

Lemma 4.1. *Let K be an admissible operator and let λ be large. Then, for all $\varphi \in L^1(\Omega)$, we have*

$$\int_0^\infty \int_0^\infty e^{-\lambda t}|\gamma_1\big(U_K(t)\varphi\big)(v)|v\,dt\,dv \leq \frac{1}{1-\|\overline{K}_\lambda\|}\|\varphi\|_1, \tag{4.1}$$

where $\overline{K}_\lambda$ is given by (3.11).

Proof. Let λ be large and let $\varphi \in W_1$. Applying the trace mapping γ_1 to (3.18), we infer that

$$\gamma_1(U_K(t)\varphi)(v) = \gamma_1(U_0(t)\varphi)(v) + \xi(1,v,t)\left[K\gamma_1(U_K(t-\tfrac{1}{v})\varphi)\right](v)$$

for all $t \geq 0$ and for almost all $v \in (0,\infty)$. Integrating it, we obtain that

$$\begin{aligned} &\int_0^\infty \int_0^\infty e^{-\lambda t}|\gamma_1\big(U_K(t)\varphi\big)(v)|v\,dt\,dv \\ &\leq \int_0^\infty \int_0^\infty e^{-\lambda t}|\gamma_1\big(U_0(t)\varphi\big)(v)|v\,dt\,dv \\ &\quad + \int_0^\infty \int_0^\infty e^{-\lambda t}\xi(1,v,t)|(K\gamma_1(U_K(t-\tfrac{1}{v})\varphi))(v)|v\,dt\,dv \\ &:= I + J. \end{aligned} \tag{4.2}$$

Thanks to Lemma 3.2, the term I becomes

$$\begin{aligned} I &= \int_0^\infty \int_0^\infty e^{-\lambda t}|\gamma_1(U_0(t)\varphi)(v)|v\,dvdt \\ &\leq \int_0^\infty \int_0^\infty |\chi(1,v,t)\varphi(1-tv,v)|vdtdv \\ &= \int_0^\infty \int_{1-tv}^1 |\varphi(\mu,v)|d\mu dt \end{aligned}$$

and therefore

$$I \leq \|\varphi\|_1. \tag{4.3}$$

For the term J we have

$$\begin{aligned} J &= \int_0^\infty \int_0^\infty e^{-\lambda t}\xi(1,v,t)|(K\gamma_1(U_K(t-\tfrac{1}{v})\varphi))(v)|v\,dt\,dv \\ &= \int_0^\infty \int_0^\infty e^{-\lambda(x+\frac{1}{v})}|(K\gamma_1(U_K(x)\varphi))(v)|v\,dx\,dv \\ &= \int_0^\infty e^{-\lambda x}\Big[\int_0^\infty e^{-\lambda/v}|(K\gamma_1(U_K(x)\varphi))(v)|v\,dv\Big]dx \\ &= \int_0^\infty e^{-\lambda x}\Big[\int_0^\infty |(\overline{K}_\lambda\gamma_1(U_K(x)\varphi))(v)|v\,dv\Big]dx \end{aligned}$$

which leads, by the boundedness of $\overline{K}_\lambda$ (Lemma 3.5), to

$$J \leq \|\overline{K}_\lambda\| \int_0^\infty \int_0^\infty e^{-\lambda x}|\gamma_1(U_K(x)\varphi)(v)|v\,dx\,dv \tag{4.4}$$

Now, (4.2) together with (4.3) and (4.4) clearly imply that

$$(1-\|\overline{K}_\lambda\|)\int_0^\infty \int_0^\infty e^{-\lambda t}|\gamma_1\big(U_K(t)\varphi\big)(v)|v\,dt\,dv \leq \|\varphi\|_1$$

and therefore (4.1) holds because of (3.12). Finally, the density of W^1 into $L^1(\Omega)$ achieves the proof. □

Now, we are ready to give the main result of this section.

Theorem 4.2. *Let K be an admissible operator and let $(K_n)_n$ be a sequence of admissible operators such that*

$$\lim_{n\to\infty} \|K_n - K\|_{\mathcal{L}(Y_1)} = 0. \tag{4.5}$$

Then, for all $t \geq 0$, we have

$$\lim_{n\to\infty} \|U_{K_n}(t) - U_K(t)\|_{\mathcal{L}(L^1(\Omega))} = 0. \tag{4.6}$$

Proof. Let λ be large and let $\varphi \in L^1(\Omega)$. In the sequel, we are going to divide this proof in two steps.

Step I. If H denotes another admissible operator, then by (3.18) it follows that

$$U_H(t)\varphi - U_K(t)\varphi := A(t)\varphi + B(t)\varphi \quad t \geq 0, \tag{4.7}$$

where

$$A(t)\varphi(\mu, v) = \xi(\mu, v, t)(H - K)\gamma_1(U_H(t - \tfrac{\mu}{v})\varphi)(v) \tag{4.8}$$

$$B(t)\varphi(\mu, v) = \xi(\mu, v, t)K\gamma_1\Big(U_H(t - \tfrac{\mu}{v})\varphi - U_K(t - \tfrac{\mu}{v})\varphi\Big)(v) \tag{4.9}$$

for almost all $(\mu, v) \in \Omega$. Furthermore, applying γ_1 to (4.7) we obtain that

$$\gamma_1\Big(U_H(t)\varphi - U_K(t)\varphi\Big) = \gamma_1\big(A(t)\varphi\big) + \gamma_1\big(B(t)\varphi\big), \tag{4.10}$$

where

$$\gamma_1\big(A(t)\varphi\big)(v) = \xi(1, v, t)(H - K)\gamma_1(U_H(t - \tfrac{1}{v})\varphi)(v) \tag{4.11}$$

$$\gamma_1\big(B(t)\varphi\big)(v) = \xi(1, v, t)K\gamma_1\Big(U_H(t - \tfrac{1}{v})\varphi - U_K(t - \tfrac{1}{v})\varphi\Big)(v) \tag{4.12}$$

for almost all $v \in (0, \infty)$. So, multiplying (4.10) by $e^{-\lambda t}$ and integrating it over $\in (0, \infty) \times (0, \infty)$, we infer that

$$\begin{aligned} &\int_0^\infty \int_0^\infty e^{-\lambda t}|\gamma_1\Big(U_H(t)\varphi - U_K(t)\varphi\Big)(v)|v\,dt\,dv \\ &\leq \int_0^\infty \int_0^\infty e^{-\lambda t}|\gamma_1\big(A(t)\big)(v)|v\,dt\,dv + \int_0^\infty \int_0^\infty e^{-\lambda t}|\gamma_1\big(B(t)\big)(v)|v\,dt\,dv \\ &:= I_1 + I_2. \end{aligned} \tag{4.13}$$

