Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 54, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE OF SOLUTIONS FOR MULTI-POINT NONLINEAR DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDERS WITH INTEGRAL BOUNDARY CONDITIONS

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Abstract. In this article, we study the multi-point boundary-value problem of nonlinear fractional differential equation

$$D_{0+}^{\alpha}u(t) = f(t, u(t)), \quad 1 < \alpha \le 2, \ t \in [0, T], \ T > 0,$$

$$I_{0+}^{2-\alpha}u(t)|_{t=0} = 0, \quad D_{0+}^{\alpha-2}u(T) = \sum_{i=1}^{m} a_i I_{0+}^{\alpha-1}u(\xi_i),$$

where D_{0+}^{α} and I_{0+}^{α} are the standard Riemann-Liouville fractional derivative and fractional integral respectively. Some existence and uniqueness results are obtained by applying some standard fixed point principles. Several examples are given to illustrate the results.

1. Introduction

The study of fractional differential equations ranges from the theoretical aspects of existence and uniqueness of solutions to the analytic and numerical methods for finding solutions. Fractional differential equations appear naturally in a number of fields such as physics, polymer rheology, regular variation in thermodynamics, biophysics,blood flow phenomena, aerodynamics, electro-dynamics of complex medium, viscoelasticity, Bodes analysis of feedback amplifiers, capacitor theory, electrical circuits, electron-analytical chemistry, biology, control theory, fitting of experimental data, etc. An excellent account in the study of fractional differential equations can be found in [13, 14, 16, 17]. Boundary value problems for fractional differential equations have been discussed in [1, 8, 11, 12, 15, 19, 20, 21].

Integral boundary conditions have various applications in applied fields such as blood flow problems, chemical engineering, thermo-elasticity, underground water flow, population dynamics, and so forth. For a detailed description of the integral boundary conditions, we refer the reader to a recent paper [6]. For more details of nonlocal and integral boundary conditions, see [7, 9, 10] and references therein.

2000 *Mathematics Subject Classification.* 34B15.
Key words and phrases. Fractional differential equation; boundary value problem; fixed point theorem; existence and uniqueness.

Submitted November 27, 2011. Published April 5, 2012.

Ahmada and Nieto [1] considered the anti-periodic fractional boundary value problem given

$$
\begin{gathered}
{}^cD^q u(t) = f(t, u(t)), \quad 1 < \alpha \le 2, \\
u(0) = -u(T), \quad {}^cD^p u(0) = {}^c D^p u(T),
\end{gathered}
$$

where ${}^cD^q$ is the standard Caputo fractional derivative. Using of some existence and uniqueness results are obtained by applying some standard fixed point principles.

Ahmada and Nieto [3] considered the fractional integro-differential equation with integral boundary conditions

$$
\begin{gathered}
{}^cD^q x(t) = f(t, x(t), (\chi x)(t)), \quad 1 < q \le 2, \ t \in (0,1), \\
\alpha x(0) + \beta x'(0) = \int_0^1 q_1(x(s))ds, \quad \alpha x(1) + \beta x'(1) = \int_0^1 q_2(x(s))ds,
\end{gathered}
$$

where ${}^cD^q$ is the standard Caputo fractional derivative,

$$
(\chi x)(t) = \int_0^t \gamma(t,s)x(s)ds.
$$

Some existence and uniqueness results are obtained by applying standard fixed point principles.

In this paper, we investigate the existence and uniqueness of solutions for the fractional boundary-value problem

$$
D^\alpha_{0+}u(t) = f(t, u(t)), \quad 1 < \alpha \le 2, \ t \in [0,T], \ T > 0, \tag{1.1}
$$

$$
I^{2-\alpha}_{0+}u(t)|_{t=0} = 0, \quad D^{\alpha-2}_{0+}u(T) = \sum_{i=1}^m a_i I^{\alpha-1}_{0+}u(\xi_i), \tag{1.2}
$$

where $0 < \xi_i < T$, $T > 0$, $a_i \in \mathbb{R}$, $m \ge 2$, D^α_{0+} and I^α_{0+} are the standard Riemann-Liouville fractional derivative and fractional integral respectively, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous.

