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HIGHER ORDER VIABILITY PROBLEM IN BANACH SPACES

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Abstract. We show the existence of viable solutions to the differential inclusion

$$\begin{gathered} x^{(k)}(t) \in F(t, x(t)) \\ x(0) = x_0, \quad x^{(i)}(0) = y_0^i, \quad i = 1, \dots, k-1, \\ x(t) \in K \quad \text{on } [0, T], \end{gathered}$$

where $k \geq 1$, K is a closed subset of a separable Banach space and $F(t, x)$ is an integrable bounded multifunction with closed values, (strongly) measurable in t and Lipschitz continuous in x.

1. Introduction

The aim of this paper is to establish the existence of local solutions of the higher-order viability problem

$$\begin{gathered} x^{(k)}(t) \in F(t, x(t)) \quad \text{a.e on } [0, T] \\ x(0) = x_0 \in K, x^{(i)}(0) = y_0^i \in \Omega_i, \quad i = 1, \dots, k-1, \\ x(t) \in K \quad \text{on } [0, T]. \end{gathered} \tag{1.1}$$

where K is a closed subset of a separable Banach space E, $F : [0, 1] \times K \to 2^E$ is a measurable multifunction with respect to the first argument and Lipschitz continuous with respect to the second argument, $\Omega_1, \dots, \Omega_{k-1}$ are open subsets of E and $(x_0, y_0^1, \dots, y_0^{k-1})$ is given in $K \times \prod_{i=1}^{k-1} \Omega_i$.

As regards the existence result of such problems, we refer to the work of Marco and Murillo [6], in the case when F is a convex and compact valued-multifunction in finite-dimensional space.

First-order viability problems with the non-convex Carathéodory Lipschitzean right-hand side in Banach spaces have been studied by Duc Ha [3]. The author established a multi-valued version of Larrieu's work [4], assuming the tangential condition:

$$\liminf_{h \to 0^+} \frac{1}{h} d\Big(x + \int_t^{t+h} F(s, x) ds, K\Big) = 0,$$

where K is the viability set and $d(.,.)$ denotes the Hausdorff's excess.

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Lupulescu and Necula [5] extended the result of Duc Ha [3] to first-order functional differential inclusions with the non-convex Carathéodory Lipschitzean right-hand side in Banach space. The authors used the same kind of tangential conditions that in Duc Ha [3].

Recently, Aitalioubrahim and Sajid [1] proved the existence of viable solution to the following second-order differential inclusions with the non-convex Carathéodory Lipschitzean right-hand side in Banach space E:

$$\begin{gathered}\ddot{x}(t) \in F(t, x(t), \dot{x}(t)) \quad \text{a.e.};\\ (x(0), \dot{x}(0)) = (x_0, y_0);\\ (x(t), \dot{x}(t)) \in K \times \Omega;\end{gathered} \tag{1.2}$$

where K (resp. Ω) is a closed subset (resp. an open subset) of E. The authors introduced the tangential condition:

$$\liminf_{h\to 0^+} \frac{1}{h^2} d\Big(x + hy + \frac{h}{2}\int_t^{t+h} F(s, x, y)ds, K\Big) = 0. \tag{1.3}$$

In this paper we extend this result to the higher-order case with the tangential condition:

$$\liminf_{h\to 0^+} \frac{1}{h^k} d\Big(x + \sum_{i=1}^{k-1} \frac{h^i}{i!} y^i + \frac{h^{k-1}}{k!}\int_t^{t+h} F(s, x)ds, K\Big) = 0.$$

2. Preliminaries and statement of the main result

Let E be a separable Banach space with the norm $\|.\|$. For measurability purpose, E (resp. $U \subset E$) is endowed with the σ-algebra $B(E)$ (resp. $B(U)$) of Borel subsets for the strong topology and $[0, 1]$ is endowed with Lebesgue measure and the σ-algebra of Lebesgue measurable subsets. For $x \in E$ and $r > 0$ let $B(x, r) := \{y \in E; \|y - x\| < r\}$ be the open ball centered at x with radius r and $\bar{B}(x, r)$ be its closure and put $B = B(0, 1)$. For $x \in E$ and for nonempty sets A, B of E we denote $d(x, A) := \inf\{\|y - x\|; y \in A\}$, $d(A, B) := \sup\{d(x, B); x \in A\}$ and $H(A, B) = \max\{d(A, B), d(B, A)\}$. A multifunction is said to be measurable if its graph is measurable. For more detail on measurability theory, we refer the reader to the book of Castaing-Valadier [2].

Let us recall the following Lemmas that will be used in the sequel. For the proofs, we refer the reader to [8].

Lemma 2.1. *Let Ω be a nonempty set in E. Assume that $F : [a, b] \times \Omega \to 2^E$ is a multifunction with nonempty closed values satisfying:*

- *For every $x \in \Omega$, $F(., x)$ is measurable on $[a, b]$;*
- *For every $t \in [a, b]$, $F(t, .)$ is (Hausdorff) continuous on Ω.*

Then for any measurable function $x(.) : [a, b] \to \Omega$, the multifunction $F(., x(.))$ is measurable on $[a, b]$.

Lemma 2.2. *Let $G : [a, b] \to 2^E$ be a measurable multifunction and $y(.) : [a, b] \to E$ a measurable function. Then for any positive measurable function $r(.) : [a, b] \to \mathbb{R}^+$, there exists a measurable selection $g(.)$ of G such that for almost all $t \in [a, b]$*

$$\|g(t) - y(t)\| \le d\big(y(t), G(t)\big) + r(t).$$

Before stating our main result, for any integer $n \geq 2$, we recall the tangent set of nth order denoted by $A_K^n(x_0, x_1, \dots, x_{n-1})$ introduced by Marco and Murillo [7, Def. 3.1] as follows.