Firstly, thanks to (4.11) the term I_1 becomes

$$\begin{aligned} I_1 &= \int_0^\infty \int_0^\infty e^{-\lambda t}\xi(1, v, t)|(H - K)\gamma_1(U_H(t - \tfrac{1}{v})\varphi)(v)|v\,dt\,dv \\ &= \int_0^\infty \int_0^\infty e^{-\lambda(x + \frac{1}{v})}|(H - K)\gamma_1(U_H(x)\varphi)(v)|v\,dx\,dv \\ &= \int_0^\infty e^{-\lambda x}[\int_0^\infty |(H - K)\gamma_1(U_H(x)\varphi)(v)|v\,dv]dx \\ &\leq \|H - K\| \int_0^\infty \int_0^\infty e^{-\lambda x}|\gamma_1(U_H(x)\varphi)(v)|v\,dx\,dv \end{aligned}$$

which leads, by Lemma 4.1 (with H instead K), to

$$I_1 \leq \frac{\|H - K\|}{1 - \|\overline{H}_\lambda\|}\|\varphi\|_1. \tag{4.14}$$

Thanks to (4.12), the term I_2 becomes

$$
\begin{aligned}
I_2 &= \int_0^\infty \int_0^\infty e^{-\lambda t}|\gamma_1\big(B(t)\big)(v)|v\,dt\,dv \\
&= \int_0^\infty \int_0^\infty e^{-\lambda t}\xi(1,v,t)\Big|K\gamma_1\Big(U_H(t-\tfrac{1}{v})\varphi - U_K(t-\tfrac{1}{v})\varphi\Big)(v)\Big|v\,dt\,dv \\
&= \int_0^\infty \int_0^\infty e^{-\lambda(x+\frac{1}{v})}\Big|K\gamma_1\Big(U_H(x)\varphi - U_K(x)\varphi\Big)(v)\Big|v\,dx\,dv \\
&= \int_0^\infty e^{-\lambda x}[\int_0^\infty |\overline{K}_\lambda\gamma_1\Big(U_H(x)\varphi - U_K(x)\varphi\Big)(v)|v\,dv]dx \\
&\le \|\overline{K}_\lambda\| \int_0^\infty \int_0^\infty e^{-\lambda x}|\gamma_1\Big(U_H(x)\varphi - U_K(x)\varphi\Big)(v)|v\,dx\,dv
\end{aligned}
$$

which leads, by the boundedness of $\overline{K}_\lambda$ (Lemma 3.5), to

$$
I_2 \le \|\overline{K}_\lambda\| \int_0^\infty \int_0^\infty e^{-\lambda x}|\gamma_1\Big(U_H(x)\varphi - U_K(x)\varphi\Big)(v)|v\,dx\,dv. \tag{4.15}
$$

Now, (4.13) together with (4.14) and (4.15) clearly imply that

$$
(1-\|\overline{K}_\lambda\|)\int_0^\infty \int_0^\infty e^{-\lambda t}|\gamma_1\Big(U_H(t)\varphi - U_K(t)\varphi\Big)(v)|v\,dt\,dv \le \frac{\|H-K\|\|\varphi\|_1}{1-\|\overline{H}_\lambda\|}
$$

and therefore, (3.12) leads to

$$
\int_0^\infty \int_0^\infty e^{-\lambda t}|\gamma_1\Big(U_H(t)\varphi - U_K(t)\varphi\Big)(v)|v\,dt\,dv \le \frac{\|H-K\|\|\varphi\|_1}{(1-\|\overline{K}_\lambda\|)(1-\|\overline{H}_\lambda\|)}. \tag{4.16}
$$

On the other hand, by (4.7) it follows that

$$
\begin{aligned}
\|U_H(t)\varphi - U_K(t)\varphi\|_1 &\le \int_\Omega |A(t)\varphi(\mu,v)|\,d\mu\,dv + \int_\Omega |B(t)\varphi(\mu,v)|\,d\mu\,dv \\
&:= J_1 + J_2.
\end{aligned} \tag{4.17}
$$

for all $t \ge 0$. So, using (4.8) the term J_1 becomes

$$
\begin{aligned}
J_1 &= \int_\Omega \xi(\mu,v,t)|(H-K)\gamma_1(U_H(t-\tfrac{\mu}{v})\varphi)(v)|\,d\mu\,dv \\
&\le \int_0^\infty \int_0^1 e^{\lambda\frac{\mu}{v}}\xi(\mu,v,t)|(H-K)\gamma_1(U_H(t-\tfrac{\mu}{v})\varphi)(v)|\,d\mu\,dv \\
&\le e^{\lambda t}\int_0^\infty \int_0^\infty e^{-\lambda x}|(H-K)\gamma_1(U_H(x)\varphi)(v)|v\,dx\,dv \\
&\le e^{\lambda t}\int_0^\infty \Big[\int_0^\infty e^{-\lambda x}|(H-K)\gamma_1(U_H(x)\varphi)(v)|v\,dv\Big]dx \\
&\le e^{\lambda t}\|H-K\|\int_0^\infty \int_0^\infty e^{-\lambda x}|\gamma_1(U_H(x)\varphi)(v)|v\,dx\,dv
\end{aligned}
$$

which leads, by Lemma 4.1 (with H instead K), to

$$
J_1 \le e^{\lambda t}\frac{\|H-K\|}{1-\|\overline{H}_\lambda\|}\|\varphi\|_1. \tag{4.18}
$$

