Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 231, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

SOLUTIONS TO A PARTIAL INTEGRO-DIFFERENTIAL PARABOLIC SYSTEM ARISING IN THE PRICING OF FINANCIAL OPTIONS IN REGIME-SWITCHING JUMP DIFFUSION MODELS

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Abstract. We study a complex system of partial integro-differential equations (PIDE) of parabolic type modeling the option pricing problem in a regime-switching jump diffusion model. Under suitable conditions, we prove the existence of solutions of the PIDE system in a general domain by using the method of upper and lower solutions.

1. Introduction

The problem of pricing derivatives in financial mathematics often leads to studying partial differential and/or integral equations. The typical differential equations obtained are of parabolic type. In recent years, the complexity of the equations studied has increased, due to the inclusion of stochastic volatility, stochastic interest rate, and jumps in the mathematical models governing the dynamics of the underlying asset prices. The integral terms in a partial differential equation with integral terms (henceforth PIDE) come from modeling jumps in the underlying asset prices.

Florescu and Mariani [4] considered a continuous time asset price model containing both stochastic volatility and discontinuous jumps. In this model, the volatility is driven by a second correlated Brownian motion and the jump is modeled by a compound Poisson process. Standard risk-neutral pricing principle is used to obtain a single second-order partial integro-differential equation (PIDE) for the prices of European options written on the asset. Motivated by this financial mathematics problem, a general integro-differential parabolic problem is posed and studied in the cited work [4]. The existence of solution is proved by employing a method of upper and lower solutions and a diagonal argument. Moreover, the proof can provide an approximation method for numerically finding the solution of the general type PIDE which was later implemented in [5]. In the current work we are discussing a more general model capable of producing realistic paths. The resulting option

2000 *Mathematics Subject Classification.* 35K99, 35F99, 45K05, 45K99.
Key words and phrases. Partial integro-differential equations; option pricing; regime-switching jump diffusion; upper and lower solutions.

Submitted December 15, 2011. Published December 21, 2012.

price may be found as the solution of a system of PIDE's, which to our knowledge, have never been studied before by the method employed in this work.

The main result of this paper is Theorem 3.2 which provides conditions on the integral terms in the PIDE system which guarantee the existence of the solution to this system. The emphasis in this work is on the applied mathematical methods rather than the stochastic process due to the technical nature of this result.

2. Motivating the PIDE System under Study

In this section, we introduce and motivate the regime-switching jump diffusion model, the option pricing problem, and the resulting system of partial integro-differential equations we will study in the next section.

2.1. **About the suitability of the stochastic model postulated.** From the beginning of the 20-th century starting with Louis Jean-Baptiste Alphonse Bachelier (1870-1946) researchers have been looking for mathematical models which are capable of capturing the main features of an observed price path. The most famous attempt is the Black-Scholes-Merton model [2, 11] which influenced so much of the literature on asset pricing. Of course, the model is now known to be too simple for high frequency data and many attempts have been made in the last 20 years to capture the complexity exhibited by the evolution of asset prices. In recent years, considerable attention has been drawn to regime-switching models in financial mathematics aiming to include the influence of macroeconomic factors on the individual asset price behavior. See, for example [6, 9, 10]. In this setting, asset prices are dictated by a number of stochastic differential equations coupled by a finite-state Markov chain, which represents various randomly changing economical factors. Mathematically, the regime-switching models generalize the traditional models in such a way that various coefficients in the models depend on the Markov chain. Consequently, a system (not a single one) of coupled PDEs (or PIDEs) is obtained for option prices.

To further illustrate the motivation of this study, in Figure 1 we present the one day evolution of high frequency data (all trades) for a particular equity gathered from a single exchange. This image or sample path is generally representative for many traded assets in any markets during any given day. Looking at the image we recognize several characteristics which can be captured by using a regime-switching jump diffusion model. The price path seems to jump in several places during the day (either up or down) and in between these jumps it seems to follow processes with perhaps different parameters. For example, the variability at the beginning of the day seems to be larger than the variability in the middle of the day. As described next, in a regime-switching jump diffusion model the process jumps at random times by a random amount and, in between jumps, the process could follow diffusions with distinct coefficients. We believe such a model is appropriate for describing the observed features of the asset price during the day.

2.2. **Regime-switching jump diffusion model.** We assume that all the stochastic processes in this paper are defined on some underlying complete probability space $(\mathcal{S}, \mathcal{F}, \mathcal{P})$. Let B_t be a one-dimensional standard Brownian motion. Let α_t be a continuous-time Markov chain with state space $\mathcal{M} := \{1, \dots, m\}$. Let $Q = (q_{ij})_{m\times m}$ be the intensity matrix (or the generator) of α_t. In this context the generator q_{ij}, $i, j = 1, \dots, m$ satisfy:

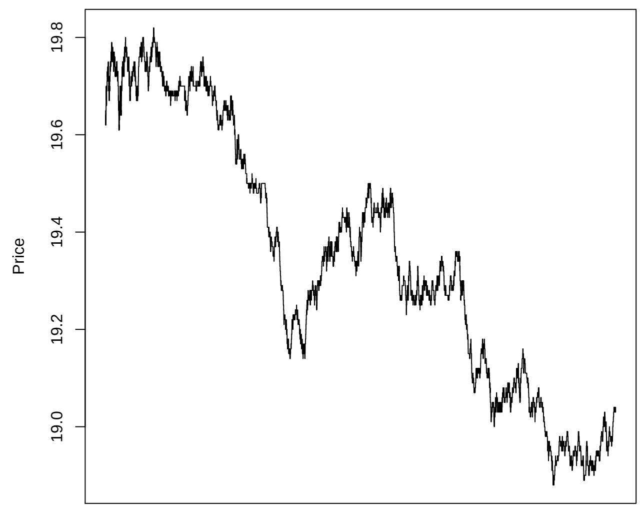

FIGURE 1. Tick data for one trading day and a certain equity

(I) $q_{ij} \geq 0$ if $i \neq j$;
(II) $q_{ii} = -\sum_{j \neq i} q_{ij}$ for each $i = 1, \dots, m$.

We assume that the Brownian motion B_t and the Markov chain α_t are independent.

Let N_t be a Cox process (a specialized non-homogeneous Poisson process) with regime-dependent intensity λ_{α_t}. Thus, when the current state is $\alpha_t = i$, the time until the next jump is given by an exponential random variable with mean $1/\lambda_i$. N_t models the number of the jumps in the asset price up to time t. Let the jump sizes be given by a sequence of iid random variables $Y_i, i = 1, 2, \dots,$ with probability density $g(y)$. Assume that the jump sizes Y_i, $i = 1, 2, \dots,$ are independent of B_t and α_t.

We model the time evolution of the asset price S_t by using the regime-switching jump diffusion:

$$\frac{dS_t}{S_t} = \mu_{\alpha_t} dt + \sigma_{\alpha_t} dB_t + dJ_t, \quad t \geq 0, \tag{2.1}$$

where μ_{α_t} and σ_{α_t} are the appreciation rate and the volatility rate of the asset S_t, respectively. J_t is the jump component given by

$$J_t = \sum_{k=1}^{N_t} (Y_k - 1). \tag{2.2}$$

The $Y_i - 1$ values represent the percentage of the asset price by which the process jumps. Note that, in between switching times the process follows a regular jump diffusion with constant coefficients. However, the coefficients are switching as governed by the corresponding state of the Markov chain. In the model setting (2.1) the volatility is modeled as a finite-state stochastic Markov chain σ_{α_t}. As further reference for the model usefulness, (2.1) may be considered as a discrete approximation of a continuous-time diffusion model for the stochastic volatility (e.g. the Heston's model). See Liu [9] and references therein for more details.

2.3. **The option pricing problem.** Given that the asset price process follows the hypothesized model (2.1) we look into the problem of derivative pricing written on the corresponding asset. To this end denote r_{α_t} the risk-free interest rate corresponding to the state α_t of the Markov chain.

We consider an European type option written on the asset S_t with maturity $T < \infty$. Let $V_i(S,t)$ denote the option value functions at time to maturity t, when the asset price $S_t = S$ and the regime $\alpha_t = i$ (assuming that the regime α_t is observable). Under these assumptions the value functions $V_i(S,t)$, $i = 1,\dots,m$, satisfy the system of PIDEs

$$\begin{aligned} &\frac{1}{2}\sigma_i^2 S^2 \frac{\partial^2 V_i}{\partial S^2} + (r_i - \lambda_i \kappa) S \frac{\partial V_i}{\partial S} - r_i V_i - \frac{\partial V_i}{\partial t} \\ &+ \lambda_i E[V_i(SY,t) - V_i(S,t)] + \sum_{j\neq i} q_{ij}[V_j - V_i] = 0, \end{aligned} \tag{2.3}$$

where we use the notation $\kappa = E[Y-1] = \int (y-1)g(y)dy$. Recalling that $q_{ii} = -\sum_{j\neq i} q_{ij}$ and using the density $g(y)$, we can rewrite (2.3) as

$$\begin{aligned} &\frac{1}{2}\sigma_i^2 S^2 \frac{\partial^2 V_i}{\partial S^2} + (r_i - \lambda_i \kappa) S \frac{\partial V_i}{\partial S} - (r_i + \lambda_i - q_{ii}) V_i - \frac{\partial V_i}{\partial t} \\ &= -\lambda_i \int V_i(Sy,t) g(y) dy - \sum_{j\neq i} q_{ij} V_j. \end{aligned} \tag{2.4}$$

Standard risk-neutral pricing principle is used for the derivation of equation (2.3) from the dynamics (2.1) (not presented here), we refer for instance to [6, 7].

Such types of systems are complicated and hard to approach. In [4] we analyze a single PIDE which appears when the process exhibit jumps and has stochastic volatility. The approach was further implemented and an algorithm to calculate the solution was provided in [5]. The current problem is more complex by involving a system of PIDE's. However, note that the system is coupled only through the final term in the equation (2.4), the rest of the terms in each equation i are in the respective $V_i(\cdot,\cdot)$. This fact provides hope that an existence proof (and a potential solving algorithm) may be provided in the current situation as well.

As a historical note William Feller (1906-1970) and his students developed the semigroup theory for Markov Processes and there is a well known direct link through them with the resulting PDE's for option pricing (see e.g., [3] or [13] for excellent reviews of this connection). However, they worked with diffusion processes (and later jump diffusion processes) characterizing Markov processes and these models lead to simple PIDE's.