For $y \in E$,we say that $y \in A_K^n(x_0, x_1, \dots, x_{n-1})$ if

$$\liminf_{h\to 0^+} \frac{n!}{h^k} d\Big(\sum_{i=0}^{n-1} \frac{h^i}{i!} x_i + \frac{h^n}{n!} y, K\Big) = 0.$$

Let $gr(A_K^n)$ be its graph.

Assume that the following hypotheses hold:

(H1) K is a nonempty closed subset in E and for $i = 1, \dots, k-1$, Ω_i is a nonempty open subset in E, such that $K \times \prod_{i=1}^{k-1} \Omega_i \subset gr(A_K^n)$.

(H2) $F : [0,1] \times K \to 2^E$ is a set valued map with nonempty closed values satisfying

(i) For each $x \in K$, $t \mapsto F(t,x)$ is measurable.

(ii) There is a function $m \in L^1([0,1], \mathbb{R}^+)$ such that for all $t \in [0,1]$ and for all $x_1, x_2 \in K$

$$H(F(t,x_1), F(t,x_2)) \leq m(t)\|x_1 - x_2\|$$

(iii) For all bounded subset S of K, there is a function $g_S \in L^1([0,1], \mathbb{R}^+)$ such that for all $t \in [0,1]$ and for all $x \in S$

$$\|F(t,x)\| := \sup_{z\in F(t,x)} \|z\| \leq g_S(t)$$

(H3) (**Tangential condition**) For every $(t, x, (y^1, \dots, y^{k-1})) \in [0,1] \times K \times \prod_{i=1}^{k-1} \Omega_i$,

$$\liminf_{h\to 0^+} \frac{1}{h^k} d\Big(x + \sum_{i=1}^{k-1} \frac{h^i}{i!} y^i + \frac{h^{k-1}}{k!} \int_t^{t+h} F(s,x) ds, K\Big) = 0.$$

Theorem 2.3. *If assumptions* (H1)–(H3) *are satisfied, then there exist $T > 0$ and an absolutely continuous function $x(.) : [0,T] \to E$, for which $x^{(i)}(.) : [0,T] \to E$, for all $i = 1, \dots, k-1$, is also absolutely continuous, such that $x(.)$ is solution of* (1.1).

3. Proof of the main result

Let $r > 0$ and $\bar{B}(y_0^i, r) \subset \Omega_i$ for $i = 1, \dots, k-1$. Choose $g \in L^1([0,1], \mathbb{R}^+)$ such that

$$\|F(t,x)\| \leq g(t) \quad \forall (t,x) \in [0,1] \times (K \cap B(x_0, r)). \tag{3.1}$$

Let $T_1 > 0$ and $T_2 > 0$ be such that

$$\int_0^{T_1} m(t) dt < 1, \tag{3.2}$$

$$\int_0^{T_2} \Big(g(t) + (k-1)r + 1 + \sum_{i=1}^{k-1} \|y_0^i\|\Big) dt < \frac{r}{2}. \tag{3.3}$$

For $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ such that

$$\Big|\int_{t_1}^{t_2} g(\tau) d\tau\Big| < \varepsilon \quad \text{if } |t_1 - t_2| < \eta(\varepsilon). \tag{3.4}$$

Set

$$T = \min\{T_1, T_2, 1\}, \tag{3.5}$$

$$\alpha = \min\Big\{T, \frac{1}{2}\eta(\frac{\varepsilon}{4}), \frac{\varepsilon}{4}\Big\}. \tag{3.6}$$

We will used the following Lemma to prove the main result.

Lemma 3.1. *If assumptions* (H1)–(H3) *are satisfied, then for all* $\varepsilon > 0$ *and all* $y(.) \in L^1([0,T], E)$*, there exists* $f \in L^1([0,T], E)$*,* $z(.) : [0,T] \to E$ *differentiable and a step function* $\theta : [0,T] \to [0,T]$ *such that*

- $f(t) \in F(t, z(\theta(t)))$ *for all* $t \in [0,T]$*;*
- $\|f(t) - y(t)\| \le d(y(t), F(t, z(\theta(t)))) + \varepsilon$ *for all* $t \in [0,T]$*;*
- $\big\|z^{(k-1)}(t) - y_0^{k-1} - \int_0^t f(\tau)d\tau\big\| \le \varepsilon$ *for all* $t \in [0,T]$*.*

Proof. Let $\varepsilon > 0$ and $y(.) \in L^1([0,T], E)$ be fixed. For $(0, x_0, (y_0^1, \dots, y_0^{k-1})) \in [0,T] \times K \times \prod_{i=1}^{k-1} \Omega_i$, by (H3),

$$\liminf_{h\to 0^+} \frac{1}{h^k} d\Big(x_0 + \sum_{i=1}^{k-1} \frac{h^i}{i!} y_0^i + \frac{h^{k-1}}{k!} \int_0^h F(s, x_0)ds, K\Big) = 0.$$

Hence, there exists $0 < h \le \alpha$ such that

$$d\Big(x_0 + \sum_{i=1}^{k-1} \frac{h^i}{i!} y_0^i + \frac{h^{k-1}}{k!} \int_0^h F(s, x_0)ds, K\Big) < \frac{\alpha h^k}{4k!}.$$

Put

$$h_0 := \max\Big\{h \in]0, \alpha] : d\Big(x_0 + \sum_{i=1}^{k-1} \frac{h^i}{i!} y_0^i + \frac{h^{k-1}}{k!} \int_0^h F(s, x_0)ds, K\Big) < \frac{\alpha h^k}{4k!}\Big\}.$$