Using (4.9) the term J_2 becomes

$$\begin{aligned}
J_2 &= \int_\Omega \xi(\mu,v,t)\Big|K\gamma_1\Big(U_H(t-\tfrac{\mu}{v})\varphi - U_K(t-\tfrac{\mu}{v})\varphi\Big)(v)\Big|\,d\mu\,dv \\
&\le \int_0^\infty \int_0^1 e^{\lambda\frac{\mu}{v}}\xi(\mu,v,t)\Big|K\gamma_1\Big(U_H(t-\tfrac{\mu}{v})\varphi - U_K(t-\tfrac{\mu}{v})\varphi\Big)(v)\Big|\,d\mu\,dv \\
&\le e^{\lambda t}\int_0^\infty \int_0^\infty e^{-\lambda x}|K\gamma_1\Big(U_H(x)\varphi - U_K(x)\varphi\Big)(v)|v\,dx\,dv \\
&\quad\times e^{\lambda t}\int_0^\infty e^{-\lambda x}\Big[\int_0^\infty |K\gamma_1\Big(U_H(x)\varphi - U_K(x)\varphi\Big)(v)|v\,dv\Big]dx \\
&\le e^{\lambda t}\|K\|\int_0^\infty \int_0^\infty e^{-\lambda x}\big|\gamma_1\Big(U_H(x)\varphi - U_K(x)\varphi\Big)(v)\big|v\,dx\,dv
\end{aligned}$$

which leads, by (4.16), to

$$J_2 \le e^{\lambda t}\frac{\|K\|\|H-K\|}{(1-\|\overline{K}_\lambda\|)(1-\|\overline{H}_\lambda\|)}\|\varphi\|_1. \tag{4.19}$$

Now, (4.17) together with (4.18) and (4.19) lead to

$$\|U_H(t)\varphi - U_K(t)\varphi\|_1 \le \frac{e^{\lambda t}(\|K\|+1)\|H-K\|}{(1-\|\overline{K}_\lambda\|)(1-\|\overline{H}_\lambda\|)}\|\varphi\|_1$$

and therefore, for all $t \ge 0$, we have

$$\|U_H(t) - U_K(t)\|_{\mathcal{L}(L^1(\Omega))} \le \frac{e^{\lambda t}(\|K\|+1)\|H-K\|}{(1-\|\overline{K}_\lambda\|)(1-\|\overline{H}_\lambda\|)}. \tag{4.20}$$

Step II. Now, let $(K_n)_n \subset \mathcal{L}(Y_1)$ be a sequence of admissible operators such that (4.5) holds. As, we have

$$\big|\|\overline{K_{n_\lambda}}\| - \|\overline{K}_\lambda\|\big| \le \|\overline{K_{n_\lambda}} - \overline{K}_\lambda\| \le \|K_n - K\|$$

for all integers $n \ge 1$, it follows that

$$\lim_{n\to\infty}\|\overline{K_{n_\lambda}}\| = \|\overline{K}_\lambda\|. \tag{4.21}$$

On the other hand, putting $H = K_n$ into (4.20) we obtain that

$$\|U_{K_n}(t) - U_K(t)\|_{\mathcal{L}(L^1(\Omega))} \le \frac{e^{\lambda t}(\|K\|+1)\|K_n - K\|}{(1-\|\overline{K}_\lambda\|)(1-\|\overline{K_n}_\lambda\|)} \tag{4.22}$$

for all integers $n \ge 1$. Passing now at the limit $n \to \infty$ into (4.22) and using (4.5) and (4.21), we finally infer that (4.6). The proof is now achieved. $\square$

We complete this section by the following domination result.

Theorem 4.3. *Let K and H be two admissible operators. If H is a positive operator and*

$$|K\psi| \le H|\psi| \tag{4.23}$$

for all $\psi \in Y_1$, then

$$|\big[U_K(t) - U_0(t)\big]\varphi| \le \big[U_H(t) - U_0(t)\big]|\varphi| \quad t \ge 0 \tag{4.24}$$

for all $\varphi \in L^1(\Omega)$.

Proof. Let λ be large. Firstly, note that (4.23) obviously implies that $|\overline{K}_\lambda\psi| \le \overline{H}_\lambda|\psi|$ for all $\psi \in Y_1$ and by induction il follows that

$$|\overline{K}_\lambda^n\psi| \le \overline{H}_\lambda^n|\psi|. \tag{4.25}$$

Next, let $\varphi \in L^1(\Omega)$. Due to (3.20) and (4.23) we clearly get that

$$\begin{aligned}|(\lambda - T_K)^{-1}\varphi| &\le |\varepsilon_\lambda K(I-\overline{K}_\lambda)^{-1}\gamma_1(\lambda - T_0)^{-1}\varphi| + |(\lambda - T_0)^{-1}\varphi| \\ &\le \varepsilon_\lambda H|(I-\overline{K}_\lambda)^{-1}\gamma_1(\lambda - T_0)^{-1}\varphi| + |(\lambda - T_0)^{-1}\varphi|\end{aligned}$$

which leads, by (3.12) and (4.25), to

$$\begin{aligned}|(\lambda - T_K)^{-1}\varphi| &\le \varepsilon_\lambda H\Big|\sum_{n\ge 0}\overline{K}_\lambda^n\gamma_1(\lambda - T_0)^{-1}\varphi\Big| + |(\lambda - T_0)^{-1}\varphi| \\ &\le \varepsilon_\lambda H\sum_{n\ge 0}|\overline{K}_\lambda^n\gamma_1(\lambda - T_0)^{-1}\varphi| + |(\lambda - T_0)^{-1}\varphi| \\ &\le \varepsilon_\lambda H\sum_{n\ge 0}\overline{H}_\lambda^n\gamma_1|(\lambda - T_0)^{-1}\varphi| + |(\lambda - T_0)^{-1}\varphi| \\ &= \varepsilon_\lambda H(I-\overline{H}_\lambda)^{-1}\gamma_1|(\lambda - T_0)^{-1}\varphi| + |(\lambda - T_0)^{-1}\varphi|.\end{aligned}$$

Thanks to the positivity of the semigroup $U_0 = (U_0(t))_{t\ge 0}$ (Lemma 3.2), we infer that of the operator $(\lambda - T_0)^{-1}$ and therefore

$$\begin{aligned}|(\lambda - T_K)^{-1}\varphi| &\le \varepsilon_\lambda H(I-H_\lambda)^{-1}\gamma_1(\lambda - T_0)^{-1}|\varphi| + (\lambda - T_0)^{-1}|\varphi| \\ &= (\lambda - T_H)^{-1}|\varphi|.\end{aligned}$$