In the case presented here, while the regime switching is governed by a continuous-time Markov chain and while each process being switched is indeed a continuous-time Markov process (jump diffusion), the overall structure may not be described by a simple Markov process with a diffusion + density type infinitesimal generator. Instead, the resulting overall Markov process is complex and produces the type of coupled systems of PIDE's studied in this paper. The work we present proves an existence of solution theorem for such systems. This system is very different from the work published in Pitt's dissertation in 1967 [12] and naturally the analysis follows different techniques, thus our proof (about a different problem) is different than the analysis done by Pitt, that was later extended to time dependent coefficients on a simpler Markov process.

3. The General PIDE System

To obtain a solution to the system (2.4) we formulate the problem using more general terms. This will provide a universal approach to the kind of PIDE systems arising when solving complex option pricing problems.

We first recall that the Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

becomes a heat type equation after performing the classical (Euler type) change of variable: $S = Ee^x$ and $t = T - \frac{2\tau}{\sigma^2}$, where E, T, σ are constants, see for example [14]. From now on, we assume that this classical change of variable for Black-Scholes type equations was performed.

To this end, let $\Omega \subset \mathbb{R}^d$ be an unbounded smooth domain, and we consider a collection of m functions $u_i(x,t)$, $i = 1,\dots,m$, where $x = (x_1, x_2, \dots, x_d)$ ($u_i : \mathbb{R}^d \times [0,T] \to \mathbb{R}$). Let the operator $\mathcal{L}_i$ be defined by:

$$\mathcal{L}_i u_i = \sum_{j=1}^{d}\sum_{k=1}^{d} a^i_{jk}(x,t)\frac{\partial u_i}{\partial x_j \partial x_k} + \sum_{j=1}^{d} b^i_j(x,t)\frac{\partial u_i}{\partial x_j} + c^i(x,t)u_i,\ i = 1,\dots,m, \tag{3.1}$$

where the coefficients a^i_{jk}, b^i_j and c^i, $i \in \{1,\dots,m\}$; $j,k \in \{1,\dots,d\}$ belong to the Hölder space $C^{\delta,\delta/2}(\overline{\Omega} \times [0,T])$ and satisfy the following conditions:

- There exist two constants Λ_1, Λ_2 with $0 < \Lambda_1 \le \Lambda_2 < \infty$ such that

$$\Lambda_1|v|^2 \le \sum_{j=1}^{d}\sum_{k=1}^{d} a^i_{jk}(x,t)v_j v_k \le \Lambda_2|v|^2 \quad \text{for } v = (v_1,\dots,v_d)^T \in \mathbb{R}^d. \tag{3.2}$$

- There exists a constant $C > 0$ such that

$$|b^i_j(x,t)| \le C. \tag{3.3}$$

- The functions

$$c^i(x,t) \le 0. \tag{3.4}$$

This general formulation encompasses all models presented including as degenerate cases the diffusion model of Black Scholes and the jump diffusion of Merton. The conditions are needed to ensure the existence of solution for a system of the type (2.4). Generally, these conditions are satisfied by most option pricing equations arising in finance.

The generalized problem corresponding to the system of PIDE's in equation (2.4) on an unbounded smooth domain Ω is:

$$\begin{gathered}
\mathcal{L}_i u_i - \frac{\partial u_i}{\partial t} = \mathcal{G}_i(t,u_i) - \sum_{j\neq i} q_{ij}u_j \quad \text{in } \Omega \times (0,T) \\
u_i(x,0) = u_{i,0}(x) \quad \text{on } \Omega \times \{0\} \\
u_i(x,t) = h_i(x,t) \quad \text{on } \partial\Omega \times (0,T)
\end{gathered} \tag{3.5}$$

for $i = 1, \dots, m$, where $\mathcal{G}_i$, are continuous integral operators. We assume that the boundary conditions $u_{i,0} \in C^{2+\delta}(\overline{\Omega})$, and $h_i \in C^{2+\delta,1+\delta/2}(\overline{\Omega} \times [0,T])$ satisfy the compatibility condition

$$h_i(x,0) = u_{i,0}(x), \quad \text{for any } x \in \partial\Omega,\ i = 1, \dots, m. \tag{3.6}$$

We note that as applied to problem (2.4) the operators $\mathcal{L}_i$ and $\mathcal{G}_i$ differ in the parameter values only, not in functional form. However, the general problem formulation as described above contains the case when the option is written on a basket of assets (not only a single stock) which are all modeled by different jump-diffusion type processes and they are all dependent on the same regime-switching Markov process α_t.

The goal is to establish the existence of a solution to the system (3.5) using the method of upper and lower solutions.

Definition 3.1. A collection of m smooth functions $u = \{u_i, 1 \le i \le m\}$ is called an upper (lower) solution of problem (3.5) if:

$$\begin{gathered} \mathcal{L}_i u_i - \frac{\partial u_i}{\partial t} \le (\ge) \mathcal{G}_i(t,u_i) - \sum_{j \ne i} q_{ij} u_j \quad \text{in } \Omega \times (0,T) \\ u_i(x,0) \ge (\le) u_{i,0}(x) \quad \text{on } \Omega \times \{0\} \\ u_i(x,t) \ge (\le) h_i(x,t) \quad \text{on } \partial\Omega \times (0,T) \end{gathered} \tag{3.7}$$

for $i = 1, \dots, m$.