2. Preliminaries

For the convenience of the reader, we present here some necessary basic knowledge and definitions for fractional calculus theory, that can be found in the recent literature.

Definition 2.1. The fractional integral of order $\alpha > 0$ of a function $y : (0,\infty) \to R$ is given by

$$
I^\alpha_{0+}y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds,
$$

provided the right side is pointwise defined on $(0,\infty)$, where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2. The fractional derivative of order $\alpha > 0$ of a function $y : (0,\infty) \to R$ is given by

$$
D^\alpha_{0+}y(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dt})^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}} ds,
$$

where $n = [\alpha] + 1$, provided the right side is pointwise defined on $(0,\infty)$.

Lemma 2.3. *Let $\alpha > 0$ and $u \in C(0,1) \cap L^1(0,1)$. Then fractional differential equation $D^\alpha_{0+}u(t) = 0$ has*
$$u(t) = c_1t^{\alpha-1} + c_2t^{\alpha-2} + \dots + c_Nt^{\alpha-N}, \quad c_i \in \mathbb{R},\ N = [\alpha] + 1,$$
as unique solution.

Lemma 2.4. *Assume that $u \in C(0,1) \cap L^1(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L^1(0,1)$. Then*
$$I^\alpha_{0+}D^\alpha_{0+}u(t) = u(t) + c_1t^{\alpha-1} + c_2t^{\alpha-2} + \dots + c_Nt^{\alpha-N},$$
for some $c_i \in \mathbb{R}, i = 1, 2, \dots, N$, where N is the smallest integer grater than or equal to α.

Definition 2.5. For $n \in N$, we denote by $AC^n[0,1]$ the space of functions $u(t)$ which have continuous derivatives up to order $n-1$ on $[0,1]$ such that $u^{(n-1)}(t)$ is absolutely continuous: $AC^n[0,1] =\{u|[0,1] \to R$ and $(D^{(n-1)})u(t)$ is absolutely continuous in $[0,1]\}$.

Lemma 2.6 ([13])**.** *Let $\alpha > 0$, $n = [\alpha] + 1$. Assume that $u \in L^1(0,1)$ with a fractional integration of order $n-\alpha$ that belongs to $AC^n[0,1]$. Then the equality*
$$(I^\alpha_{0+}D^\alpha_{0+}u)(t) = u(t) - \sum_{i=1}^{n} \frac{((I^{n-\alpha}_{0+}u)(t))^{n-i}|_{t=0}}{\Gamma(\alpha-i+1)}t^{\alpha-i}$$
holds almost everywhere on $[0,1]$.

Lemma 2.7 ([13])**.**
(i) *Let $k \in N, \alpha > 0$. If $D^\alpha_{a+}y(t)$ and $(D^{\alpha+k}_{a+}y)(t)$ exist, then*
$$(D^kD^\alpha_{a+})y(t) = (D^{\alpha+k}_{a+}y)(t);$$
(ii) *If $\alpha > 0, \beta > 0, \alpha + \beta > 1$, then*
$$(I^\alpha_{a+}I^\alpha_{a+})y(t) = (I^{\alpha+\beta}_{a+}y)(t)$$
satisfies at any point on $[a,b]$ for $y \in L_p(a,b)$ and $1 \le p \le \infty$;
(iii) *Let $\alpha > 0$ and $y \in C[a,b]$. Then $(D^\alpha_{a+}I^\alpha_{a+})y(t) = y(t)$ holds on $[a,b]$;*
(iv) *Note that for $\lambda > -1, \lambda \ne \alpha - 1, \alpha - 2, \dots, \alpha - n$, we have*
$$D^\alpha t^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)}t^{\lambda-\alpha},$$
$$D^\alpha t^{\alpha-i} = 0, i = 1, 2, \dots, n$$