This leads, by induction, to

$$|(\lambda(\lambda - T_K)^{-1})^n\varphi| \le (\lambda(\lambda - T_H)^{-1})^n|\varphi|$$

for all integers n. Putting now $\lambda = \frac{n}{t}$ ($t > 0$), we obtain that

$$|[\frac{n}{t}(\frac{n}{t} - T_K)^{-1}]^n\varphi| \le [\frac{n}{t}(\frac{n}{t} - T_H)^{-1}]^n|\varphi|$$

and therefore

$$|U_K(t)\varphi| \le U_H(t)|\varphi| \quad t \ge 0 \tag{4.26}$$

because of Exponential formula. Finally, (3.18) together with (4.23) and (4.26) imply that

$$\begin{aligned}|U_K(t)\varphi - U_0(t)\varphi|(\mu,v) &= |\xi(\mu,v,t)K(\gamma_1 U_K(t-\tfrac{\mu}{v})\varphi)(v)| \\ &\le \xi(\mu,v,t)H|(\gamma_1 U_K(t-\tfrac{\mu}{v})\varphi)(v)| \\ &\le \xi(\mu,v,t)H\gamma_1|(U_K(t-\tfrac{\mu}{v})\varphi)(v)| \\ &\le \xi(\mu,v,t)H\gamma_1(U_H(t-\tfrac{\mu}{v})|\varphi|)(v) \\ &= \Big(U_H(t)|\varphi| - U_0(t)|\varphi|\Big)(\mu,v)\end{aligned}$$

for almost all $(\mu, v) \in \Omega$. The proof is now achieved. $\square$

5. Asynchronous exponential growth

In this section, we prove that the generated semigroup $V_K = (V_K(t))_{t\geq 0}$ possesses the asynchronous exponential growth property by applying Lemma 1.1. However, one of the most difficulties to apply Lemma 1.1 is to compute the essential type $\omega_{\rm ess}(V_K)$ (given by (2.2)) of the generated semigroup $V_K = (V_K(t))_{t\geq 0}$ whose explicit form is unfortunately not available.

In the sequel, we are going to circumvent this difficulty by proving some useful results. Before we start, let us recall that all rank one or finite rank operators are compact which leads to their admissibility because of Definition 3.4. Therefore, all the semigroups of this work exist.

Lemma 5.1. *Let K be the following rank one operator in Y_1; i.e.,*

$$K\psi = h\int_0^\infty k(v')\psi(v')v'dv', \quad h\in Y_1, \quad k\in L^\infty(0,\infty).$$

Then we have

$$U_K(t) = U_0(t) + \sum_{m=1}^\infty U_m(t) \quad t\geq 0,$$

where $U_0(t)$ is given by (3.3) and $U_m(t)$ is defined by

$$\begin{aligned}
&U_1(t)\varphi(\mu,v)\\
&= \xi(\mu,v,t)h(v)\int_0^\infty k(v_1)\chi\left(1,v_1,t-\tfrac{\mu}{v}\right)\varphi\left(1-(t-\tfrac{\mu}{v})v_1,v_1\right)v_1dv_1
\end{aligned}$$

and, for $m\geq 2$, by

$$\begin{aligned}
&U_m(t)\varphi(\mu,v)\\
&= \xi(\mu,v,t)h(v)\underbrace{\int_0^\infty\cdots\int_0^\infty}_{m\ times}\prod_{j=1}^{m-1}h(v_j)\prod_{j=1}^{m}k(v_j)\\
&\quad\times\xi\Big(1,v_{m-1},t-\frac{\mu}{v}-\sum_{i=1}^{(m-2)}\frac{1}{v_i}\Big)\chi\Big(1,v_m,t-\frac{\mu}{v}-\sum_{i=1}^{(m-1)}\frac{1}{v_i}\Big)\\
&\quad\times\varphi\Big(1-\Big(t-\frac{\mu}{v}-\sum_{i=1}^{m-1}\frac{1}{v_i}\Big)v_m,v_m\Big)v_1v_2\cdots v_mdv_1\cdots dv_m.
\end{aligned}$$

for all $\varphi\in L^1(\Omega)$. Furthermore, for all $t\geq 0$ we have

$$\lim_{N\to\infty}\|U_K(t)-U_0(t)-\sum_{m=1}^N U_m(t)\|_{\mathcal{L}(L^1(\Omega))} = 0. \tag{5.1}$$

Proof. Let $\varphi\in L^1(\Omega)$. By (3.18), it is easy to check, by induction, that for all integer $N\geq 1$ we have

$$U_K(t) = U_0(t) + \sum_{m=1}^N U_m(t) + R_N(t) \tag{5.2}$$

where the rest $R_{N+1}(t)$ is given by

$$R_N(t)\varphi(\mu,v)$$

$$
\begin{aligned}
&= \xi(\mu, v, t) h(v) \underbrace{\int_0^\infty \cdots \int_0^\infty}_{(N+1)\text{ times}} \\
&\quad \times \prod_{j=1}^{N} h(v_j) \prod_{j=1}^{N+1} k(v_j) \xi\Big(1, v_N, t - \frac{\mu}{v} - \sum_{i=1}^{(N-1)} \frac{1}{v_i}\Big) \\
&\quad \times \gamma_1\Big(U_K\Big(t - \frac{\mu}{v} - \sum_{i=1}^{N} \frac{1}{v_i}\Big)\varphi\Big)(v_{N+1}) v_1 v_2 \cdots v_{N+1} dv_1 \cdots dv_{N+1}.
\end{aligned}
$$

Now, let us prove (5.1). Let λ be large. Then we have

$$
\begin{aligned}
&\|R_N(t)\varphi\|_1 \\
&\le \int_\Omega |R_{N+1}(t)\varphi(\mu, v)| e^{\lambda \frac{\mu}{v}}\, d\mu\, dv \\
&= \int_\Omega \Big| \xi(\mu, v, t) h(v) \underbrace{\int_0^\infty \cdots \int_0^\infty}_{(N+1)\text{ times}} e^{\lambda \frac{\mu}{v}} \prod_{j=1}^{N} h(v_j) \prod_{j=1}^{N+1} k(v_j) \\
&\quad \times \xi\Big(1, v_N, t - \frac{\mu}{v} - \sum_{i=1}^{(N-1)} \frac{1}{v_i}\Big) \gamma_1\Big(U_K\Big(t - \frac{\mu}{v} - \sum_{i=1}^{N} \frac{1}{v_i}\Big)\varphi\Big)(v_{N+1}) \\
&\quad \times v_1 v_2 \cdots v_{N+1} dv_1 \cdots dv_{N+1} \Big| \, d\mu\, dv.
\end{aligned}
$$