Our main result is stated in the following theorem.

Theorem 3.2. *Let the operators $\mathcal{L}_i$ and $\mathcal{G}_i$, $1 \le i \le m$ be as defined above. Assume that either:*

- *for each $1 \le i \le m$, $\mathcal{G}_i$ is non-increasing with respect to u_i, or*
- *for each $1 \le i \le m$, there exists a continuous and increasing one-dimensional function f_i such that $\mathcal{G}_i(t,u_i) - f_i(u_i)$ is non-increasing with respect to u_i.*

Furthermore, assume there exist a lower solution $\alpha = \{\alpha_i, 1 \le i \le m\}$ and an upper solution $\beta = \{\beta_i, 1 \le i \le m\}$ of problem (3.5) satisfying $\alpha \le \beta$ componentwise (i.e., $\alpha_i \le \beta_i$, $1 \le i \le m$) in $\Omega \times (0,T)$. Then (3.5) admits a solution u such that $\alpha \le u \le \beta$ in $\Omega \times (0,T)$.

3.1. **The method of upper and lower solutions.** In this section we present a proof of our main result, Theorem 3.2. To this end, we first solve an analogous problem in a bounded domain and then extend the solution to the unbounded domain $\Omega \times (0,T)$. We note that we need this extension since in general option problems are solved on $(S_1, \dots, S_d, t) \in (0,\infty)^d \times [0,T]$. Please also note that the theory may be used just as well for perpetual options (when $T = \infty$).

Lemma 3.3. *Let U be a smooth and bounded subset of Ω. Then, there exists a unique collection of functions $\varphi_U = \{\varphi_{U,i}, 1 \le i \le m\}$ with $\varphi_{U,i} \in C^{2+\delta,1+\delta/2}(\overline{U} \times [0,T])$ such that*

$$\begin{gathered} \mathcal{L}_i \varphi_{U,i} - \frac{\partial \varphi_{U,i}}{\partial t} = 0, \quad (x,t) \in U \times (0,T), \\ \varphi_{U,i}(x,0) = u_{i,0}(x), \quad x \in U, \\ \varphi_{U,i}(x,t) = h_i(x,t), \quad (x,t) \in \partial U \times [0,T], \end{gathered} \tag{3.8}$$

for $i = 1, \dots, m$. Moreover, if α and β are respectively a lower and an upper solution of this reduced problem (3.8) *with $\alpha \le \beta$ in $U \times (0, T)$, then*

$$\alpha(x,t) \le \varphi_U(x,t) \le \beta(x,t), \quad (x,t) \in \overline{U} \times [0,T]. \tag{3.9}$$

Proof. Note that the homogeneous system (3.8) is decoupled. Thus, solving the system means solving the individual PDE's. Applying Florescu and Mariani [4, Lemma 2.1] to each of the m component equations, we obtain the expected result. □

The next result is crucial and it is the lemma that makes the transition from simple PDE's to a complex system of PIDE's on a bounded domain.

Lemma 3.4. *Let $U \in \mathbb{R}^d$ be a smooth and bounded domain. Let $0 < \tilde{T} < T$. Let φ_U be defined as in Lemma 3.3. Assume α and β are respectively a lower and an upper solution of the initial problem* (3.5) *on the bounded domain $\overline{U} \times [0, \tilde{T}]$ with $\alpha \le \beta$. Then the problem*

$$\begin{gathered} \mathcal{L}_i u_i - \frac{\partial u_i}{\partial t} = \mathcal{G}_i(t, u_i) - \sum_{j \ne i} q_{ij} u_j \quad \textit{in } U \times (0, \tilde{T}) \\ u_i(x, 0) = u_{i,0}(x) \quad \textit{on } U \times \{0\} \\ u_i(x,t) = \varphi_{U,i}(x,t) \quad \textit{on } \partial U \times (0, \tilde{T}) \end{gathered} \tag{3.10}$$

for $i = 1, \dots, m$, admits at least one solution u such that $\alpha(x,t) \le u(x,t) \le \beta(x,t)$ for $x \in U$, $0 \le t \le \tilde{T}$.

Proof. Suppose first that for each $1 \le i \le m$, $\mathcal{G}_i$ is non-increasing with respect to u_i. Let $V = U \times (0, \tilde{T})$. In this proof we use the following result provided by the existence and uniqueness of the solution for homogeneous PDE's (Lemma 3.3) and the extension to non-homogeneous PDE's (this is a standard extension, see for example [8]):

Given a collection of m functions with $w_i \in W_p^{2,1}(V)$, the problem

$$\begin{gathered} \mathcal{L}_i v_i - \frac{\partial v_i}{\partial t} = \mathcal{G}_i(t, w_i) - \sum_{j \ne i} q_{ij} w_j \quad \text{in } U \times (0, \tilde{T}) \\ v_i(x, 0) = u_{i,0}(x) \quad \text{on } U \times \{0\} \\ v_i(x,t) = \varphi_{U,i}(x,t) \quad \text{on } \partial U \times (0, \tilde{T}) \end{gathered} \tag{3.11}$$

for $i = 1, \dots, m$, has a unique solution $v = \{v_i, 1 \le i \le m\}$ with $v_i \in W_p^{2,1}(V)$.