In view of Lemma 2.2, there exists a function $f_0 \in L^1([0, h_0], E)$ such that $f_0(t) \in F(t, x_0)$ and

$$\|f_0(t) - y(t)\| \le d(y(t), F(t, x_0)) + \varepsilon, \quad \forall t \in [0, h_0].$$

Then

$$d\Big(x_0 + \sum_{i=1}^{k-1} \frac{h_0^i}{i!} y_0^i + \frac{h_0^{k-1}}{k!} \int_0^{h_0} f_0(s)ds, K\Big) < \frac{\alpha h_0^k}{4k!}.$$

So, there exists $x_1 \in K$ such that

$$\begin{aligned} &\frac{k!}{h_0^k} \|x_1 - \Big(x_0 + \sum_{i=1}^{k-1} \frac{h_0^i}{i!} y_0^i + \frac{h_0^{k-1}}{k!} \int_0^{h_0} f_0(s)ds\Big)\| \\ &\le \frac{k!}{h_0^k} d\Big(x_0 + \sum_{i=1}^{k-1} \frac{h_0^i}{i!} y_0^i + \frac{h^{k-1}}{k!} \int_0^h f_0(s)ds, K\Big) + \frac{\alpha}{4}, \end{aligned}$$

hence

$$\|\frac{x_1 - x_0 - \sum_{i=1}^{k-1} \frac{h_0^i}{i!} y_0^i}{-} \frac{1}{h_0} \int_0^{h_0} f_0(s)ds\| < \alpha.$$

Set

$$u_0 = \frac{x_1 - x_0 - \sum_{i=1}^{k-1} \frac{h_0^i}{i!} y_0^i}{\frac{h_0^k}{k!}},$$

then

$$x_1=\Big(x_0+\sum_{i=1}^{k-1}\frac{h_0^i}{i!}y_0^i+\frac{h_0^k}{k!}u_0\Big)\in K,\quad u_0\in\frac{1}{h_0}\int_0^{h_0}f_0(s)ds+\alpha B.$$

For $i=1,\dots,k-1$, put

$$y_1^i=\sum_{j=i}^{k-1}\frac{h_0^{j-i}}{(j-i)!}y_0^j+\frac{h_0^{k-i}}{(k-i)!}u_0.$$

Since $f_0(t)\in F(t,x_0)$ for all $t\in[0,h_0]$ and by (3.1), (3.3) and (3.6), we have

$$\begin{aligned}\|x_1-x_0\|&=\Big\|\sum_{i=1}^{k-1}\frac{h_0^i}{i!}y_0^i+\frac{h_0^k}{k!}u_0\Big\|\\&\le h_0\sum_{i=1}^{k-1}\|y_0^i\|+\int_0^{h_0}g(s)ds+h_0\alpha\\&\le\int_0^{h_0}\Big(g(s)+1+\sum_{i=1}^{k-1}\|y_0^i\|\Big)ds<\frac{r}{2}.\end{aligned}$$

Then $x_1\in B(x_0,r)$. For $i=1,\dots,k-2$, we have

$$\begin{aligned}\|y_1^i-y_0^i\|&\le\sum_{j=i+1}^{k-1}\frac{h_0^{j-i}}{(j-i)!}\|y_0^j\|+\frac{h_0^{k-i}}{(k-i)!}\|u_0\|\\&\le h_0\sum_{j=i+1}^{k-1}\|y_0^j\|+\int_0^{h_0}g(s)ds+h_0\alpha\\&\le\int_0^{h_0}\Big(g(s)+1+\sum_{j=i+1}^{k-1}\|y_0^j\|\Big)ds<\frac{r}{2},\end{aligned}$$

and

$$\begin{aligned}\|y_1^{k-1}-y_0^{k-1}\|&\le h_0\|u_0\|\\&\le\int_0^{h_0}g(s)ds+h_0\alpha\\&\le\int_0^{h_0}(g(s)+1)ds<\frac{r}{2}.\end{aligned}$$

Then $y_1^i\in B(y_0^i,r)$ for all $i=1,\dots,k-1$. We reiterate this process for constructing sequences $h_q,t_q,x_q,y_q^1,\dots,y_q^{k-1},f_q$ and u_q satisfying for some rank $m\ge 1$ the following assertions:

(a) For all $q\in\{0,\dots,m-1\}$,

$$h_q:=\max\Big\{h\in]0,\alpha]:d\Big(x_q+\sum_{i=1}^{k-1}\frac{h^i}{i!}y_q^i+\frac{h^{k-1}}{k!}\int_{t_q}^{t_{q+1}}F(s,x_q)ds,K\Big)<\frac{\alpha h^k}{4k!}\Big\};$$

(b) $t_0=0$, $t_{m-1}<T\le t_m$ with $t_q=\sum_{j=0}^{q-1}h_j$ for all $q\in\{1,\dots,m\}$;

(c) For all $q \in \{1, \dots, m\}$ and for all $j \in \{1, \dots, k-2\}$

$$x_q = x_0 + \sum_{j=1}^{k-1} \sum_{i=0}^{q-1} \frac{h_i^j}{j!} y_i^j + \sum_{i=0}^{q-1} \frac{h_i^k}{k!} u_i, \quad x_q \in K \cap B(x_0, r),$$

$$y_{q+1}^{k-1} := y_q^{k-1} + h_q u_q = y_0^{k-1} + \sum_{i=0}^{q} h_i u_i, \quad y_q^{k-1} \in B(y_0^{k-1}, r),$$

$$y_q^j = y_0^j + \sum_{l=j+1}^{k-1} \sum_{i=0}^{q-1} \frac{h_i^{l-j}}{(l-j)!} y_i^l + \sum_{i=0}^{q-1} \frac{h_i^{k-j}}{(k-j)!} u_i, \quad y_q^j \in B(y_0^j, r);$$

