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EXISTENCE OF POSITIVE SOLUTIONS FOR SINGULAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH INTEGRAL BOUNDARY CONDITIONS

JINGFU JIN, XIPING LIU, MEI JIA

Abstract. This article shows the existence of a positive solution for the singular fractional differential equation with integral boundary condition

$$\begin{gathered}
{}^{C}D^{p}u(t) = \lambda h(t) f(t,u(t)), \quad t \in (0,1),\\
u(0) - au(1) = \int_0^1 g_0(s)u(s)\,\mathrm{d}s,\\
u'(0) - b\,{}^{C}D^{q}u(1) = \int_0^1 g_1(s)u(s)\,\mathrm{d}s,\\
u''(0) = u'''(0) = \dots = u^{(n-1)}(0) = 0,
\end{gathered}$$

where λ is a parameter and the nonlinear term is allowed to be singular at $t = 0, 1$ and $u = 0$. We obtain an explicit interval for λ such that for any λ in this interval, existence of at least one positive solution is guaranteed. Our approach is by a fixed point theory in cones combined with linear operator theory.

1. Introduction

We consider the singular integral boundary-value problem involving Caputo fractional derivative:

$$\begin{gathered}
{}^{C}D^{p}u(t) = \lambda h(t) f(t,u(t)), \quad t \in (0,1),\\
u(0) - au(1) = \int_0^1 g_0(s)u(s)\,\mathrm{d}s,\\
u'(0) - b\,{}^{C}D^{q}u(1) = \int_0^1 g_1(s)u(s)\,\mathrm{d}s,\\
u''(0) = u'''(0) = \dots = u^{(n-1)}(0) = 0,
\end{gathered}\tag{1.1}$$

where ${}^{C}D$ is the standard Caputo derivative, $n \ge 3$ is an integer, $p \in (n-1, n)$, $0 < q < 1$, $0 < a < 1$, $0 < b < \Gamma(2-q)$ are real numbers. $f \in C([0,1] \times$

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$(0, +\infty), [0, +\infty))$, and $f(t, u)$ may be singular at $u = 0$. g_0, $g_1 \in C[0, 1]$ are given functions. $h \in C((0, 1), [0, +\infty))$, $h(t)$ is allowed to be singular at $t = 0, 1$.

There exist a great number of important applications using fractional differential equations in many areas, such as physics, mechanics, chemistry, engineering, etc. Due to this, the study of related problems has attracted much attention of the researchers, especially most recently [2, 3, 21, 1, 20, 5, 7, 14, 15, 16, 19]. Also, as another important factor, singularity is sometimes inevitable in the mathematical models of modern science and technology areas. Among the studies of the existence of positive solutions for singular boundary-value problems, extensive work has been done for the singular integer order differential equations with integral boundary conditions; see [10, 11, 18, 9, 13, 12, 17, 8], and the references therein. On the other hand, for fractional differential equations, however, most results on singular boundary-value problems are only restricted to the two-point boundary conditions [2, 3, 21, 1, 20, 5]. For example, with the assumptions of $1 < \alpha < 2$ and $f(t, x, y)$ is singular at $x = 0$, [1] discussed existence and multiplicity of positive solutions for the two-point boundary-value problem (1.2):

$$\begin{gathered} D^{\alpha}_{0+}u(t) + f(t, u(t), D^{\mu}u(t)) = 0, \quad t \in (0, 1), \\ u(0) = u(1) = 0, \end{gathered} \tag{1.2}$$

where $\mu > 0$ and $\alpha - \mu \geq 1$, D^{α}_{0+} is the standard Riemann-Liouville derivative.

When $2 < \alpha < 3$, the two-point boundary-value problems (1.3) and (1.4) are studied in [2] and [20] respectively:

$$\begin{gathered} D^{\alpha}_{0+}u(t) + f(t, u(t)) = 0, \quad t \in (0, 1), \\ u(0) = u'(1) = u''(0) = 0, \end{gathered} \tag{1.3}$$

$$and \begin{gathered} D^{\alpha}u(t) + f(t, u(t), u'(t), D^{\mu}u(t)) = 0, \\ u(0) = 0,\ u'(0) = u'(1) = 0. \end{gathered} \tag{1.4}$$

In (1.3), f is assumed to be singular at $t = 0$, and D^{α}_{0+} is the standard Caputo derivative. In (1.4), $f(t, x, y, z)$ may be singular at the value 0 of all variables x, y, z and $D^{\alpha}u(t)$ is the standard Riemann-Liouville fractional derivative.