Lemma 2.8. *For any $y(t) \in C[0,1]$, the linear fractional boundary-value problem*
$$\begin{gathered} D^\alpha_{0+}u(t) = y(t), \quad 1 < \alpha \le 2,\ t \in [0,T], \\ I^{2-\alpha}_{0+}u(t)|_{t=0} = 0, \quad D^{\alpha-2}_{0+}u(T) = \sum_{i=1}^{m} a_iI^{\alpha-1}_{0+}u(\xi_i), \end{gathered} \tag{2.1}$$
has unique solution
$$\begin{aligned} u(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds \\ &\quad + \frac{t^{\alpha-1}}{\Gamma(\alpha)(T-A)}\Big[\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)}\int_0^{\xi_i}(\xi_i-s)^{2\alpha-2}y(s)ds - \int_0^T(T-s)y(s)ds\Big], \end{aligned} \tag{2.2}$$

where $A = \sum_{i=1}^{m} a_i \xi_i^{2\alpha-2}/\Gamma(2\alpha-1)$ *and* $T \neq A$.

Proof. By Lemma 2.4. the solution of (2.1) can be written as

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds.$$

From $I_{0+}^{2-\alpha} u(t)|_{t=0} = 0$, and by Lemmas 2.6 and 2.7, we know that $c_2 = 0$, and

$$D_{0+}^{\alpha-2} u(t) = c_1 t \Gamma(\alpha) + I_{0+}^{2} y(t),$$
$$I_{0+}^{\alpha-1} u(t) = c_1 \frac{\Gamma(\alpha)}{\Gamma(2\alpha-1)} t^{2\alpha-2} + I_{0+}^{\alpha-1} I_{0+}^{\alpha} y(t),$$

from $D_{0+}^{\alpha-2} u(T) = \sum_{i=1}^{m} a_i I_{0+}^{\alpha-1} u(\xi_i)$, we have

$$c_1 = \frac{1}{\Gamma(\alpha)(T-A)} \Big[\frac{\sum_{i=1}^{m} a_i}{\Gamma(2\alpha-1)} \int_0^{\xi_i} (\xi_i - s)^{2\alpha-2} y(s) ds - \int_0^T (T-s) y(s) ds \Big],$$

where $A = \sum_{i=1}^{m} a_i \xi_i^{2\alpha-2}/\Gamma(2\alpha-1)$ and $T \neq A$, so

$$\begin{aligned} u(t) = & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds \\ & \frac{t^{\alpha-1}}{\Gamma(\alpha)(T-A)} \Big[\frac{\sum_{i=1}^{m} a_i}{\Gamma(2\alpha-1)} \int_0^{\xi_i} (\xi_i - s)^{2\alpha-2} y(s) ds - \int_0^T (T-s) y(s) ds \Big]. \end{aligned}$$

The proof is complete. □

3. Existence and uniqueness of solutions

Let $E = C([0,T], R)$ denote the Banach space of all continuous functions from $[0,T] \to R$ endowed with the norm defined by $\|x\| = sup\{|x(t)|, t \in [0,T]\}$. Now we state some known fixed point theorems which are needed to prove the existence of solutions for (1.1)–(1.2).

Theorem 3.1 ([18])**.** *Let X be a Banach space. Assume that $T : X \to X$ is a completely continuous operator and the set $V = \{u \in X | u = \mu T u, 0 < \mu < 1\}$ is bounded. Then T has a fixed point in X.*

Theorem 3.2. [18] *Let X be a Banach space. Assume that Ω is an open bounded subset of X with $\theta \in \Omega$and let $T : \bar{\Omega} \to X$ be a completely continuous operator such that*

$$\|Tu\| \leq \|u\|, \forall u \in \partial\Omega.$$

Then T has a fixed point in $\bar{\Omega}$.

We define, in relation to (2.2), an operator $P : E \to E$, as

$$\begin{aligned} (Pu)(t) = & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(t, u(s)) ds \\ & + \frac{t^{\alpha-1}}{\Gamma(\alpha)(T-A)} \Big(\frac{\sum_{i=1}^{m} a_i}{\Gamma(2\alpha-1)} \int_0^{\xi_i} (\xi_i - s)^{2\alpha-2} f(t, u(s)) ds \\ & - \int_0^T (T-s) f(t, u(s)) ds \Big). \end{aligned} \tag{3.1}$$

Observe that this equation has a solution if and only if the operator P has a fixed point.