The idea in the proof of the lemma is to construct a convergent sequence of functions and show that the limit is a solution to the general system (3.10).

To this end we use an inductive construction starting with $u^0 = \alpha$ and constructing a sequence of solutions $\{u^n, n = 0, 1, 2, \dots\}$ such that $u^{n+1} = \{u_i^{n+1}, 1 \le i \le m\}$ is the unique solution of the problem

$$\begin{gathered} \mathcal{L}_i u_i^{n+1} - \frac{\partial u_i^{n+1}}{\partial t} = \mathcal{G}_i(t, u_i^n) - \sum_{j \ne i} q_{ij} u_j^n \quad \text{in } U \times (0, \tilde{T}) \\ u_i^{n+1}(x, 0) = u_{i,0}(x) \quad \text{on } U \times \{0\} \\ u_i^{n+1}(x,t) = \varphi_{U,i}(x,t) \quad \text{on } \partial U \times (0, \tilde{T}) \end{gathered} \tag{3.12}$$

for $i = 1, \dots, m$.

We claim that componentwise,

$$\alpha \le u^n \le u^{n+1} \le \beta \quad \text{in } \overline{U} \times [0, \tilde{T}],\ \forall n \in \mathbb{N}. \tag{3.13}$$

Using the maximum principle we can show that $u^1 \ge \alpha$, (i.e., $u_i^1 \ge \alpha_i$, for all $1 \le i \le m$). If we assume this is not true, there would exist an index $1 \le i_0 \le m$ and a point $(x_0, t_0) \in \overline{U} \times [0, \tilde{T}]$ such that $u_{i_0}^1(x_0, t_0) < \alpha_{i_0}(x_0, t_0)$. Since $u_{i_0}^1|_{\partial\overline{U}\times[0,\tilde{T}]} \ge \alpha_{i_0}|_{\partial\overline{U}\times[0,\tilde{T}]}$, we deduce that $(x_0, t_0) \in U \times (0, \tilde{T})$ (interior of the domain) and furthermore we may assume that (x_0, t_0) is a maximum point of $\alpha_{i_0} - u_{i_0}^1$ since both functions are smooth. Since the point is a maximum, it follows that $\nabla(\alpha_{i_0} - u_{i_0}^1)(x_0, t_0) = 0$, $\triangle(\alpha_{i_0} - u_{i_0}^1)(x_0, t_0) < 0$ and $\frac{\partial(\alpha_{i_0} - u_{i_0}^1)}{\partial t}(x_0, t_0) = 0$. Since $\mathcal{L}_{i_0}$ is strictly elliptic (by the conditions imposed on its coefficients), we have

$$\mathcal{L}_{i_0}(\alpha_{i_0} - u_{i_0}^1)(x_0, t_0) < 0. \tag{3.14}$$

On the other hand, in view of the definition (3.7) for the lower solution α and the way u^1 is constructed in (3.12), we have

$$\begin{aligned} \mathcal{L}_{i_0} u_{i_0}^1(x_0, t_0) - \frac{\partial u_{i_0}^1}{\partial t}(x_0, t_0) &= \mathcal{G}_{i_0}(t, \alpha_{i_0})(x_0, t_0) - \sum_{j \ne i_0} q_{i_0 j} \alpha_j(x_0, t_0) \\ &\le \mathcal{L}_{i_0}\alpha_{i_0}(x_0, t_0) - \frac{\partial \alpha_{i_0}}{\partial t}(x_0, t_0), \end{aligned} \tag{3.15}$$

resulting in $\mathcal{L}_{i_0}(\alpha_{i_0} - u_{i_0}^1)(x_0, t_0) \ge 0$, a contradiction with (3.14). Therefore, we must have $u^1 \ge \alpha$.

Next, since for each $1 \le i \le m$, $\mathcal{G}_i$ is non-increasing with respect to u_i, and $q_{ij} \ge 0$ whenever $i \ne j$, we have for each $1 \le i \le m$ that

$$\mathcal{L}_i u_i^1 - \frac{\partial u_i^1}{\partial t} = \mathcal{G}_i(t, \alpha_i) - \sum_{j \ne i} q_{ij}\alpha_j \ge \mathcal{G}_i(t, \beta_i) - \sum_{j \ne i} q_{ij}\beta_j \ge \mathcal{L}_i\beta_i - \frac{\partial \beta_i}{\partial t}. \tag{3.16}$$