In the literature, results on singular integral boundary-value problems of the fractional differential equations are relatively rare. In this paper, we first give the Green function of boundary-value problem (BVP) (1.1) and prove some of its properties. Then, applying a fixed-point theorem with linear operator theory analysis, we obtain some sufficient conditions on the existence of positive solutions of (1.1). An explicit interval for λ is derived such that for any λ in this interval, the existence of at least one positive solution is guaranteed.

2. Preliminaries

In this section, we introduce definitions and preliminary facts which are used throughout this paper.

Definition 2.1 ([19])**.** The fractional integral of order $\alpha > 0$ of a function $y : (0, +\infty) \to \mathbb{R}$ is given by

$$I^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} y(s)\, \mathrm{d}s.$$

provided that the right side is point wise defined on $(0, +\infty)$, and Γ denotes the Gamma function.

Definition 2.2 ([19])**.** The fractional Caputo derivative of order $\alpha > 0$ for a function $x : (0, +\infty) \to \mathbb{R}$ is given by

$$ {}^{C}D^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{x^{(n)}(s)}{(t-s)^{\alpha+1-n}} \,\mathrm{d}s, $$

where $n = [\alpha] + 1$, provided the right integral converges.

Lemma 2.3. *Suppose that $y \in C[0,1]$ and $n \geq 3$ is an integer, $p \in (n-1, n)$, $0 < q < 1$, $0 < a < 1$, $0 < b < \Gamma(2-q)$. Then the integral boundary-value problem*

$$ \begin{gathered} {}^{C}D^{p} u(t) = y(t), \quad t \in (0,1), \\ u(0) - au(1) = \int_0^1 g_0(s) u(s) \,\mathrm{d}s, \\ u'(0) - b\,{}^{C}D^{q} u(1) = \int_0^1 g_1(s) u(s) \,\mathrm{d}s, \\ u''(0) = u'''(0) = \dots = u^{(n-1)}(0) = 0 \end{gathered} \tag{2.1} $$

is equivalent to the fractional integral equation

$$ u(t) = \int_0^1 G(t,s) y(s) \,\mathrm{d}s + \int_0^1 \Phi(t,s) u(s) \,\mathrm{d}s, \tag{2.2} $$

where

$$ G(t,s) = \begin{cases} \frac{(t-s)^{p-1}}{\Gamma(p)} + \frac{a\Gamma(p-q)(\Gamma(2-q)-b)(1-s)^{p-1} + b\Gamma(2-q)\Gamma(p)(a+t-at)(1-s)^{p-q-1}}{(1-a)(\Gamma(2-q)-b)\Gamma(p-q)\Gamma(p)}, \\ \quad \text{if } 0 \leq s \leq t \leq 1, \\ \frac{a\Gamma(p-q)(\Gamma(2-q)-b)(1-s)^{p-1} + b\Gamma(2-q)\Gamma(p)(a+t-at)(1-s)^{p-q-1}}{(1-a)(\Gamma(2-q)-b)\Gamma(p-q)\Gamma(p)}, \\ \quad \text{if } 0 \leq t \leq s \leq 1. \end{cases} \tag{2.3} $$

and

$$ \Phi(t,s) = \frac{(a+t-at)\Gamma(2-q) g_1(s)}{(1-a)(\Gamma(2-q)-b)} + \frac{g_0(s)}{1-a}. \tag{2.4} $$

Proof. From ${}^{C}D^{p} u(t) = y(t)$, $t \in (0,1)$ and the boundary conditions $u''(0) = u'''(0) = \dots = u^{(n-1)}(0) = 0$, we have

$$ \begin{aligned} u(t) &= I_t^p y(t) + u(0) + u'(0)t + \frac{u''(0)}{2!} t^2 + \dots + \frac{u^{(n-1)}(0)}{(n-1)!} t^{n-1} \\ &= \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} y(s) \,\mathrm{d}s + u(0) + u'(0)t. \end{aligned} $$

By properties of the Caputo derivative, we get

$$ \begin{aligned} {}^{C}D^{q} u(t) &= I_t^{p-q} y(t) + {}^{C}D^{q}(u(0) + u'(0)t) \\ &= \frac{\int_0^t (t-s)^{p-q-1} y(s) \,\mathrm{d}s}{\Gamma(p-q)} + \frac{u'(0) t^{1-q}}{\Gamma(2-q)}. \end{aligned} $$