Theorem 3.3. *Assume that there exists a positive constant L_1 such that $|f(t,u)| \le L_1$ for $t \in [0,T], u \in E$. Then (1.1)-(1.2) has at least one solution.*

Proof. We show, as a first step, that the operator P is completely continuous. Clearly, continuity of the operator P follows from the continuity of f. Let $\Omega \subset E$ be bounded. Then, $\forall u \in \Omega$ together with the assumption $|f(t,u)| \le L_1$, we obtain

$$\begin{aligned}
(Pu)(t) &\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(t,u(s))|ds \\
&\quad + \frac{t^{\alpha-1}}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)}\int_0^{\xi_i}(\xi_i-s)^{2\alpha-2}|f(t,u(s))|ds \\
&\quad - \int_0^T (T-s)|f(t,u(s))|ds\Big) \\
&\le L_1\Big[\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds \\
&\quad + \frac{t^{\alpha-1}}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)}\int_0^{\xi_i}(\xi_i-s)^{2\alpha-2}ds - \int_0^T (T-s)ds\Big)\Big] \\
&\le L_1\Big[\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{T^{\alpha-1}}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^m a_i\xi^{2\alpha-1}}{\Gamma(2\alpha)} - \frac{T^2}{2}\Big)\Big],
\end{aligned}$$

which implies

$$\|Pu\| \le L_1\Big[\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{T^{\alpha-1}}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^m a_i\xi^{2\alpha-1}}{\Gamma(2\alpha)} - \frac{T^2}{2}\Big)\Big] < \infty.$$

Hence, $T(\Omega)$ is uniformly bounded.

For any $t_1, t_2 \in [0,T], u \in \Omega$, we have

$$\begin{aligned}
&|(Pu)(t_1) - (Pu)(t_2)| \\
&= \Big|\int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s))ds \\
&\quad + \frac{t_1^{\alpha-1}}{\Gamma(\alpha)(T-A)}\Big(\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)}\int_0^{\xi_i}(\xi_i-s)^{2\alpha-2}f(s,u(s))ds \\
&\quad - \int_0^T (T-s)^f(s,u(s))ds\Big) - \int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s))ds - \frac{t_2^{\alpha-1}}{\Gamma(\alpha)(T-A)} \\
&\quad \times \Big(\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)}\int_0^{\xi_i}(\xi_i-s)^{2\alpha-2}f(s,u(s))ds - \int_0^T (T-s)f(s,u(s))ds\Big)\Big| \\
&\le L_1\Big|\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}ds \\
&\quad + \frac{t_1^{\alpha-1}-t_2^{\alpha-1}}{\Gamma(\alpha)(T-A)}\Big(\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)}\int_0^{\xi_i}(\xi_i-s)^{2\alpha-2}ds \\
&\quad - \int_0^T (T-s)ds\Big) - \int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}ds\Big| \\
&\le L_1\Big[\Big|\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}ds - \int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}ds\Big|
\end{aligned}$$

$$
+\Big|\frac{t_1^{\alpha-1}-t_2^{\alpha-1}}{\Gamma(\alpha)(T-A)}\Big(\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)}\int_0^{\xi_i}(\xi_i-s)^{2\alpha-2}ds-\int_0^T(T-s)ds\Big)\Big|\Big]
\\ \to 0 \quad \text{as } t_1\to t_2.
$$

Thus, by the Arzela-Ascoli theorem, $P(\Omega)$ is equicontinuous. Consequently, the operator P is compact.