By the change of variables $x = t - \frac{\mu}{v} - \sum_{i=1}^{N} \frac{1}{v_i}$ and $v dx = -d\mu$, we infer that

$$
\begin{aligned}
&\|R_N(t)\varphi\|_1 \\
&\le \int_0^\infty \int_0^t \Big| h(v) \underbrace{\int_0^\infty \cdots \int_0^\infty}_{(N+1)\text{ times}} e^{\lambda(t - x - \sum_{i=1}^{N} \frac{1}{v_i})} \\
&\times \prod_{j=1}^{N} h(v_j) \prod_{j=1}^{N+1} k(v_j) \gamma_1\left(U_K(x)\varphi\right)(v_{N+1}) v_1 v_2 \cdots v_{N+1} dv_1 \cdots dv_{N+1} \Big| v\, dx\, dv
\end{aligned}
$$

which leads to

$$
\begin{aligned}
\|R_N(t)\varphi\|_1 &\le e^{\lambda t} \Big[\Big(\int_0^\infty |h(v)| v\, dv \Big) \Big(\operatorname{ess\,sup}_{v \in (0,\infty)} |k(v)| \Big) \Big] \\
&\quad \times \Big[\Big(\int_0^\infty e^{-\lambda/v} |h(v)| v\, dv \Big) \Big(\operatorname{ess\,sup}_{v \in (0,\infty)} |k(v)| \Big) \Big]^N \\
&\quad \times \int_0^\infty \int_0^t e^{-\lambda x} |\gamma_1\left(U_K(x)\varphi\right)(v_{N+1})| v_{N+1}\, dx\, dv_{N+1}.
\end{aligned}
$$

Hence

$$
\|R_N(t)\varphi\|_1 \le e^{\lambda t} \|K\| \|\overline{K}_\lambda\|^N \int_0^\infty \int_0^\infty e^{-\lambda x} |\gamma_1(U_K(x)\varphi)(v)| v\, dx\, dv
$$

which implies, by (4.1), that

$$\|R_N(t)\varphi\|_1 \le \frac{e^{\lambda t}\|K\|\|\overline{K}_\lambda\|^N}{1-\|\overline{K}_\lambda\|}\|\varphi\|_1 \tag{5.3}$$

and therefore

$$\lim_{N\to\infty}\|R_N(t)\|_{\mathcal{L}(L^1(\Omega))}=0$$

because of (3.12). Now the proof is complete. □

The second useful result concerns the linear operator

$$\mathbb{U}_K(t,s) := \Big[U_K(t)-U_0(t)\Big]U_0(s)\Big[U_K(t)-U_0(t)\Big] \tag{5.4}$$

which is clearly bounded from $L^1(\Omega)$ into itself for all $t\ge 0$ and all $s\ge 0$ because of Lemmas 3.2 and 3.6. One of the most important properties of this operator is as follows.

Proposition 5.2. *Let K be a compact operator from Y_1 into itself. Then $\mathbb{U}_K(t,s)$ is a weakly compact operator into $L^1(\Omega)$ for all $t>0$ and all $s\ge 0$.*

Proof. Let $\varphi\in L^1(\Omega)$, $t>0$ and $s\ge 0$. We divide this proof in several steps.

Step 1. Let us consider the boundary operator

$$K\psi = h\int_0^\infty k(v')\psi(v')v'dv' \quad h\in C_c(J)\ k\in L^\infty(0,\infty). \tag{5.5}$$

By Lemma 5.1, the operator $U_K(t)$ is expressed as follows

$$U_K(t)=U_0(t)+\sum_{m=1}^{\infty}U_m(t). \tag{5.6}$$

Let us show that $U_m(t)$ is weakly compact into $L^1(\Omega)$ for all $m\ge 2$.

So, as $h\in C_c(J)$, there exist a and b $(0<a<b<\infty)$ such that (supp) $h\subset(a,b)$.

Let $n=[tb]+2$, where $[tb]$ denotes the integer part of tb and let m be any integer such that $m\ge n$. So, for all $v_i\in(a,b)$, $i=1\cdots(m-1)$, we have

$$\begin{aligned}\Big(t-\frac{\mu}{v}-\sum_{i=1}^{(m-2)}\frac{1}{v_i}\Big)v_{m-1} &\le \Big(t-\frac{\mu}{v}-\sum_{i=1}^{(n-2)}\frac{1}{v_i}\Big)v_{m-1}\\ &\le \Big(t-\frac{(n-2)}{b}\Big)v_{m-1}\\ &\le \Big(t-\frac{(n-2)}{b}\Big)b<1\end{aligned}$$

which leads to

$$\xi\Big(1,v_{m-1},t-\frac{\mu}{v}-\sum_{i=1}^{(m-2)}\frac{1}{v_i}\Big)=0,$$

where the function ξ is defined by (3.19). Therefore, $U_m(t)=0$ for all integers $m\ge n=[bt]+2$ and hence (5.6) becomes a finite sum that is

$$U_K(t)=U_0(t)+\sum_{m=1}^{[bt]+1}U_m(t). \tag{5.7}$$

Next, for all m such that $2 \le m \le [bt]+1$, the change of variables

$$x = 1 - \Big(t - \frac{\mu}{v} - \sum_{i=1}^{(m-1)} \frac{1}{v_i}\Big) v_m$$
$$v_{m-1}^2 dx = -v_m dv_{m-1}$$

allows us to write

$$\begin{aligned} |U_m(t)\varphi|(\mu, v) &\le \frac{m^3}{t^3}\|h\|_\infty \|h\|_{Y_1}^{m-2}\|k\|_\infty^m \xi(\mu, v, t)|h(v)| \int_\Omega |\varphi(x, v_m)|\,dx\,dv_m \\ &:= C_m(t)\mathbb{I}\otimes\mathbb{I}\varphi(\mu, v), \end{aligned} \tag{5.8}$$

where the operator $\mathbb{I}\otimes\mathbb{I}$ is defined by

$$\mathbb{I}\otimes\mathbb{I}\varphi(\mu, v) = \xi(\mu, v, t)|h(v)| \int_\Omega |\varphi(x, v_m)|\,dx\,dv_m \tag{5.9}$$

and the constant $C_m(t)$ by

$$C_m(t) = \frac{m^3}{t^3}\|h\|_\infty \|h\|_{Y_1}^{m-2}\|k\|_\infty^m.$$

As we have

$$\int_\Omega \xi(\mu, v, t)|h(v)|\,d\mu\,dv = t\|h\|_{Y_1} < \infty$$

it follows that $\mathbb{I}\otimes\mathbb{I}$ is a rank one operator into $L^1(\Omega)$ and therefore compact. Due to (5.8), it follows that

$$0 \le U_m(t) + C_m(t)\mathbb{I}\otimes\mathbb{I} \le 2C_m(t)\mathbb{I}\otimes\mathbb{I}$$

which implies, by the second point of Lemma 2.2, that the operator $U_m(t)+C_m(t)\mathbb{I}\otimes\mathbb{I}$ is a weakly compact into $L^1(\Omega)$ and therefore

$$U_m(t) = \Big(U_m(t) + C_m(t)\mathbb{I}\otimes\mathbb{I}\Big) - C_m(t)\mathbb{I}\otimes\mathbb{I}$$

is a weakly compact operator into $L^1(\Omega)$ for all m ($2 \le m \le [bt]+1$).