Then

$$ u(1) = \frac{1}{\Gamma(p)} \int_0^1 (1-s)^{p-1} y(s) \,\mathrm{d}s + u(0) + u'(0), $$

and
$$
{}^{C}D^{q}u(1)=\frac{\int_0^1(1-s)^{p-q-1}y(s)\,\mathrm{d}s}{\Gamma(p-q)}+\frac{u'(0)}{\Gamma(2-q)}.
$$
By the boundary conditions $u(0)-au(1)=\int_0^1 g_0(s)u(s)\,\mathrm{d}s$ and $u'(0)-b\,{}^{C}D^{q}u(1)=\int_0^1 g_1(s)u(s)\,\mathrm{d}s$, we have
$$
u(0)-\frac{a}{\Gamma(p)}\int_0^1(1-s)^{p-1}y(s)\,\mathrm{d}s-au(0)-au'(0)=\int_0^1 g_0(s)u(s)\,\mathrm{d}s,
$$
and
$$
u'(0)-\frac{b\int_0^1(1-s)^{p-q-1}y(s)\,\mathrm{d}s}{\Gamma(p-q)}-\frac{bu'(0)}{\Gamma(2-q)}=\int_0^1 g_1(s)u(s)\,\mathrm{d}s.
$$
Hence,
$$
\begin{aligned}
u'(0)&=\frac{b\Gamma(2-q)}{(\Gamma(2-q)-b)\Gamma(p-q)}\int_0^1(1-s)^{p-q-1}y(s)\,\mathrm{d}s\\
&\quad+\frac{\Gamma(2-q)}{\Gamma(2-q)-b}\int_0^1 g_1(s)u(s)\,\mathrm{d}s,
\end{aligned}
$$
and
$$
\begin{aligned}
u(0)&=\frac{1}{1-a}\int_0^1 g_0(s)u(s)\,\mathrm{d}s+\frac{a\int_0^1(1-s)^{p-1}y(s)\,\mathrm{d}s}{(1-a)\Gamma(p)}\\
&\quad+\frac{ab\Gamma(2-q)\int_0^1(1-s)^{p-q-1}y(s)\,\mathrm{d}s}{(1-a)(\Gamma(2-q)-b)\Gamma(p-q)}+\frac{a\Gamma(2-q)\int_0^1 g_1(s)u(s)\,\mathrm{d}s}{(1-a)(\Gamma(2-q)-b)}.
\end{aligned}
$$
We can easily obtain
$$
\begin{aligned}
u(t)&=\frac{1}{\Gamma(p)}\int_0^t(t-s)^{p-1}y(s)\,\mathrm{d}s+\frac{bt\Gamma(2-q)\int_0^1(1-s)^{p-q-1}y(s)\,\mathrm{d}s}{(\Gamma(2-q)-b)\Gamma(p-q)}\\
&\quad+\frac{a\int_0^1(1-s)^{p-1}y(s)\,\mathrm{d}s}{(1-a)\Gamma(p)}+\frac{ab\Gamma(2-q)\int_0^1(1-s)^{p-q-1}y(s)\,\mathrm{d}s}{(1-a)(\Gamma(2-q)-b)\Gamma(p-q)}\\
&\quad+\frac{1}{1-a}\int_0^1 g_0(s)u(s)\,\mathrm{d}s+\frac{t\Gamma(2-q)\int_0^1 g_1(s)u(s)\,\mathrm{d}s}{\Gamma(2-q)-b}\\
&\quad+\frac{a\Gamma(2-q)\int_0^1 g_1(s)u(s)\,\mathrm{d}s}{(1-a)(\Gamma(2-q)-b)}\\
&=\int_0^1 G(t,s)y(s)\,\mathrm{d}s+\int_0^1\Phi(t,s)u(s)\,\mathrm{d}s.
\end{aligned}
$$
The proof is complete. □

Denote
$$
k_1=\frac{(\Gamma(2-q)-b)\Gamma(p-q)+b\Gamma(2-q)\Gamma(p)}{(1-a)(\Gamma(2-q)-b)\Gamma(p-q)\Gamma(p)},
$$
$$
k_2=\frac{ab\Gamma(2-q)}{(1-a)(\Gamma(2-q)-b)\Gamma(p-q)}.
$$

Lemma 2.4. *The function $G(t,s)$ in Lemma 2.3 satisfies the following conditions:*

(i) *$G(t,s)$ is continuous on $[0,1]\times[0,1]$;*
(ii) *$G(t,s)\le k_1(1-s)^{p-q-1}$, for any $(t,s)\in[0,1]\times[0,1]$;*

(iii) $G(t,s) \geq k_2(1-s)^{p-q-1}$, *for any* $(t,s) \in [0,1]\times[0,1]$.