Next, we consider the set $V=\{u\in E: u=\mu Pu, 0<\mu<1\}$, and show that it is bounded. Let $u\in V$; then $u=\mu Pu, 0<\mu<1$. For any $t\in[0,T]$, we have

$$
\begin{aligned}
u(t)=&\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(t,u(s))ds\\
&+\frac{t^{\alpha-1}}{\Gamma(\alpha)(T-A)}\Big(\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)}\int_0^{\xi_i}(\xi_i-s)^{2\alpha-2}f(t,u(s))ds\\
&-\int_0^T(T-s)f(t,u(s))ds\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
|u(t)|&=\mu|Pu|\\
&\le\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(t,u(s))|ds\\
&\quad+\frac{t^{\alpha-1}}{\Gamma(\alpha)(T-A)}\Big(\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)}\int_0^{\xi_i}(\xi_i-s)^{2\alpha-2}|f(t,u(s))|ds\\
&\quad-\int_0^T(T-s)|f(t,u(s))|ds\Big)\\
&\le L_1\Big[\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds\\
&\quad+\frac{t^{\alpha-1}}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)}\int_0^{\xi_i}(\xi_i-s)^{2\alpha-2}ds-\int_0^T(T-s)ds\Big)\Big]\\
&\le\max_{t\in[0,T]}\Big\{L_1\Big[\frac{|t^\alpha|}{\Gamma(\alpha+1)}+\frac{|t^{\alpha-1}|}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^m a_i\xi^{2\alpha-1}}{\Gamma(2\alpha)}-\frac{T^2}{2}\Big)\Big]\Big\}=M.
\end{aligned}
$$

Thus, $\|u\|\le M$. So, the set V is bounded. Thus, by the conclusion of Theorem 3.1, the operator P has at least one fixed point, which implies that (1.1)-(1.2) has at least one solution. □

Theorem 3.4. *Let $lim_{x\to0}\frac{f(t,x)}{x}=0$. Then* (1.1)-(1.2) *has at least one solution.*

Proof. Since $\lim_{x\to0}\frac{f(t,x)}{x}=0$, there exists a constant $r>0$ such that $|f(t,x)|\le\varepsilon|x|$ for $0<|x|<r$, where $\varepsilon>0$ is such that

$$
\max_{t\in[0,T]}\Big\{\frac{|t^\alpha|}{\Gamma(\alpha+1)}+\frac{|t^{\alpha-1}|}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^m a_i\xi^{2\alpha-1}}{\Gamma(2\alpha)}-\frac{T^2}{2}\Big)\Big\}\varepsilon\le1, \tag{3.2}
$$

Define $\Omega_1=\{x\in E:\|x\|<r\}$ and take $x\in E$ such that $\|x\|=r$; that is, $x\in\Omega_1$. As before, it can be shown that T is completely continuous and

$$
|(Tx)(t)|\le max_{t\in[0,T]}\Big\{\frac{|t^\alpha|}{\Gamma(\alpha+1)}+\frac{|t^{\alpha-1}|}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^m a_i\xi^{2\alpha-1}}{\Gamma(2\alpha)}-\frac{T^2}{2}\Big)\Big\}\varepsilon\|x\|,
$$

which, in view of (3.2), yields $\|Tx\| \leq \|x\|, x \in \partial\Omega_1$. Therefore, by Theorem 3.2, the operator T has at least one fixed point, which in turn implies that (1.1)-(1.2) has at least one solution. $\square$

For the next theorem we use the following two assumptions:

(H1) there exist positive functions L , such that
$$|f(t,x) - f(t,y)| \leq L|x-y|, \quad \forall t \in [0,T], x, y \in \mathbb{R},$$

(H2) The function L satisfies
$$2L \leq \Big[\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{T^{\alpha-1}}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^m a_i \xi_i^{2\alpha-1}}{\Gamma(2\alpha)} - \frac{T^2}{2}\Big)\Big]^{-1}.$$

Theorem 3.5. *Assume thatUnder assumptions* (H1), (H2), *Problem* (1.1)–(1.2)*) has a unique solution in* $C[0,T]$.