(d) For all $t \in [t_q, t_{q+1}]$ and for all $q \in \{0, \dots, m-1\}$,

$$u_q \in \frac{1}{h_q} \int_{t_q}^{t_{q+1}} f_q(s) ds + \alpha B, \quad f_q(t) \in F(t, x_q),$$
$$\|f_q(t) - y(t)\| \le d(y(t), F(t, x_q)) + \varepsilon.$$

It is easy to see that for $q = 1$ the assertions (a)-(d) are fulfilled. Let now $q \ge 2$. Assume that (a)-(d) are satisfied for any $j = 1, \dots, q$. If, $T \le t_{q+1}$, then we take $m = q + 1$ and so the process of iterations is stopped and we get (a)-(d) satisfied with $t_{m-1} < T \le t_m$. In the other case, i.e, $t_{q+1} < T$, we define $y_{q+1}^1, \dots, y_{q+1}^{k-1}$ and x_{q+1} as follows

$$x_{q+1} := x_q + \sum_{i=1}^{k-1} \frac{h_q^i}{i!} y_q^i + \frac{h_q^k}{k!} u_q = \Big(x_0 + \sum_{j=1}^{k-1} \sum_{i=0}^{q} \frac{h_i^j}{j!} y_i^j + \sum_{i=0}^{q} \frac{h_i^k}{k!} u_i \Big) \in K,$$

$$y_{q+1}^{k-1} := y_q^{k-1} + h_q u_q = y_0^{k-1} + \sum_{i=0}^{q} h_i u_i,$$

$$y_{q+1}^j := \sum_{l=j}^{k-1} \frac{h_q^{l-j}}{(l-j)!} y_q^l + \frac{h_q^{k-j}}{(k-j)!} u_q = y_0^j + \sum_{l=j+1}^{k-1} \sum_{i=0}^{q} \frac{h_i^{l-j}}{(l-j)!} y_i^l + \sum_{i=0}^{q} \frac{h_i^{k-j}}{(k-j)!} u_i$$

for $j = 1, \dots, k-2$. By (3.1), (3.3) and (3.6), we have

$$\begin{aligned}
\|x_{q+1} - x_0\| &\le \sum_{j=1}^{k-1} \sum_{i=0}^{q} \frac{h_i^j}{j!} \|y_i^j\| + \sum_{i=0}^{q} \frac{h_i^k}{k!} \|u_i\| \\
&\le \sum_{j=1}^{k-1} \sum_{i=0}^{q} h_i (r + \|y_0^j\|) + \sum_{i=0}^{q} \|h_i u_i\| \\
&\le \sum_{i=0}^{q} h_i \Big((k-1) r + \sum_{j=1}^{k-1} \|y_0^j\| \Big) + \sum_{i=0}^{q} \Big(\int_{t_i}^{t_{i+1}} \|f_i(t)\| dt + \alpha h_i \Big) \\
&\le \int_0^{t_{q+1}} \Big(g(t) + 1 + (k-1) r + \sum_{j=1}^{k-1} \|y_0^j\| \Big) dt < r,
\end{aligned}$$

which ensures that $x_{q+1} \in K \cap B(x_0, r)$. For all $j = 1, \dots, k-2$, we have

$$\|y_{q+1}^j - y_0^j\| \le \sum_{l=j+1}^{k-1} \sum_{i=0}^{q} \frac{h_i^{l-j}}{(l-j)!} \|y_i^l\| + \sum_{i=0}^{q} \frac{h_i^{k-j}}{(k-j)!} \|u_i\|$$

$$
\begin{aligned}
&\leq \sum_{l=j+1}^{k-1}\sum_{i=0}^{q} h_i(r+\|y_0^l\|)+\sum_{i=0}^{q}\|h_iu_i\| \\
&\leq \sum_{i=0}^{q}\Big(h_i\Big((k-j-1)r+\sum_{l=j+1}^{k-1}\|y_0^l\|\Big)+\int_{t_i}^{t_{i+1}} g(t)dt+\alpha h_i\Big) \\
&\leq \int_0^{t_{q+1}}\Big(g(t)+1+(k-j-1)r+\sum_{l=j+1}^{k-1}\|y_0^l\|\Big)dt<r
\end{aligned}
$$

and

$$
\begin{aligned}
\|y_{q+1}^{k-1}-y_0^{k-1}\| &\leq \sum_{i=0}^{q} h_i\|u_i\| \\
&\leq \sum_{i=0}^{q}\Big(\int_{t_i}^{t_{i+1}} g(t)dt+\alpha h_i\Big) \\
&\leq \int_0^{t_{q+1}}(g(t)+1)dt<r,
\end{aligned}
$$

which ensures that $y_{q+1}^j\in B(y_0^j,r)$ for all $j=1,\dots,k-1$.

Now, we have to prove that this iterative process is finite; i.e., there exists a positive integer m such that $t_{m-1}<T\leq t_m$. Suppose to the contrary, that is $t_q\leq T$, for all $q\geq 1$. Then the bounded increasing sequence $\{t_q\}_q$ converges to some $\bar t$ such that $\bar t\leq T$. Hence for $q>p$,

$$
\|x_q-x_p\|\leq \int_{t_p}^{t_q} g(t)dt+(t_q-t_p)\Big((k-1)r+1+\sum_{i=1}^{k-1}\|y_0^i\|\Big)\to 0 \quad \text{as } q,p\to\infty
$$

and for $j=1,\dots,k-2$,

$$
\|y_q^j-y_p^j\|\leq \int_{t_p}^{t_q} g(t)dt+(t_q-t_p)\Big((k-j-1)r+1+\sum_{l=j+1}^{k-1}\|y_0^l\|\Big)\to 0 \quad \text{as } q,p\to\infty
$$

and

$$
\|y_q^{k-1}-y_p^{k-1}\|\leq \int_{t_p}^{t_q}(g(t)+1)dt\to 0 \quad \text{as } q,p\to\infty.
$$