Proof. It is easy to check that (i) holds and $G(t,s) \geq 0$ on $[0,1]\times[0,1]$.

(ii) For $0 \leq s \leq t \leq 1$, denote

$$\begin{aligned} &G_1(t,s)\\ &= \frac{a\Gamma(p-q)(\Gamma(2-q)-b)(1-s)^{p-1} + b\Gamma(2-q)\Gamma(p)(a+t-at)(1-s)^{p-q-1}}{(1-a)(\Gamma(2-q)-b)\Gamma(p-q)\Gamma(p)}\\ &\quad + \frac{(t-s)^{p-1}}{\Gamma(p)}, \end{aligned}$$

and for $0 \leq t \leq s \leq 1$, denote

$$\begin{aligned} &G_2(t,s)\\ &= \frac{a\Gamma(p-q)(\Gamma(2-q)-b)(1-s)^{p-1} + b\Gamma(2-q)\Gamma(p)(a+t-at)(1-s)^{p-q-1}}{(1-a)(\Gamma(2-q)-b)\Gamma(p-q)\Gamma(p)}. \end{aligned}$$

For $0 \leq s \leq t \leq 1$, we have

$$\begin{aligned} &(1-a)(\Gamma(2-q)-b)\Gamma(p-q)\Gamma(p)G_1(t,s)\\ &\leq (1-a)(\Gamma(2-q)-b)\Gamma(p-q)(1-s)^{p-1} + a\Gamma(p-q)(\Gamma(2-q)-b)(1-s)^{p-1}\\ &\quad + b\Gamma(2-q)\Gamma(p)(a+t-at)(1-s)^{p-q-1}\\ &\leq (1-s)^{p-q-1}[(\Gamma(2-q)-b)\Gamma(p-q)(1-s)^q + b\Gamma(2-q)\Gamma(p)(a+t-at)]\\ &\leq (1-s)^{p-q-1}[(\Gamma(2-q)-b)\Gamma(p-q) + b\Gamma(2-q)\Gamma(p)]. \end{aligned}$$

Hence, $G_1(t,s) \leq k_1(1-s)^{p-q-1}$, for any $0 \leq s \leq t \leq 1$.

For $0 \leq t \leq s \leq 1$, we have

$$\begin{aligned} &(1-a)(\Gamma(2-q)-b)\Gamma(p-q)\Gamma(p)G_2(t,s)\\ &= (1-s)^{p-q-1}[a\Gamma(p-q)(\Gamma(2-q)-b)(1-s)^q + b\Gamma(2-q)\Gamma(p)(a+t-at)]\\ &\leq (1-s)^{p-q-1}[(\Gamma(2-q)-b)\Gamma(p-q) + b\Gamma(2-q)\Gamma(p)]. \end{aligned}$$

Hence, $G_2(t,s) \leq k_1(1-s)^{p-q-1}$.

Therefore, $G(t,s) \leq k_1(1-s)^{p-q-1}$, for any $(t,s) \in [0,1]\times[0,1]$.

(iii) It is easy to see for $(t,s) \in [0,1]\times[0,1]$,

$$\begin{aligned} (1-a)(\Gamma(2-q)-b)\Gamma(p-q)\Gamma(p)G(t,s) &\geq b\Gamma(2-q)\Gamma(p)(a+t-at)(1-s)^{p-q-1}\\ &\geq ab\Gamma(2-q)\Gamma(p)(1-s)^{p-q-1}. \end{aligned}$$

Therefore, $G(t,s) \geq k_2(1-s)^{p-q-1}$, for any $(t,s) \in [0,1]\times[0,1]$. □

Denote

$$m_0 = \min_{t,s\in[0,1]} \Phi(t,s), \quad M_0 = \max_{t,s\in[0,1]} \Phi(t,s).$$

Let $E = C[0,1]$ be the Banach space with the norm $\|u\| = \max_{0\leq t\leq 1}|u(t)|$, $P = \{u \in E : u(t) \geq 0\}$ and $K = \{u \in P : u(t) \geq \frac{k_2(1-M_0)}{k_1}\|u\|\}$ be cones of E.