Proof. Let us set $\sup_{t\in[0,T]} |f(t,0)| = M_1$, and choose
$$r \geq 2M_1\Big[\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{T^{\alpha-1}}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^m a_i \xi_i^{2\alpha-1}}{\Gamma(2\alpha)} - \frac{T^2}{2}\Big)\Big]$$
Then we show that $PBr \subset Br$, where $Br = \{u \in E : \|u\| \leq r\}$. For $u \in Br$, we have
$$\begin{aligned}
&\|(Pu)(t)\| \\
&= \sup_{t\in[0,T]} \Big| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))ds \\
&\quad + \frac{t^{\alpha-1}}{\Gamma(\alpha)(T-A)}\Big(\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)}\int_0^{\xi_i}(\xi_i-s)^{2\alpha-2} f(s,u(s))ds \\
&\quad - \int_0^T (T-s)f(s,u(s))ds\Big)\Big| \\
&\leq \sup_{t\in[0,T]} \Big[\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s)|ds \\
&\quad + \frac{t^{\alpha-1}}{\Gamma(\alpha)(T-A)}\Big(\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)}\int_0^{\xi_i}(\xi_i-s)^{2\alpha-2}|f(s,u(s)|ds \\
&\quad - \int_0^T (T-s)|f(s,u(s))|ds\Big)\Big] \\
&\leq \sup_{t\in[0,T]} \Big[\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(|f(s,u(s)-f(s,0)| + |f(s,0)|)ds \\
&\quad + \frac{t^{\alpha-1}}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)}\int_0^{\xi_i}(\xi_i-s)^{2\alpha-2}(|f(s,u(s)-f(s,0)| + |f(s,0)|)ds \\
&\quad - \int_0^T (T-s)(|f(s,u(s)-f(s,0)| + |f(s,0)|)ds\Big)\Big] \\
&\leq \sup_{t\in[0,T]} \Big[(Lr+M_1)\Big(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds \\
&\quad + \frac{t^{\alpha-1}}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)}\int_0^{\xi_i}(\xi_i-s)^{2\alpha-2}ds - \int_0^T (T-s)ds\Big)\Big)\Big]
\end{aligned}$$

$$\le (Lr+M_1)\Big[\frac{T^\alpha}{\Gamma(\alpha+1)}+\frac{T^{\alpha-1}}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^m a_i\xi_i^{2\alpha-1}}{\Gamma(2\alpha)}-\frac{T^2}{2}\Big)\Big]\le r$$

Taking the maximum over the interval $[0,T]$, we obtain $\|(Pu)(t)\|\le r$.

In view of (H1), for every $t\in[0,T]$, we have

$$\begin{aligned}
&\|(Px)(t)-(Py)(t)\|\\
&=\sup_{t\in[0,T]}\Big|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(f(t,x)-f(t,y)ds\\
&\quad+\frac{t^{\alpha-1}}{\Gamma(\alpha)(T-A)}\Big(\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)}\int_0^{\xi_i}(\xi_i-s)^{2\alpha-2}(f(t,x)-f(t,y)ds\\
&\quad-\int_0^T(T-s)(f(t,x)-f(t,y)ds\Big)\Big|\\
&\le\sup_{t\in[0,T]}\Big[\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|(f(t,x)-f(t,y)|ds\\
&\quad+\frac{t^{\alpha-1}}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)}\int_0^{\xi_i}(\xi_i-s)^{2\alpha-2}|(f(t,x)-f(t,y)|ds\\
&\quad-\int_0^T(T-s)|(f(t,x)-f(t,y)|ds\Big)\Big]\\
&\le\sup_{t\in[0,T]}\Big[L\|x-y\|\Big(\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds\\
&\quad+\frac{t^{\alpha-1}}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)}\int_0^{\xi_i}(\xi_i-s)^{2\alpha-2}ds-\int_0^T(T-s)ds\Big)\Big)\Big]\\
&\le L\|x-y\|\Big[\frac{T^\alpha}{\Gamma(\alpha+1)}+\frac{T^{\alpha-1}}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^m a_i\xi_i^{2\alpha-1}}{\Gamma(2\alpha)}-\frac{T^2}{2}\Big)\Big]=A\|x-y\|,
\end{aligned}$$

where

$$A=L\Big[\frac{T^\alpha}{\Gamma(\alpha+1)}+\frac{T^{\alpha-1}}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^m a_i\xi_i^{2\alpha-1}}{\Gamma(2\alpha)}-\frac{T^2}{2}\Big)\Big],$$

which depends only on the parameters involved in the problem. As $A<1$, T is therefore a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle (the Banach fixed point theorem). □