Again, by the maximum principle we obtain that $u^1 \le \beta$. The proof of this is identical with one above. If the inequality did not hold there would exist an index $1 \le i_0 \le m$ and a point $(x_0, t_0) \in \overline{U} \times [0, \tilde{T}]$ such that $u_{i_0}^1(x_0, t_0) > \beta_{i_0}(x_0, t_0)$. Since $u_{i_0}^1|_{\partial\overline{U}\times[0,\tilde{T}]} \le \beta_{i_0}|_{\partial\overline{U}\times[0,\tilde{T}]}$, we deduce that $(x_0, t_0) \in U \times (0, \tilde{T})$ and furthermore we may assume that (x_0, t_0) is a maximum point of $u_{i_0}^1 - \beta_{i_0}$. It follows that $\nabla(u_{i_0}^1 - \beta_{i_0})(x_0, t_0) = 0$, $\triangle(u_{i_0}^1 - \beta_{i_0})(x_0, t_0) < 0$ and $\frac{\partial(u_{i_0}^1 - \beta_{i_0})}{\partial t}(x_0, t_0) = 0$. Since $\mathcal{L}_{i_0}$ is strictly elliptic, we have

$$\mathcal{L}_{i_0}(u_{i_0}^1 - \beta_{i_0})(x_0, t_0) < 0. \tag{3.17}$$

On the other hand, (3.16) implies that at the maximum point (x_0, t_0), $\mathcal{L}_{i_0}(u_{i_0}^1 - \beta_{i_0})(x_0, t_0) \ge 0$, a contradiction with (3.17).

In the general induction step, given $\alpha \le u^{n-1} \le u^n \le \beta$, we can use a similar argument to show that $\alpha \le u^n \le u^{n+1} \le \beta$. First, we claim that $u^n \le u^{n+1}$. If this is not true, there exists an index $1 \le i_0 \le m$ and a point $(x_0, t_0) \in U \times (0, \tilde{T})$ such that

$$\mathcal{L}_{i_0}(u_{i_0}^n - u_{i_0}^{n+1})(x_0, t_0) < 0. \tag{3.18}$$

On the other hand, from the way the sequence is defined in (3.12) and the fact that, $\mathcal{G}_i$ is non-increasing with respect to u_i for each $1 \le i \le m$ and $q_{ij} \ge 0$ for $i \ne j$, we have

$$\mathcal{L}_i u_i^{n+1} - \frac{\partial u_i^{n+1}}{\partial t} = \mathcal{G}_i(t, u_i^n) - \sum_{j\neq i} q_{ij} u_j^n \leq \mathcal{G}_i(t, u_i^{n-1}) - \sum_{j\neq i} q_{ij} u_j^{n-1} = \mathcal{L}_i u_i^n - \frac{\partial u_i^n}{\partial t}. \tag{3.19}$$

It follows that at the maximum point (x_0, t_0), $\mathcal{L}_{i_0}(u_{i_0}^n - u_{i_0}^{n+1})(x_0, t_0) \geq 0$, a contradiction with (3.18). In a similar way, we can show that $u^{n+1} \leq \beta$.

We now define:

$$u(x,t) = \lim_{n\to\infty} u^n(x,t), \tag{3.20}$$

or componentwise,

$$u_i(x,t) = \lim_{n\to\infty} u_i^n(x,t), \quad \forall (x,t) \in \overline{U} \times [0, \tilde{T}],\ i = 1, \dots m. \tag{3.21}$$

Since $u^n \leq \beta$ and $\beta \in L^p(V)$, by the Lebesgue's dominated convergence theorem, we obtain that $\{u_i^n\}_{n=1}^\infty$ is a convergent sequence, therefore a Cauchy sequence in the complete space $L^p(V)$ for each $i = 1, \dots m$. Using the results in [8, Chapter 7], the $W_p^{2,1}$-norm of the difference $u_i^n - u_i^m$ can be controlled by its L^p-norm and the L^p-norm of its image under the operator $\mathcal{L}_i - \frac{\partial}{\partial t}$. Using these results, there exists a constant $C > 0$ such that

$$\begin{aligned}
&\|u_i^n - u_i^m\|_{W_p^{2,1}(V)} \\
&= \|D^2(u_i^n - u_i^m)\|_{L^p(V)} + \|(u_i^n - u_i^m)_t\|_{L^p(V)} \\
&\leq C\Big(\|\mathcal{L}_i(u_i^n - u_i^m) - \frac{\partial(u_i^n - u_i^m)}{\partial t}\|_{L^p(V)} + \|u_i^n - u_i^m\|_{L^p(V)}\Big).
\end{aligned} \tag{3.22}$$

By construction,

$$\mathcal{L}_i(u_i^n - u_i^m) - \frac{\partial(u_i^n - u_i^m)}{\partial t} = \mathcal{G}_i(\cdot, u_i^{n-1}) - \mathcal{G}_i(\cdot, u_i^{m-1}) - \sum_{j\neq i} q_{ij}(u_j^{n-1} - u_j^{m-1}). \tag{3.23}$$

Since $\mathcal{G}_i$ is a completely continuous operator, there is a constant $C_1 > 0$ such that,

$$\begin{aligned}
&\|\mathcal{G}_i(\cdot, u_i^{n-1}) - \mathcal{G}_i(\cdot, u_i^{m-1}) - \sum_{j\neq i} q_{ij}(u_j^{n-1} - u_j^{m-1})\|_{L^p(V)} \\
&\leq C_1 \sum_{j=1}^m \|u_j^{n-1} - u_j^{m-1}\|_{L^p(V)}.
\end{aligned} \tag{3.24}$$

Combining (3.22), (3.23), (3.24), it follows that $\{u_i^n\}_{n=1}^\infty$ is a Cauchy sequence in $W_p^{2,1}(V)$ for each $i = 1, \dots m$. Hence $u_i^n \to u_i$ in the $W_p^{2,1}$-norm, and thus $u = \{u_i, 1 \leq i \leq m\}$ is a strong solution of the problem (3.10).