Denote $K_r = \{u \in K : \|u\| < r\}$, $\partial K_r = \{u \in K : \|u\| = r\}$, and $\overline{K}_{r,R} = \{u \in K : r \leq \|u\| \leq R\}$, where $0 < r < R < +\infty$.

Lemma 2.5 ([6, 4])**.** *Let K be a positive cone in real Banach space E, $0 < r < R < +\infty$, and let $S : \overline{K}_{r,R} \to K$ be a completely continuous operator and such that*

(i) $\|Su\| \leq \|u\|$ *for* $u \in \partial K_R$;

(ii) *There exists* $e \in \partial K_r$ *such that* $u \neq Su + me$ *for any* $u \in \partial K_r$*, and* $m > 0$.
Then S *has a fixed point in* $\overline{K}_{r,R}$.

Remark 2.6. If (i) and (ii) are satisfied for $u \in \partial K_r$ and $e \in \partial K_R$, respectively. Then Lemma 2.5 is still true.

Define a linear operator $A : E \to E$, by

$$Au(t) = \int_0^1 \Phi(t,s)u(s)\,\mathrm{d}s. \tag{2.5}$$

Lemma 2.7. *Suppose* $0 \le m_0 \le M_0 < 1$ *holds. Then*

(i) A *is a bounded linear operator;*
(ii) $A(P) \subset P$*;*
(iii) $(I - A)$ *is invertible and* $\|(I - A)^{-1}\| \le \frac{1}{1-M_0}$.

Proof. (i) It is easy to see that A is a linear operator with

$$|Au(t)| = \Big|\int_0^1 \Phi(t,s)u(s)\,\mathrm{d}s\Big| \le M_0\|u\|.$$

Therefore, $\|A\| \le M_0 < 1$. It follows that A is a bounded linear operator.

(ii) For each $u \in P$, we have $u \in C([0,1])$, $u(t) \ge 0$. Since $\Phi(t,s)$ is continuous and nonnegative, it is easy to check that $Au \in C([0,1])$, $Au(t) \ge 0$. This implies that $A(P) \subset P$.

(iii) We have proved in (i) that $\|A\| \le M_0 < 1$, which implies that $(I - A)^{-1}$ is invertible.

To find the expression for $(I - A)^{-1}$, we use the theory of Fredholm integral equations. We have $u(t) = (I - A)^{-1}v(t)$ if and only if $u(t) = v(t) + Au(t)$ for each $t \in [0,1]$. The definition of the operator A implies that

$$u(t) = v(t) + \int_0^1 \Phi(t,s)u(s)\,\mathrm{d}s. \tag{2.6}$$

The condition $\|A\| \le M_0 < 1$ implies that 1 is not an eigenvalue of the kernel $\Phi(t,s)$.

Hence, (2.6) has a unique solution $u \in E$, for each $v \in E$. By successive substitutions in (2.6), we obtain

$$u(t) = v(t) + \int_0^1 \rho(t,s)v(s)\,\mathrm{d}s, \tag{2.7}$$

where the resolvent kernel $\rho(t,s)$ is given by

$$\rho(t,s) = \sum_{j=1}^{\infty} \Phi_j(t,s),$$

where $\Phi_1(t,s) = \Phi(t,s)$, $\Phi_j(t,s) = \int_0^1 \Phi(t,\tau)\Phi_{j-1}(\tau,s)\,\mathrm{d}\tau$, $(j = 2,3,\dots)$. Since $0 \le m_0 \le \Phi(t,s) \le M_0 < 1$, we have $m_0^j \le \Phi_j(t,s) \le M_0^j$, $(j = 1,2,3,\dots)$. Hence, we have

$$\frac{m_0}{1-m_0} \le \rho(t,s) \le \frac{M_0}{1-M_0}, \tag{2.8}$$

and $\rho(t,s)$ is continuous on $[0,1] \times [0,1]$. In view of (2.7) and (2.8), we obtain

$$|(I - A)^{-1}v(t)| \le |v(t)| + \int_0^1 |\rho(t,s)v(s)|\,\mathrm{d}s \le (1 + \frac{M_0}{1-M_0})\|v\| = \frac{1}{1-M_0}\|v\|.$$