Example 3.6. Consider the following three-point nonlinear differential equations

$$D_{0+}^{3/2}u(t)=f(t,u(t)),\quad 0<t<1,\tag{3.3}$$

$$I_{0+}^{2-\alpha}u(t)|_{t=0}=0,\quad D_{0+}^{\alpha-2}u(T)=\sum_{i=1}^m a_iI_{0+}^{\alpha-1}u(\xi_i),\tag{3.4}$$

where $f(t,u)=e^{-2sin^2(u(t))}[3+5\sin(2t)+4ln(5+2\cos^2(u(t)))]/(2+\cos t)$, $a_1=4$, $a_2=2$, $\xi_1=1/2$, $\xi_2=1/4$, $T=1$ we have $A=\sum_{i=1}^m a_i\xi_i^{2\alpha-2}/\Gamma(2\alpha-1)=5/2\neq T=1$.

Clearly $L_1=4+2ln7$, and the hypothesis of Theorem 3.3 holds. Therefore, the conclusion of Theorem 3.3 applies to (3.3)–(3.4). Then, there exists at least one solution.

Example 3.7. Consider the problem

$$D_{0+}^{3/2}u(t) = f(t,u(t)), \quad 0 < t < 1, \tag{3.5}$$

$$I_{0+}^{2-\alpha}u(t)|_{t=0} = 0, \quad D_{0+}^{\alpha-2}u(T) = \sum_{i=1}^{m} a_i I_{0+}^{\alpha-1}u(\xi_i), \tag{3.6}$$

where $f(t,u) = (8+2u^3(t))^{1/3} + (2t-1)(2u - 2\sin(u(t))) - 2$, $a_1 = 1/2$, $a_2 = 1/3$, $\xi_1 = 1/3$, $\xi_2 = 1/4$, $T = 2$ we have $A = \sum_{i=1}^m a_i\xi_i^{2\alpha-2}/\Gamma(2\alpha-1) = 1/4 \neq T = 2$. Clearly $\lim_{u\to 0}\frac{f(t,u)}{u} = 0$. It can easily be verified that all the assumptions of Theorem 3.4 hold. Consequently, (3.5)-(3.6) has at least one solution.

Example 3.8. Consider the three-point nonlinear differential equation

$$D_{0+}^{3/2}u(t) + f(t,u(t)) = 0, \quad 0 < t < 1, \tag{3.7}$$

$$I_{0+}^{2-\alpha}u(t)|_{t=0} = 0, \quad D_{0+}^{\alpha-2}u(T) = \sum_{i=1}^{m} a_i I_{0+}^{\alpha-1}u(\xi_i), \tag{3.8}$$

where $f(t,u) = \frac{1}{(2t+8)^2}\frac{8\|u\|}{1+\|u\|}, a_1 = 2, a_2 = 3, \xi_1 = 1/2, \xi_2 = 1/3, T = 2$ we have $A = \sum_{i=1}^m a_i\xi_i^{2\alpha-2}/\Gamma(2\alpha-1) = 1 \neq T = 2$. Clearly, $L = 1/8$ as

$$|f(t,u) - f(t,v)| \le 1/8\|u-v\|.$$

Further,

$$L\Big[\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{T^{\alpha-1}}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^m a_i\xi_i^{2\alpha-1}}{\Gamma(2\alpha)} - \frac{T^2}{2}\Big)\Big] \approx 0.3 < 1.$$

Thus, all the assumptions of Theorem 3.5 are satisfied. Hence, (3.7)-(3.8) has a unique solution on $[0,1]$.

Acknowledgements. The authors want to thank the anonymous referee for his or her valuable comments and suggestions. This study was supported by grants 10771212 from the NNSF of China, and 2010LKSX09 from the Fundamental Research Funds for the Central Universities.

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