Now suppose the condition on $\mathcal{G}_i(t, u_i)$ is that for each $1 \leq i \leq m$, there exists a continuous and increasing function f_i such that $\mathcal{G}_i(t, u_i) - f_i(u_i)$ is non-increasing with respect to u_i. Starting with $\tilde{u}^0 = 0$, we define recursively a sequence $\{\tilde{u}^n, n = 0, 1, \dots\}$ such that $\tilde{u}^{n+1} = \{\tilde{u}_i^{n+1} \in W_p^{2,1}(V), 1 \leq i \leq m\}$ is the unique solution of the problem

$$\begin{gathered}
\mathcal{L}_i \tilde{u}_i^{n+1} - \frac{\partial \tilde{u}_i^{n+1}}{\partial t} - f_i(\tilde{u}_i^{n+1}) = \mathcal{G}_i(t, \tilde{u}_i^n) - f_i(\tilde{u}_i^n) - \sum_{j\neq i} q_{ij}\tilde{u}_j^n \quad \text{in } U \times (0, \tilde{T}) \\
\tilde{u}_i^{n+1}(x, 0) = u_{i,0}(x) \quad \text{on } U \times \{0\} \\
\tilde{u}_i^{n+1}(x,t) = \varphi_{U,i}(x,t) \quad \text{on } \partial U \times (0, \tilde{T})
\end{gathered} \tag{3.25}$$

for $i = 1, \dots, m$. The same arguments as before may be repeated almost verbatim to show that

$$0 \le \tilde{u}^n \le \tilde{u}^{n+1} \le \beta \quad \text{in } \overline{U} \times [0, \tilde{T}],\ \forall n \in \mathbb{N}. \tag{3.26}$$

This will imply that, $\{\tilde{u}_i^n\}_{n=1}^{\infty}$ is a Cauchy sequence in $W_p^{2,1}(V)$ for each $i = 1, \dots m$. If we denote with $\tilde{u}_i = \lim_{n\to\infty} \tilde{u}_i^n$. Then $\tilde{u} = \{\tilde{u}_i, 1 \le i \le m\}$ is a strong solution of problem (3.10). Note that the function f is continuous and thus the solution of the modified problem (3.25) also solves the original system. □

Finally, all that remains is to extend the solution to the original unbounded domain.

Proof of Theorem 3.2. We first approximate the unbounded domain Ω by a non-decreasing sequence $(\Omega_N)_{N\in\mathbb{N}}$ of bounded smooth sub-domains of Ω, which can be chosen in such a way that $\partial\Omega$ is also the union of the non-decreasing sequence $\partial\Omega_N \cap \partial\Omega$.

In view of Lemma 3.4, we define $u^N = \{u_i^N, 1 \le i \le m\}$ as a solution of the problem

$$\begin{gathered} \mathcal{L}_i u_i - \frac{\partial u_i}{\partial t} = \mathcal{G}_i(t, u_i) - \sum_{j\ne i} q_{ij} u_j \quad \text{in } \Omega_N \times (0, T - \frac{1}{N}) \\ u_i(x,0) = u_{i,0}(x) \quad \text{on } \Omega_N \times \{0\} \\ u_i(x,t) = h_i(x,t) \quad \text{on } \partial\Omega_N \times (0, T - \frac{1}{N}) \end{gathered} \tag{3.27}$$

for $i = 1, \dots, m$, such that $0 = \alpha \le u^N \le \beta$ in $\Omega_N \times (0, T - \frac{1}{N})$. Define $V_N = \Omega_N \times (0, T - \frac{1}{N})$ and choose $p > d$. For $M > N$, we have:

$$\begin{aligned} &\|D^2(u_i^M)\|_{L^p(V_N)} + \|(u_i^M)_t\|_{L^p(V_N)} \\ &\le C_1 \Big(\|\mathcal{L}_i u_i^M - \frac{\partial u_i^M}{\partial t}\|_{L^p(V_N)} + \|u_i^M\|_{L^p(V_N)} \Big) \\ &\le C_1 \Big(\|\mathcal{G}_i(t, u_i^M) - \sum_{j\ne i} q_{ij} u_j^M \|_{L^p(V_N)} + \|\beta\|_{L^p(V_N)} \Big) \le C, \end{aligned} \tag{3.28}$$

for some constant C depending only on N.

By Morrey embedding theorem, $W_p^{2,1}(V_N) \hookrightarrow C(\overline{V}_N)$ (see e. g. [1]), there exists a subsequence that converges uniformly on $\overline{V}_N$.