That is, $\|(I-A)^{-1}\| \le 1/(1-M_0)$. □

Define a nonlinear operator $T: E \to E$, by

$$Tu(t) = \lambda \int_0^1 G(t,s)h(s)f(s,u(s))\,\mathrm{d}s. \tag{2.9}$$

In view of (2.5), (2.9), and Lemma 2.3, we can easily prove that the existence of solutions to (1.1) is equivalent to the existence of solutions to the equation

$$u(t) = Tu(t) + Au(t), \quad t \in [0,1]. \tag{2.10}$$

It follows from Lemma 2.7 that u is a solution of (2.10) if and only if u is a solution of $u(t) = (I-A)^{-1}Tu(t)$. That is, u is a fixed point of the operator $S := (I-A)^{-1}T$. By (2.7) and (2.9), we have

$$\begin{aligned}(Su)(t) &= \lambda \int_0^1 G(t,s)h(s)f(s,u(s))\,\mathrm{d}s \\ &\quad + \lambda \int_0^1 \rho(t,s) \int_0^1 G(s,\tau)h(\tau)f(\tau,u(\tau))\,\mathrm{d}\tau\,\mathrm{d}s.\end{aligned} \tag{2.11}$$

We can prove the following lemma.

Lemma 2.8. *A function u is a solution of* (1.1) *if and only if u is a fixed point of the operator S.*

We denote

$$L = \int_0^1 (1-s)^{p-q-1}h(s)\,\mathrm{d}s.$$

and assume the following conditions hold

(H) $h \in C((0,1),[0,+\infty))$, $\int_0^1 h(s)\,\mathrm{d}s < +\infty$ and $0 < L < +\infty$.

To overcome the singularity, we consider the following approximating equation of (2.11) with boundary condition of (1.1),

$$\begin{aligned}(S_n u)(t) &= \lambda \int_0^1 G(t,s)h(s)f_n(s,u(s))\,\mathrm{d}s \\ &\quad + \lambda \int_0^1 \rho(t,s) \int_0^1 G(s,\tau)h(\tau)f_n(\tau,u(\tau))\,\mathrm{d}\tau\,\mathrm{d}s.\end{aligned} \tag{2.12}$$

where n is a positive integer and $f_n(t,u) = f(t,\max\{1/n,u\})$.

Lemma 2.9. *Suppose $0 \le m_0 \le M_0 < 1$ and* (H) *holds. Then for each positive integer n, we have*

(i) *For any $0 < r \le R < +\infty$, the operator $S_n : \overline{K}_{r,R} \to P$ is completely continuous;*
(ii) *$S_n(\overline{K}_{r,R}) \subset K$.*

Proof. (i) Suppose $D \subset \overline{K}_{r,R}$ is a bounded set. Then there exists $r_1 > 0$ such that $\|u\| \le r_1$ for any $u \in D$. Denote

$$M_1 = \max\{f(t,\max\{1/n,u\}) : (t,u) \in [0,1] \times [\frac{1}{n}, \frac{1}{n} + r_1]\}.$$

By (2.8) and Lemma 2.4, for any $u \in D$ and $t \in [0,1]$, we have

$|S_n u(t)|$

$$
\begin{aligned}
&= |\lambda \int_0^1 G(t,s)h(s)f_n(s,u(s))\,\mathrm{d}s + \lambda \int_0^1 \rho(t,s) \int_0^1 G(s,\tau)h(\tau)f_n(\tau,u(\tau))\,\mathrm{d}\tau\,\mathrm{d}s| \\
&\le \lambda k_1 \int_0^1 (1-s)^{p-q-1}h(s)f_n(s,u(s))\,\mathrm{d}s \\
&\quad + \frac{M_0\lambda k_1}{1-M_0}\int_0^1 (1-s)^{p-q-1}h(s)f_n(s,u(s))\,\mathrm{d}s \\
&= \frac{\lambda k_1}{1-M_0}\int_0^1 (1-s)^{p-q-1}h(s)f(s,\max\{1/n,u(s)\})\,\mathrm{d}s \\
&\le \frac{\lambda k_1 M_1}{1-M_0}\int_0^1 (1-s)^{p-q-1}h(s)\,\mathrm{d}s \\
&= \frac{\lambda k_1 M_1 L}{1-M_0}.
\end{aligned}
$$

Therefore, $S_n(D)$ is uniformly bounded.