Therefore, the sequences $\{x_q\}_q$ and $\{y_q^j\}_q$, for all $j=1,\dots,k-1$, are Cauchy sequences and hence, they converge to some $\bar x\in K$ and $\bar y^j\in\Omega_j$ respectively. Hence, as $(\bar t,\bar x,(\bar y^1,\dots,\bar y^{k-1}))\in[0,T]\times K\times\prod_{i=1}^{k-1}\Omega_i$, by (H3), there exist $h\in]0,\alpha]$ and an integer $q_0\geq 1$ such that for all $q\geq q_0$ and for all $j=1,\dots,k-1$

$$
\begin{gathered}
d\Big(\bar x+\sum_{j=1}^{k-1}\frac{h^j}{j!}\bar y^j+\frac{h^{k-1}}{k!}\int_{\bar t}^{\bar t+h} F(s,\bar x)ds,K\Big)\leq \frac{h^k\alpha}{8(k+5)(k!)};\\
\|x_q-\overline{x}\|\leq \frac{h^k\alpha}{8(k+5)(k!)};\\
\|y_q^j-\overline{y}^j\|\leq \frac{h^{k-j}\alpha j!}{8(k+5)(k!)};\\
\bar t-t_q<\min\{\eta(\frac{h\alpha}{8(k+5)}),h\};
\end{gathered}
\tag{3.7}
$$

$$\int_{\bar t}^{\bar t+h} m(t)\|x_q-\bar x\|dt \le \frac{h\alpha}{8(k+5)}.$$

Let $q > q_0$ be given. For an arbitrary measurable selection ϕ_q of $F(t, x_q)$ on $[0, \bar t + h]$, there exists a measurable selection ϕ of $F(t, \bar x)$ on $[0, \bar t + h]$ such that

$$\|\phi_q(t)-\phi(t)\| \le d(\phi_q(t), F(t,\bar x)) + \frac{\alpha}{2(k+5)} \le m(t)\|x_q-\bar x\| + \frac{\alpha}{8(k+5)}. \quad (3.8)$$

Relations (3.7) and (3.8) imply

$$\begin{aligned}
& d\Big(x_q + \sum_{j=1}^{k-1} \frac{h^j}{j!} y_q^j + \frac{h^{k-1}}{k!}\int_{t_q}^{t_q+h} \phi_q(s)ds, K\Big) \\
& \le \|x_q-\bar x\| + \sum_{j=1}^{k-1} \frac{h^j}{j!}\|y_q^j-\bar y^j\| + d\Big(\bar x + \sum_{j=1}^{k-1}\frac{h^j}{j!}\bar y^j + \frac{h^{k-1}}{k!}\int_{\bar t}^{\bar t+h}\phi(s)ds, K\Big) \\
& \quad + \frac{h^{k-1}}{k!}\int_{t_q}^{\bar t}\|\phi_q(s)\|ds + \frac{h^{k-1}}{k!}\int_{\bar t}^{t_q+h}\|\phi_q(s)-\phi(s)\|ds + \frac{h^{k-1}}{k!}\int_{t_q+h}^{\bar t+h}\|\phi(s)\|ds \\
& \le \|x_q-\bar x\| + \sum_{j=1}^{k-1} \frac{h^j}{j!}\|y_q^j-\bar y^j\| + \frac{h^{k-1}}{k!}\int_{t_q}^{\bar t} g(s)ds \\
& \quad + d\Big(\bar x + \sum_{j=1}^{k-1}\frac{h^j}{j!}\bar y^j + \frac{h^{k-1}}{k!}\int_{\bar t}^{\bar t+h}\phi(s)ds, K\Big) + \frac{h^{k-1}}{k!}\int_{\bar t}^{\bar t+h} m(s)\|x_q-\bar x\|ds \\
& \quad + \frac{h^k\alpha}{8(k+5)(k!)} + \frac{h^{k-1}}{k!}\int_{t_q+h}^{\bar t+h} g(s)ds \\
& \le \frac{h^k\alpha}{8(k+5)(k!)} + (k-1)\frac{h^k\alpha}{8(k+5)(k!)} + \frac{h^k\alpha}{8(k+5)(k!)} + \frac{h^k\alpha}{8(k+5)(k!)} \\
& + \frac{h^k\alpha}{8(k+5)(k!)} + \frac{h^k\alpha}{8(k+5)(k!)} + \frac{h^k\alpha}{8(k+5)(k!)} < \frac{h^k\alpha}{4k!}.
\end{aligned}$$

Since ϕ_q is an arbitrary measurable selection of $F(t, x_q)$ on $[0, \bar t + h]$ it follows that

$$d\Big(x_q + \sum_{j=1}^{k-1}\frac{h^j}{j!}y_q^j + \frac{h^{k-1}}{k!}\int_{t_q}^{t_q+h} F(t,x_q)ds, K\Big) < \frac{h^k\alpha}{4k!}.$$

On the other hand, by (3.7), we have $t_{q+1} \le \bar t < t_q+h$ and hence $h > t_{q+1}-t_q = h_q$. Thus, there exists $h > h_q$ (for all $q \ge q_0$) such that $0 < h \le \alpha$ and

$$d\Big(x_q + \sum_{i=1}^{k-1}\frac{h^i}{i!}y_q^i + \frac{h^{k-1}}{k!}\int_{t_q}^{t_q+h} F(t,x_q)ds, K\Big) < \frac{h^k\alpha}{4k!}.$$

This contradicts the definition of h_q. Therefore, there is an integer $m \ge 1$ such that $t_{m-1} < T \le t_m$ and for which the assertions (a)-(d) are fulfilled.