Now, we apply the well known Cantor diagonal argument: for $N = 1$, we extract a subsequence of $u_i^M|_{\overline{\Omega}_1 \times [0,T-1]}$ (still denoted $\{u_i^M\}$ for notational simplicity) that converges uniformly to some function u_{i1} over $\overline{\Omega}_1 \times [0, T-1]$. Next, we extract a subsequence of $u_i^M|_{\overline{\Omega}_2 \times [0,T-\frac{1}{2}]}$ for $M \ge 2$ (still denoted $\{u_i^M\}$) that converges uniformly to some function u_{i2} over $\overline{\Omega}_2 \times [0, T-\frac{1}{2}]$, and so on. As the families $\{\Omega_N\}$ and $\{\partial\Omega_N \cap \partial\Omega\}$ are non-decreasing, it is clear that $u_{iN}(x,0) = u_{iN}(x)$ for $x \in \Omega_N$, and that $u_{iN}(x,t) = h(x,t)$ for $x \in \partial\Omega \cap \partial\Omega_N$ and $t \in (0, T - \frac{1}{N})$. Moreover, as $u_{i(N+1)}$ is constructed as the limit of a subsequence of $u_i^M|_{\overline{\Omega}_{N+1} \times [0,T-\frac{1}{N+1}]}$, which converges uniformly to some function u_{iN} over $\overline{\Omega}_N \times [0, T - \frac{1}{N}]$, it follows that $u_{i(N+1)}|_{\overline{\Omega}_N \times [0,T-\frac{1}{N}]} = u_{iN}$ for every N.

Thus, the diagonal subsequence (still denoted $\{u_i^M\}$) converges uniformly over compact subsets of $\Omega \times (0,T)$ to the function u_i defined as $u_i = u_{iN}$ over $\overline{\Omega}_N \times$

$[0, T - \frac{1}{N}]$. For $V = U \times (0, \tilde{T})$, $U \subset \Omega$ and $\tilde{T} < T$, taking $M, N \geq N_V$ for some N_V large enough we have that

$$\begin{aligned} &\|u_i^M - u_i^N\|_{W_p^{2,1}(V)} \\ &= \|D^2(u_i^M - u_i^N)\|_{L^p(V)} + \|(u_i^M - u_i^N)_t\|_{L^p(V)} \\ &\leq C\Big(\|\mathcal{L}_i(u_i^M - u_i^N) - \frac{\partial(u_i^M - u_i^N)}{\partial t}\|_{L^p(V)} + \|u_i^M - u_i^N\|_{L^p(V)}\Big). \end{aligned} \tag{3.29}$$

By construction,

$$\mathcal{L}_i(u_i^M - u_i^N) - \frac{\partial(u_i^M - u_i^N)}{\partial t} = \mathcal{G}_i(\cdot, u_i^{M-1}) - \mathcal{G}_i(\cdot, u_i^{N-1}) - \sum_{j \neq i} q_{ij}(u_j^{M-1} - u_j^{N-1}). \tag{3.30}$$

As in the proof of Lemma 3.4, since $\mathcal{G}_i$ is continuous, $\alpha \leq u^N \leq \beta$, using the Lebesgue's dominated convergence theorem, it follows that $\{u_i^N\}$ is a Cauchy sequence in $W_p^{2,1}(V)$ for each $i = 1, \dots m$. Hence $u_i^N \to u_i$ in the $W_p^{2,1}(V)$-norm, and then $u = \{u_i, 1 \leq i \leq m\}$ is a strong solution in V. It follows that u satisfies the equation on $\Omega \times (0, T)$. Furthermore, it is clear that $u_i(x, 0) = u_{i,0}(x)$. For $M > N$ we have that $u_i^M(x, t) = u_i^N(x, t) = h_i(x, t)$ for $x \in \partial\Omega_N \cap \partial\Omega$, $t \in (0, T - \frac{1}{N})$. It then follows that u satisfies the boundary conditions $u_i(x, t) = h_i(x, t)$, $1 \leq i \leq m$ on $\partial\Omega \times [0, T)$. This completes the proof. □

4. Conclusion

In this paper we provided an existence proof of the solution of a system of PIDE's coupled in a very specific way. This coupling type arises in regime-switching models when the assets are all changing their stochastic dynamics according to the same continuous-time Markov chain α_t with intensity matrix $Q = (q_{ij})_{m \times m}$. The proof of our main result, Theorem 3.2, uses a construction that may be used in a numerical scheme implementing a PDE solver.

Theorem 3.2 is directly applicable to our motivating system (2.4), noticing that in this case $\mathcal{G}_i(t, u) = -\lambda_i \int u_i(Sy, t) g(y) dy$ is a non-increasing continuous operator in u_i and that $\alpha = \{\alpha_i(S, t) = 0,\ 1 \leq i \leq m\}$ is a lower solution of the option problem since the boundary conditions $u_{i,0}$ and h_i are nonnegative functions (represent the monetary value of the option on the boundaries). The upper solution also exists in these cases but its specific form depends on the jump distribution $g(y)$ and needs to be derived in each case. Note that the construction in Theorem 3.2 does not use the upper solution at all but its existence guarantees the convergence of the final solution. For specific examples of upper solutions as depending on the distribution $g(y)$ we refer to [4] and [5].

We want to add a remark about the general nature of Theorem 3.2. The result is applicable whenever the jump-diffusion process and the regime switching may be thought of as Markovian. In particular, a simple generalization is to make the distribution of jumps dependent on the state of the regime as in $g_{\alpha_t}(\cdot)$. This is directly solvable with the theory presented. As mentioned in the paper, options written on a basket of stocks which all follow different jump-diffusions but they are all dependent on the same regime switching process α_t also solve a system of PIDE's of the type analyzed in Theorem 3.2. Finally, the case when the assets are characterized using different switching regimes (correlated) is an example of a more complex case worthy of further investigation.

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