We can also prove that $S_n(D)$ is equicontinuous. For $t_1, t_2 \in [0,1]$ and $u \in D$, we have

$$
\begin{aligned}
&|(S_nu)(t_1) - (S_nu)(t_2)| \\
&= |\lambda \int_0^1 (G(t_1,s) - G(t_2,s))h(s)f_n(s,u(s))\,\mathrm{d}s \\
&\quad + \lambda \int_0^1 (\rho(t_1,s) - \rho(t_2,s)) \int_0^1 G(s,\tau)h(\tau)f_n(\tau,u(\tau))\,\mathrm{d}\tau\,\mathrm{d}s| \\
&\le \lambda M_1\Big(\int_0^1 |G(t_1,s) - G(t_2,s)|h(s)\,\mathrm{d}s \\
&\quad + k_1 \int_0^1 |\rho(t_1,s) - \rho(t_2,s)| \int_0^1 (1-s)^{p-q-1}h(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\Big) \\
&\le \lambda M_1\Big(\int_0^1 |G(t_1,s) - G(t_2,s)|h(s)\,\mathrm{d}s + k_1 L \int_0^1 |\rho(t_1,s) - \rho(t_2,s)|\,\mathrm{d}s\Big).
\end{aligned}
$$

Since $G(t,s)$ and $\rho(t,s)$ are continuous on $[0,1]\times[0,1]$, we can get $G(t,s)$ and $\rho(t,s)$ are uniformly continuous on $[0,1]\times[0,1]$, it follows that $|(S_nu)(t_2) - (S_nu)(t_1)| \to 0$ as $|t_2 - t_1| \to 0$. Hence, $S_n(D)$ is equicontinuous. Using the Ascoli-Arzela's theorem, $S_n(D)$ is relatively compact. Therefore, $S_n : \overline{K}_{r,R} \to P$ is compact.

Now we show that S_n is continuous. Suppose $u, u_m \in D$, $(m = 1,2,3,\dots)$ with $\|u_m - u\| \to 0$ as $m \to \infty$. Then there exists $r_2 > 0$ such that $\|u_m\| < r_2$ and $\|u\| < r_2$.

For $t \in [0,1]$,

$$
\begin{aligned}
&|(S_nu_m)(t) - (S_nu)(t)| \\
&= |\lambda \int_0^1 G(t,s)h(s)\big(f_n(s,u_m(s)) - f_n(s,u(s))\big)\,\mathrm{d}s \\
&\quad + \lambda \int_0^1 \rho(t,s) \int_0^1 G(s,\tau)h(\tau)\big(f_n(\tau,u_m(\tau)) - f_n(\tau,u(\tau))\big)\,\mathrm{d}\tau\,\mathrm{d}s| \\
&\le \lambda k_1 \int_0^1 (1-s)^{p-q-1}h(s)\big|f_n(s,u_m(s)) - f_n(s,u(s))\big|\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\lambda k_1 M_0}{1-M_0}\int_0^1 (1-s)^{p-q-1}h(s)\big|f_n(s,u_m(s))-f_n(s,u(s))\big|\,\mathrm{d}s\\
&=\frac{\lambda k_1}{1-M_0}\int_0^1 (1-s)^{p-q-1}h(s)\big|f_n(s,u_m(s))-f_n(s,u(s))\big|\,\mathrm{d}s.
\end{aligned}
$$

Since $f_n(s,u)$ is continuous on $[0,1]\times[\frac{1}{n},\frac{1}{n}+r_2]$, we can get $f_n(s,u)$ is uniformly continuous on $[0,1]\times[\frac{1}{n},\frac{1}{n}+r_2]$. Hence, we have

$$\lim_{m\to\infty}\|f_n(s,u_m(s))-f_n(s,u(s))\|=0.$$

It is easy to see

$$\lim_{m\to\infty}\|S_nu_m-S_nu\|=0.$$

Therefore, S_n is continuous on P.