Now, we take $t_m = T$ and we define the function $\theta : [0, T] \to [0, T]$, $z(.) : [0, T] \to E$ and $f \in L^1([0, T], E)$ by setting for all $t \in [t_q, t_{q+1}[$

$$\theta(t) = t_q, \quad f(t) = f_q(t), \quad z(t) = x_q + \sum_{i=1}^{k-1}\frac{(t-t_q)^i}{i!}y_q^i + \frac{(t-t_q)^k}{k!}u_q.$$

Claim 3.2. *For all $q \in \{0, \dots, m\}$ we have*
$$\left\|y_q^{k-1} - y_0^{k-1} - \int_0^{t_q} f(s)ds\right\| \le \alpha t_q.$$

Proof. It is easy to see that for $q = 0$ the above assertion is fulfilled. By induction, assume that
$$\left\|y_j^{k-1} - y_0^{k-1} - \int_0^{t_j} f(s)ds\right\| \le \alpha t_j.$$
for any $j = 1, \dots, q-1$. By (d) we have
$$\begin{aligned}
&\left\|y_q^{k-1} - y_0^{k-1} - \int_0^{t_q} f(s)ds\right\| \\
&= \left\|y_{q-1}^{k-1} - y_0^{k-1} - \int_0^{t_{q-1}} f(s)ds + h_{q-1}u_{q-1} - \int_{t_{q-1}}^{t_q} f(s)ds\right\| \\
&\le \left\|y_{q-1}^{k-1} - y_0^{k-1} - \int_0^{t_{q-1}} f(s)ds\right\| + \left\|h_{q-1}u_{q-1} - \int_{t_{q-1}}^{t_q} f(s)ds\right\| \\
&\le \alpha t_{q-1} + \alpha h_{q-1} = \alpha t_{q-1} + \alpha t_q - \alpha t_{q-1} = \alpha t_q.
\end{aligned}$$
□

Now let $t \in [t_q, t_{q+1}]$, then by Claim 3.2 and the relations (d), (3.1), and (3.6), we have
$$\begin{aligned}
&\left\|z^{(k-1)}(t) - y_0^{k-1} - \int_0^t f(s)ds\right\| \\
&= \left\|y_q^{k-1} - y_0^{k-1} - \int_0^{t_q} f(s)ds + (t - t_q)u_q - \int_{t_q}^t f(s)ds\right\| \\
&\le \left\|y_q^{k-1} - y_0^{k-1} - \int_0^{t_q} f(s)ds\right\| + \|h_q u_q\| + \int_{t_q}^{t_{q+1}} g(s)ds \\
&\le \alpha t_q + 2\int_{t_q}^{t_{q+1}} g(s)ds + \alpha h_q \\
&\le \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon.
\end{aligned}$$
The proof of Lemma 3.1 is complete. □

Proof of the Theorem2.3. Let $(\varepsilon_n)_{n=1}^\infty$ be a strictly decreasing sequence of positive scalars such that $\sum_{n=1}^\infty \varepsilon_n < \infty$ and $\varepsilon_1 < 1$. In view of Lemma 3.1, we can define inductively sequences $(f_n)_{n=1}^\infty \subset L^1([0,T], E)$, $(z_n(.))_{n=1}^\infty \subset \mathcal{C}^k([0,T], E)$ and $(\theta_n)_{n=1}^\infty \subset S([0,T],[0,T])$ ($S([0,T],[0,T])$ is the space of step functions from $[0,T]$ into $[0,T]$) such that

(1) $f_n(t) \in F(t, z_n(\theta_n(t)))$ for all $t \in [0,T]$;
(2) $\|f_{n+1}(t) - f_n(t)\| \le d(f_n(t), F(t, z_{n+1}(\theta_{n+1}(t)))) + \varepsilon_{n+1}$ for all $t \in [0,T]$;
(3) $\left\|z_n^{(k-1)}(t) - y_0^{k-1} - \int_0^t f_n(\tau)d\tau\right\| \le \varepsilon_n$ for all $t \in [0,T]$.

By (1) and (2) we have
$$\begin{aligned}
\|f_{n+1}(t) - f_n(t)\| &\le H\Big(F(t, z_n(\theta_n(t))), F(t, z_{n+1}(\theta_{n+1}(t)))\Big) + \varepsilon_{n+1} \\
&\le m(t)\|z_n(\theta_n(t)) - z_{n+1}(\theta_{n+1}(t))\| + \varepsilon_{n+1}
\end{aligned}$$

$$
\begin{aligned}
&\le m(t)\Big(\|z_n(\theta_n(t)) - z_n(t)\| + \|z_n(t) - z_{n+1}(t)\| \\
&\quad + \|z_{n+1}(t) - z_{n+1}(\theta_{n+1}(t))\|\Big) + \varepsilon_{n+1}.
\end{aligned}
$$

On the other hand, for $t \in [t_q, t_{q+1}[$ we have

$$
\begin{aligned}
\|z_n(t) - z_n(\theta_n(t))\| &= \Big\| \sum_{i=1}^{k-1} \frac{(t-t_q)^i}{i!} y_q^i + \frac{(t-t_q)^k}{k!} u_q \Big\| \\
&\le \sum_{i=1}^{k-1} h_q \Big(\|y_q^i - y_0^i\| + \|y_0^i\|\Big) + \|h_q u_q\| \\
&\le \frac{\varepsilon_n}{4}\Big((k-1)r + \sum_{i=1}^{k-1}\|y_0^i\|\Big) + \alpha + \int_{t_q}^{t_{q+1}} g(s)ds \\
&\le \frac{\varepsilon_n}{4}\Big((k-1)r + \sum_{i=1}^{k-1}\|y_0^i\|\Big) + \frac{\varepsilon_n}{4} + \frac{\varepsilon_n}{4} \\
&\le \frac{\varepsilon_n}{4}\Big((k-1)r + 2 + \sum_{i=1}^{k-1}\|y_0^i\|\Big).
\end{aligned}
$$