On the other hand. A simple computation shows that

$$\begin{aligned} &U_1(t)U_0(s)U_1(t)\varphi(\mu, v) \\ &= \xi(\mu, v, t)h(v)\int_0^\infty \int_0^\infty h(v')k(v')k(v'')\chi(1, v', t+s-\tfrac{\mu}{v})\xi(1-(t+s-\tfrac{\mu}{v})v', v', t) \\ &\quad \times \chi(1, v'', 2t+s-\tfrac{\mu}{v}-\tfrac{1}{v'})\varphi\left(1-(2t+s-\tfrac{\mu}{v}-\tfrac{1}{v'})\right) v'v''\,dv'\,dv''. \end{aligned}$$

By the change of variables

$$x = 1 - \left(2t+s-\tfrac{\mu}{v}-\tfrac{1}{v}\right) v''$$
$$v'^2 dx = -v'' dv'$$

we obtain that

$$\begin{aligned} |U_1(t)U_0(s)U_1(t)\varphi|(\mu, v) &\le \frac{1}{t^3}\|k\|_\infty^2\|h\|_\infty \xi(\mu, v, t)h(v)\int_\Omega |\varphi(x, v'')|\,dx\,dv'' \\ &= C_1(t)\mathbb{I}\otimes\mathbb{I}\varphi(\mu, v) \end{aligned} \tag{5.10}$$

where the operator $\mathbb{I}\otimes\mathbb{I}$ is given by (5.9) and the constant $C_1(t)$ by

$$C_1(t) = \frac{1}{t^3}\|k\|_\infty^2\|h\|_\infty.$$

As before, we conclude through (5.10) that $U_1(t)U_0(s)U_1(t)$ is a weakly compact operator into $L^1(\Omega)$.

Now, by (5.4) and (5.7) we can write

$$\begin{aligned}\mathbb{U}_K(t,s) &= \Big[U_1(t) + \sum_{m=2}^{[bt]+1} U_m(t)\Big] U_0(s) \Big[U_1(t) + \sum_{m=2}^{[bt]+1} U_m(t)\Big] \\ &= U_1(t)U_0(s)U_1(t) + U_1(t)U_0(s)\Big[\sum_{m=2}^{[bt]+1} U_m(t)\Big] \\ &\quad + \Big[\sum_{m=2}^{[bt]+1} U_m(t)\Big] U_0(s)U_1(t) + \Big[\sum_{m=2}^{[bt]+1} U_m(t)\Big]^2\end{aligned}$$

which is the sum of weakly compact into $L^1(\Omega)$. Now we can say that:

for any boundary operator K of the form (5.5), the operator $\mathbb{U}_K(t,s)$ is a weakly compact into $L^1(\Omega)$ for all $t > 0$ and all $s \geq 0$.

Step 2. Let us consider now the rank one boundary operator

$$K\psi = h\int_0^\infty k(v')\psi(v')v'dv' \quad h \in Y_1\ k \in L^\infty(0,\infty). \tag{5.11}$$

As $h \in Y_1$, there exists a sequence $(h_n)_n$ of $C_c(J)$ converging to h into Y_1. So, let K_n be the following operator

$$K_n\psi = h_n\int_0^\infty k(v')\psi(v')v'dv'$$

obviously of the form (5.5) and for which $\mathbb{U}_{K_n}(t,s)$ is a weakly compact operator into $L^1(\Omega)$ because of the step I. Furthermore, it is easy to check that

$$\lim_{n\to\infty} \|K_n - K\|_{\mathcal{L}(Y_1)} = 0 \tag{5.12}$$

which leads, by Theorem 4.2, to

$$\lim_{n\to\infty} \|U_{K_n}(t) - U_K(t)\|_{\mathcal{L}(L^1(\Omega))} = 0 \tag{5.13}$$

and therefore, $\big(\|U_{K_n}(t)\|\big)_n$ is a bounded sequence; i.e.,

$$\|U_{K_n}(t)\| \leq M_t \quad \text{for all integer } n. \tag{5.14}$$

On the other hand, writing

$$\begin{aligned}\mathbb{U}_{K_n}(t,s) - \mathbb{U}_K(t,s) &= \Big[U_{K_n}(t) - U_K(t)\Big]U_0(s)\Big[U_{K_n}(t) - U_0(t)\Big] \\ &\quad + \Big[U_K(t) - U_0(t)\Big]U_0(s)\Big[U_{K_n}(t) - U_K(t)\Big]\end{aligned} \tag{5.15}$$

it follows that

$$\begin{aligned}\|\mathbb{U}_{K_n}(t,s) - \mathbb{U}_K(t,s)\| &\leq \Big[M_t + \|U_0(t)\|\Big]\|U_0(s)\|\|U_{K_n}(t) - U_K(t)\| \\ &\quad + \Big[\|U_K(t)\| + \|U_0(s)\|\Big]\|U_0(s)\|\|U_{K_n}(t) - U_K(t)\|\end{aligned} \tag{5.16}$$

which leads, by (5.13), to

$$\lim_{n\to\infty} \|\mathbb{U}_{K_n}(t,s) - \mathbb{U}_K(t,s)\|_{\mathcal{L}(L^1(\Omega))} = 0 \tag{5.17}$$

and therefore $\mathbb{U}_K(t,s)$ is a weakly compact operator because of the first point of Lemma 2.2. Now, we can say that:

> for any rank one operator K, the operator $\mathbb{U}_K(t,s)$ is weakly compact into $L^1(\Omega)$ for all $t>0$ and all $s\geq 0$.