Hence

$$
\|z_n(t) - z_n(\theta_n(t))\| \le \frac{\varepsilon_n}{4}\Big((k-1)r + 2 + \sum_{i=1}^{k-1}\|y_0^i\|\Big). \tag{3.9}
$$

It follows that

$$
\begin{aligned}
&\|f_{n+1}(t) - f_n(t)\| \\
&\le m(t)\Big(\frac{\varepsilon_n}{2}\Big((k-1)r + 2 + \sum_{i=1}^{k-1}\|y_0^i\|\Big) + \|z_n(.) - z_{n+1}(.)\|_\infty\Big) + \varepsilon_{n+1}.
\end{aligned} \tag{3.10}
$$

Relations (3.10) and (3.2) yield

$$
\begin{aligned}
&\|z_{n+1}^{(k-1)}(t) - z_n^{(k-1)}(t)\| \\
&\le \Big\|z_{n+1}^{(k-1)}(t) - y_0^{k-1} - \int_0^t f_{n+1}(s)ds\Big\| + \Big\|z_n^{(k-1)}(t) - y_0^{k-1} - \int_0^t f_n(s)ds\Big\| \\
&\quad + \int_0^t \|f_{n+1}(s) - f_n(s)\|ds \\
&\le \varepsilon_{n+1} + \varepsilon_n + \int_0^t m(s)\Big(\frac{\varepsilon_n}{2}\Big((k-1)r + 2 + \sum_{i=1}^{k-1}\|y_0^i\|\Big) + \|z_n(.) - z_{n+1}(.)\|_\infty\Big)ds \\
&\quad + t\varepsilon_{n+1} \\
&\le 3\varepsilon_n + \|z_n(.) - z_{n+1}(.)\|_\infty \int_0^T m(s)ds \\
&\quad + \frac{\varepsilon_n}{2}\Big((k-1)r + 2 + \sum_{i=1}^{k-1}\|y_0^i\|\Big)\int_0^T m(s)ds \\
&\le \Big((k-1)r + 5 + \sum_{i=1}^{k-1}\|y_0^i\|\Big)\varepsilon_n + \|z_n(.) - z_{n+1}(.)\|_\infty \int_0^T m(s)ds.
\end{aligned}
$$

Since $T \le 1$ for all $t \in [0,T]$, we have

$$\begin{aligned}\|z_{n+1}^{(k-2)}(t) - z_n^{(k-2)}(t)\| &\le \int_0^t \|z_{n+1}^{(k-1)}(s) - z_n^{(k-1)}(s)\| ds \\ &\le \Big((k-1)r + 5 + \sum_{i=1}^{k-1} \|y_0^i\|\Big)\varepsilon_n \\ &\quad + \|z_n(.) - z_{n+1}(.)\|_\infty \int_0^T m(s)ds.\end{aligned}$$

Then by the same reasoning, for $j = 1, \dots, k-1$, we obtain

$$\begin{aligned}\|z_{n+1}^{(j)}(t) - z_n^{(j)}(t)\| &\le \Big((k-1)r + 5 + \sum_{i=1}^{k-1} \|y_0^i\|\Big)\varepsilon_n \\ &\quad + \|z_n(.) - z_{n+1}(.)\|_\infty \int_0^T m(s)ds\end{aligned}$$

and

$$\|z_n(.) - z_{n+1}(.)\|_\infty \le \frac{\Big((k-1)r + 5 + \sum_{i=1}^{k-1} \|y_0^i\|\Big)\varepsilon_n}{1-L} \tag{3.11}$$

where $L = \int_0^T m(s)ds$. For $j = 1, \dots, k-1$ we have

$$\begin{aligned}\|z_{n+1}^{(j)}(.) - z_n^{(j)}(.)\|_\infty &\le \Big((k-1)r + 5 + \sum_{i=1}^{k-1} \|y_0^i\|\Big)\varepsilon_n + \|z_n(.) - z_{n+1}(.)\|_\infty \\ &\le \frac{\Big((k-1)r + 5 + \sum_{i=1}^{k-1} \|y_0^i\|\Big)(2-L)\varepsilon_n}{1-L}.\end{aligned}$$

Therefore, for $n < m$,

$$\|z_m(.) - z_n(.)\|_\infty \le \frac{(k-1)r + 5 + \sum_{i=1}^{k-1} \|y_0^i\|}{1-L} \sum_{i=n}^{m-1} \varepsilon_i$$

and for $j = 1, \dots, k-1$,

$$\|z_m^{(j)}(.) - z_n^{(j)}(.)\|_\infty \le \frac{\Big((k-1)r + 5 + \sum_{i=1}^{k-1} \|y_0^i\|\Big)(2-L)}{1-L} \sum_{i=n}^{m-1} \varepsilon_i.$$

Thus the sequences $\{z_n(.)\}_{n=1}^\infty$ and $\{z_n^{(j)}(.)\}_{n=1}^\infty$, for $j = 1, \dots, k-1$, converge uniformly on $[0,T]$, namely $x(.)$ and $y_j(.)$ its limits respectively. Also the relations

$$z_n(t) = x_0 + \int_0^t \dot{z}_n(s)ds$$

and

$$z_n^{(j)}(t) = y_0^j + \int_0^t z_n^{(j+1)}(s)ds \quad \text{for } j = 1, \dots, k-2$$

yield $x(t) = x_0 + \int_0^t y_1(s)ds$ and

$$y_j(t) = y_0^j + \int_0^t y_{j+1}(s)ds \quad \text{for } j = 1, \dots, k-2.$$

Thus $\dot{x}(t) = y_1(t)$ and $\dot{y}_j(t) = y_{j+1}(t)$ for all $t \in [0,T]$ and for all $j = 1,\dots,k-2$. Hence $x(0) = x_0$ and $x^{(j)}(0) = y_0^j$ for all $j = 1,\dots,k-1$. On the other hand, observe that $z_n(\theta_n(t))$ converges uniformly to $x(t)$ on $[0,T]$. Indeed, for $t \in [t_q, t_{q+1}[$ we have

$$\|z_n(\theta_n(t)) - x(t)\| \le \|z_n(t) - z_n(\theta_n(t))\| + \|z_n(t) - x(t)\|.$$