Step 3. Let us consider now the finite rank operator

$$K\psi=\sum_{i=1}^{M_K} h_i\int_0^\infty k_i(v')\psi(v')v'dv',\quad h_i\in Y_1,\quad k_i\in L^\infty(0,\infty),\quad i=1,\dots,M_K.$$

So, if we set

$$h:=\max_{i=1,\dots,M_K}|h_i|\in Y_1\quad\text{and}\quad k:=\sum_{i=1}^{M_K}|k_i|\in L^\infty(0,\infty)$$

it follows that

$$H\psi=h\int_0^\infty k(v')\psi(v')v'dv'$$

is obviously a positive operator of the form (5.11) and therefore $\mathbb{U}_H(t,s)$ is a weakly compact operator into $L^1(\Omega)$ because of the step 2. Furthermore, for all $\psi\in Y_1$ we have

$$\begin{aligned}
|K\psi| &\leq \sum_{i=1}^{M_K}|h_i|\int_0^\infty |k_i(v')||\psi(v')|v'dv'\\
&\leq \Big[\max_{i=1,..,M_K}|h_i|\Big]\int_0^\infty\Big[\sum_{i=1}^{M_K}|k_i(v')|\Big]|\psi(v')|v'dv'\\
&\leq H|\psi|
\end{aligned}$$

which leads, by Theorem 4.3 and the positivity of $U_0(s)$ (Lemma 3.2), to

$$\begin{aligned}
|\mathbb{U}_K(t,s)\varphi| &= |\Big[U_K(t)-U_0(t)\Big]U_0(s)\Big[U_K(t)-U_0(t)\Big]\varphi|\\
&\leq \Big[U_H(t)-U_0(t)\Big]|U_0(s)\Big[U_K(t)-U_0(t)\Big]\varphi|\\
&\leq \Big[U_H(t)-U_0(t)\Big]U_0(s)|\Big[U_K(t)-U_0(t)\Big]\varphi|\\
&\leq \Big[U_H(t)-U_0(t)\Big]U_0(s)\Big[U_H(t)-U_0(t)\Big]|\varphi|
\end{aligned}$$

and therefore

$$|\mathbb{U}_K(t,s)\varphi|\leq \mathbb{U}_H(t,s)|\varphi|$$

for all $\varphi\in L^1(\Omega)$. This implies

$$0\leq \mathbb{U}_K(t,s)+\mathbb{U}_H(t,s)\leq 2\mathbb{U}_H(t,s)$$

and therefore $\mathbb{U}_K(t,s)+\mathbb{U}_H(t,s)$ is a weakly compact operator into $L^1(\Omega)$ because of the second point of Lemma 2.2. Writing now

$$\mathbb{U}_K(t,s)=\Big(\mathbb{U}_K(t,s)+\mathbb{U}_H(t,s)\Big)-\mathbb{U}_H(t,s)$$

we can say that:

> for any finite rank operator K, the operator $\mathbb{U}_K(t,s)$ is weakly compact into $L^1(\Omega)$ for all $t>0$ and all $s\geq 0$.

Step 4. Let K be a compact operator into Y_1. Thanks to [5, Corollary 5.3, p.276], there exists a sequence $(K_n)_n$ of finite rank operators satisfying

$$\lim_{n\to\infty} \|K_n - K\|_{\mathcal{L}(Y_1)} = 0$$

and for which $\mathbb{U}_{K_n}(t,s)$ is a weakly compact operator into $L^1(\Omega)$ because of the step IV. Furthermore, Theorem 4.2 leads to

$$\lim_{n\to\infty} \|U_{K_n}(t) - U_K(t)\| = 0.$$

On the other hand, preceding as before by using (5.14), (5.15) and (5.16), we infer (5.17) and therefore $\mathbb{U}_K(t,s)$ is a weakly compact operator into $L^1(\Omega)$ because of the first point of Lemma 2.2. The proof is now achieved. $\square$

Now, we are finally able to compute the essential type $\omega_{\rm ess}(V_K)$ of the semigroup $V_K = (V_K(t))_{t\geq 0}$ as follows.

Theorem 5.3. *Let K be a positive compact operator from Y_1 into itself. If the hypothesis* (3.5) *holds, then we have*

$$\omega_{\rm ess}(V_K) \leq -\underline{\sigma}, \tag{5.18}$$

where $\underline{\sigma}$ is given by (3.8).

Proof. Let $t > 0$ be fixed. We divide this proof into two steps.

Step 1. Let $s \geq 0$ be given. First, due to (3.24) and (3.9) we obtain that

$$\begin{aligned} V_K(t) - V_0(t) = U_K(t) - U_0(t) + \int_0^t U_K(t-s)SV_K(s)ds \\ + \int_0^t U_0(t-s)(-S)V_0(s)ds. \end{aligned}$$

As $-S$, given by (3.6), is clearly a positive operator, then (3.21) together with the positivity of the semigroup $V_0 = (V_0(t))_{t\geq 0}$ (Lemma 3.2) imply that

$$\begin{aligned} &V_K(t) - V_0(t) \\ &\leq U_K(t) - U_0(t) + \int_0^t U_K(t-s)SV_K(s)ds + \int_0^t U_K(t-s)(-S)V_0(s)ds \\ &\leq U_K(t) - U_0(t) + \int_0^t U_K(t-s)S\Big[V_K(s) - V_0(s)\Big]ds \end{aligned}$$

which leads, by (3.25), to

$$0 \leq V_K(t) - V_0(t) \leq U_K(t) - U_0(t). \tag{5.19}$$

On the other hand, due to (3.9) together with the non-positivity of the operator S, we easily infer that $0 \leq V_0(s) \leq U_0(s)$. So, this together with (5.19) lead to

$$0 \leq \Big[V_K(t) - V_0(t)\Big]V_0(s)\Big[V_K(t) - V_0(t)\Big] \leq \mathbb{U}_K(t,s),$$

where $\mathbb{U}_K(t,s)$ is given by (5.4). Now, Proposition 5.2 together with the second point of Lemma 2.2, imply that

$$\Big[V_K(t) - V_0(t)(t)\Big]V_0(s)\Big[V_K(t) - V_0(t)\Big]$$

is a weakly compact operator into $L^1(\Omega)$, for all $t > 0$ and all $s \geq 0$.