By (3.9) and since $(z_n(.))$ converges uniformly to $x(.)$, it follows that

$$z(\theta_n(.)) \text{ converges uniformly to } x(.) \text{ on } [0,T]. \tag{3.12}$$

By construction, we have $z_n(\theta_n(t)) \in K$ for every $t \in [0,T]$ and K is closed, then $x(t) \in K$ for all $t \in [0,T]$.

Now we return to relation (3.10). By (3.11) we have

$$\begin{aligned}&\|f_{n+1}(t) - f_n(t)\| \\ &\le \Big(m(t)\Big(\frac{(k-1)r + 5 + \sum_{i=1}^{k-1}\|y_0^i\|}{1-L} + \frac{(k-1)r + 2 + \sum_{i=1}^{k-1}\|y_0^i\|}{2}\Big) + 1\Big)\varepsilon_n.\end{aligned}$$

This implies (as above) that $\{f_n(t)\}_{i=1}^n$ is a Cauchy sequence and $f_n(t)$ converges to $f(t)$. Further, since $\|f_n(t)\| \le g(t)$, by (3) and Lebesgue's theorem we have

$$y_{k-1}(t) = \lim_{n\to\infty} z_n^{(k-1)}(t) = \lim_{n\to\infty}\Big(y_0^{k-1} + \int_0^t f_n(s)ds\Big) = y_0^{k-1} + \int_0^t f(s)ds.$$

Hence $\dot{y}_{k-1}(t) = f(t)$. Finally, observe that by (1),

$$\begin{aligned} d(f(t), F(t,x(t))) &\le \|f(t) - f_n(t)\| + H\Big(F(t, z_n(\theta_n(t))), F(t,x(t))\Big) \\ &\le \|f(t) - f_n(t)\| + m(t)\|z_n(\theta_n(t)) - x(t)\|. \end{aligned}$$

Since $f_n(t)$ converges to $f(t)$ and by (3.12) the last term converges to 0. So that $x^{(k)}(t) = \dot{y}_{k-1}(t) = f(t) \in F(t,x(t))$ a.e on $[0,T]$. The proof is complete. $\square$

Remark 3.3. The tangential condition (H3) provides a sufficient condition ensuring the existence of solution to (1.1). However, this condition is not necessary at all. In fact, in the case $k = 2$, Marco and Murillo [7, Example 4.1] gave a counterexample: The multifunction $F : [0,1] \times \mathbb{R} \to 2^{\mathbb{R}}$ defined as

$$F(t,x) = [-t^{-a}, t^{-a}], \quad 0 < t \le 1, \quad F(0,x) = 0$$

with $0 < a < (3-\sqrt{3})/2$, satisfies (H2) and $x(t) = \frac{t^{2-a}}{(1-a)(2-a)}$ is a solution of

$$\begin{gathered} \ddot{x}(t) \in F(t,x(t)), \quad t \in [0,1]; \\ (x(0), \dot{x}(0)) = (0,0); \\ x(t) \in [0,2]. \end{gathered} \tag{3.13}$$

However (H3) fails on $[0,2] \times gr(A_K^1)$, because

$$\frac{1}{h^2} d\Big(\frac{h}{2}\int_t^{t+h} F(s,0)ds, [0,2]\Big) = \frac{(t+h)^{1-a} - t^{1-a}}{2(1-a)h}$$

and

$$\lim_{h\to 0^+} \frac{(t+h)^{1-a} - t^{1-a}}{2(1-a)h} = \begin{cases} +\infty & \text{if } t = 0 \\ \frac{t^{-a}}{2} & \text{if } 0 < t \le 1 \end{cases}$$

Remark 3.4. Let $F : [0,1] \times K \times \prod_{i=1}^{k-1} \Omega_i \to 2^E$. For any $(x_0, y_0^1, y_0^2 \dots, y_0^{k-1}) \in K \times \prod_{i=1}^{k-1} \Omega_i$, we can prove the existence of viable solutions of the differential inclusion

$$
\begin{gathered}
x^{(k)}(t) \in F(t, x(t), x^{(1)}(t), \dots, x^{(k-1)}(t)) \quad \text{a.e. on } [0,T] \\
x(0) = x_0 \in K, x^{(i)}(0) = y_0^i \in \Omega_i, \quad i = 1, \dots, k-1, \\
x(t) \in K \quad \text{on } [0,T],
\end{gathered}
$$

by the same technics and the same hypothesis as above except the condition (H2) part (ii) which must be replaced by: There is a function $m \in L^1([0,1], \mathbb{R}^+)$ such that for all $t \in [0,1]$, for all $x_1, x_2 \in K$ and for all $(y_1^1, \dots, y_1^{k-1}), (y_2^1, \dots, y_2^{k-1}) \in \prod_{i=1}^{k-1} \Omega_i$,

$$
H\Big(F(t, x_1, y_1^1, \dots, y_1^{k-1}), F(t, x_2, y_2^1, \dots, y_2^{k-1})\Big) \le m(t) \|y_1^{k-1} - y_2^{k-1}\|.
$$

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