Step 2. Thanks to the step above, we infer that

$$\Big[V_K(t) - V_0(t)(t)\Big] V_0(nt) \Big[V_K(t) - V_0(t)\Big]$$

is a weakly compact operator into $L^1(\Omega)$, for all integers $n \geq 0$. Therefore, for all integers $N \geq 1$, the following finite sum

$$\begin{aligned} \mathbb{V}_N(t) &:= \frac{1}{2} \sum_{n=0}^{N} \frac{1}{2^n} \Big[V_K(t) - V_0(t)\Big] V_0(nt) \Big[V_K(t) - V_0(t)\Big] \\ &= \frac{1}{2} \Big[V_K(t) - V_0(t)\Big] \Big[\sum_{n=0}^{N} \frac{1}{2^n} V_0(nt)\Big] \Big[V_K(t) - V_0(t)\Big] \end{aligned}$$

is also a weakly compact operator into $L^1(\Omega)$. Du to (3.7), it follows that $2 \in \rho(V_0(t))$ which implies that

$$\lim_{N \to \infty} \Big\| \mathbb{V}_N(t) - \Big[V_K(t) - V_0(t)\Big] (2 - V_0(t))^{-1} \Big[V_K(t) - V_0(t)\Big] \Big\| = 0$$

and therefore

$$\Big[V_K(t) - V_0(t)\Big] (2 - V_0(t))^{-1} \Big[V_K(t) - V_0(t)\Big]$$

is a weak compact operator into $L^1(\Omega)$ because of the first point of Lemma 2.2. Hence, the following operator

$$\Big(\Big[V_K(t) - V_0(t)\Big] (2 - V_0(t))^{-1} \Big)^4$$

is compact into $L^1(\Omega)$ because of the third point of Lemma 2.2, which leads, by Lemma 2.1, to

$$\omega_{\text{ess}}(V_K) = \omega_{\text{ess}}(V_0). \tag{5.20}$$

Finally, du to (3.7) together with (2.1) and (2.3) and (5.20), we clearly infer that $\omega_{\text{ess}}(V_K) = \omega_{\text{ess}}(V_0) \leq \omega(V_0) \leq -\underline{\sigma}$ and therefore (5.18) easily follows. The proof is now achieved. $\square$

Now, we are able to prove that the semigroup $V_K = (V_K(t))_{t \geq 0}$, governing the general model (1.1), (1.3), possesses Asynchronous Exponential Growth property. Before we start, note that in the case $\|K\| < 1$, the model (1.1), (1.3) is biologically uninteresting because the bacterial density is decreasing. Indeed, for all t and all s with $t > s$, (3.28) implies that

$$\|V_K(t)\varphi\|_1 = \|V_K(t-s)V_K(s)\varphi\|_1 \leq e^{-(t-s)\underline{\sigma}} \|V_K(s)\varphi\|_1 \leq \|V_K(s)\varphi\|_1$$

for all initial data $\varphi \in L^1(\Omega)$. Therefore, we well understand that $\|K\| > 1$ is closely related to an increasing number of bacteria during each mitotic. This situation is the most biologically observed for which Asynchronous Exponential Growth property is given by

Theorem 5.4. *Suppose that* (3.5) *holds and let K be a positive, irreducible and compact operator in Y_1 such that*

$$r(K_{\overline{\sigma} - \underline{\sigma}}) > 1,$$

where $\underline{\sigma}$ and $\overline{\sigma}$ are given by (3.8) *and* (3.27). *Then, there exist a rank one projector $\mathbb{P}$ in $L^1(\Omega)$ and an $\varepsilon > 0$ such that for every $\eta \in (0, \varepsilon)$, there exist $M(\eta) \geq 1$ satisfying*

$$\|e^{-\omega_0(V_K)t} V_K(t) - \mathbb{P}\|_{\mathcal{L}(L^1(\Omega))} \leq M(\eta) e^{-\eta t} \quad t \geq 0.$$

Proof. Thanks to Lemma 3.7, it follows that $V_K = (V_K(t))_{t\geq 0}$ is a positive and irreducible semigroup. Furthermore, (5.18) and (3.26) obviously lead to $\omega_{\rm ess}(V_K) < \omega(V_K)$. Now, all the conditions of Lemma 1.1 are satisfied. $\square$

When there is no bacterial mortality or bacteria loss due to causes other than division (i.e., $\sigma = 0$), then Asynchronous Exponential Growth property of the $U_K = (U_K(t))_{t\geq 0}$ can be described as follows.

Corollary 5.5. *Let K be a positive, irreducible and compact operator in Y_1 with $r(K) > 1$. Then, there exist a rank one projector $\mathbb{P}$ in $L^1(\Omega)$ and an $\varepsilon > 0$ such that for every $\eta \in (0, \varepsilon)$, there exist $M(\eta) \geq 1$ satisfying*

$$\|e^{-\omega_0(U_K)t} U_K(t) - \mathbb{P}\|_{\mathcal{L}(L^1(\Omega))} \leq M(\eta) e^{-\eta t} \quad t \geq 0.$$

The proof follows from Theorem above because $\overline{\sigma} = \underline{\sigma} = 0$.

Remark 5.6. According to a lot of modifications, we claim that all the results of this work still hold into $L^p(\Omega)$ $(p > 1)$. However, the norm of such space have no biological meaning.

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Addendum posted on June 24, 2013.

The author would like to make the following changes:

(1) Definition 3.4: (Kc) must be replaced by: "$\|K\mathbb{I}_\omega\| < 1$ *for some* $\omega > 0$ *and* $\|K\| \geq 1$. $\mathbb{I}_\omega$ *denotes the characteristic operator of the set* (ω, ∞)."

(2) Lemmas 3.5: "*K compact*" must be inserted in the preamble. Line 10 of the proof: "*Next, if (Kc) holds*" must be replaced by "*As K is compact*".

(3) Lemmas 3.5, 3.6, 3.7, 4.1 and Theorems 4.2, 4.3: "*compact*" must be inserted in the preamble (for the operators K, H and K_n).

(4) Page 13: "*In the sequel...to...exist*" (Lines 7 to 10) must be deleted.

(5) Lemma 5.1 to Corollary 5.5: "*K admissible*" must be inserted in the preamble.

(6) Lemma 3.7(3) and Theorem 5.4: "$r(K_{\overline{\sigma}-\underline{\sigma}} >$" must be replaced by "$r(\overline{K}_{\overline{\sigma}-\underline{\sigma}}) > 1$."

(7) Proof of Proposition 5.2 must start by: "*In the sequel, all one or finite rank operators must be admissible like the operator K in the preamble. As*

$$\|K_n\mathbb{I}_\omega\| \le \|K_n\mathbb{I}_\omega - K\mathbb{I}_\omega\| + \|K\mathbb{I}_\omega\| \le \|K_n - K\| + \|K\mathbb{I}_\omega\|,$$

we infer then that all approximation operators K_n of K are also admissible for large integer n."

End of addendum.

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