(ii) For any $u\in\overline{K}_{r,R}$, $t\in[0,1]$, we have

$$
\begin{aligned}
&S_nu(t)\\
&=\lambda\int_0^1 G(t,s)h(s)f_n(s,u(s))\,\mathrm{d}s+\lambda\int_0^1\rho(t,s)\int_0^1 G(s,\tau)h(\tau)f_n(\tau,u(\tau))\,\mathrm{d}\tau\,\mathrm{d}s\\
&\le\lambda k_1\int_0^1 (1-s)^{p-q-1}h(s)f_n(s,u(s))\,\mathrm{d}s\\
&\quad+\frac{M_0\lambda k_1}{1-M_0}\int_0^1 (1-s)^{p-q-1}h(s)f_n(s,u(s))\,\mathrm{d}s\\
&=\frac{\lambda k_1}{1-M_0}\int_0^1 (1-s)^{p-q-1}h(s)f_n(s,u(s))\,\mathrm{d}s.
\end{aligned}
$$

This implies that $\|S_nu\|\le\frac{\lambda k_1}{1-M_0}\int_0^1(1-s)^{p-q-1}h(s)f_n(s,u(s))\,\mathrm{d}s$.

On the other hand, for $t\in[0,1]$,

$$
\begin{aligned}
S_nu(t)&=\lambda\int_0^1 G(t,s)h(s)f_n(s,u(s))\,\mathrm{d}s\\
&\quad+\lambda\int_0^1\rho(t,s)\int_0^1 G(s,\tau)h(\tau)f_n(\tau,u(\tau))\,\mathrm{d}\tau\,\mathrm{d}s\\
&\ge\lambda k_2\int_0^1 (1-s)^{p-q-1}h(s)f_n(s,u(s))\,\mathrm{d}s\\
&\ge\lambda k_2\frac{1-M_0}{\lambda k_1}\|S_nu\|\\
&=\frac{k_2}{k_1}(1-M_0)\|S_nu\|.
\end{aligned}
$$

Therefore $S_n(\overline{K}_{r,R})\subset K$. □

3. Main results and proof

Denote

$$f^0=\limsup_{u\to0^+}\max_{t\in[0,1]}\frac{f(t,u)}{u},\quad f_\infty=\liminf_{u\to+\infty}\min_{t\in[0,1]}\frac{f(t,u)}{u},$$

and

$$f^\infty=\limsup_{u\to+\infty}\max_{t\in[0,1]}\frac{f(t,u)}{u},\quad f_0=\liminf_{u\to0^+}\min_{t\in[0,1]}\frac{f(t,u)}{u}.$$

Theorem 3.1. *Suppose* $0 \le m_0 \le M_0 < 1$ *and* (H) *holds. If*

$$0 < f^0 < \frac{1-M_0}{k_1 L} \quad \text{and} \quad 0 < \frac{1}{k_2 L} < f_\infty < +\infty, \tag{3.1}$$

then (1.1) *has at least one positive solution for* $\lambda \in (\frac{1}{k_2 L f_\infty}, \frac{1-M_0}{k_1 L f^0})$.

Proof. For $\lambda \in (\frac{1}{k_2 L f_\infty}, \frac{1-M_0}{k_1 L f^0})$, there exists $\varepsilon > 0$ such that

$$f_\infty - \varepsilon > 0, \quad \frac{1}{(f_\infty - \varepsilon) k_2 L} \le \lambda \le \frac{1-M_0}{k_1 L (f^0 + \varepsilon)}.$$

By (3.1), there exist $r > 0$ and $R_0 > 0$, such that

$$f(t,u) \le (f^0 + \varepsilon) u, \quad \text{for } t \in [0,1],\ 0 < u \le r. \tag{3.2}$$

$$f(t,u) > (f_\infty - \varepsilon) u, \quad \text{for } t \in [0,1],\ u \ge R_0. \tag{3.3}$$

For any $u \in \partial K_r$ and $n > [\frac{k_1}{r k_2 (1-M_0)}] + 1 =: n_0$, we have

$$r = \|u\| \ge u(t) \ge \frac{k_2(1-M_0)}{k_1} \|u\| = \frac{r k_2 (1-M_0)}{k_1} > \frac{1}{n}.$$

It follows that

$$f_n(t, u(t)) = f(t, \max\{1/n, u(t)\}) = f(t, u(t)) \le (f^0 + \varepsilon) u \tag{3.4}$$

from (3.2). Hence,

$$\begin{aligned}
\|S_n u\| &= \max_{t\in[0,1]} |\lambda \int_0^1 G(t,s) h(s) f_n(s, u(s))\, \mathrm{d}s \\
&\quad + \lambda \int_0^1 \rho(t,s) \int_0^1 G(s,\tau) h(\tau) f_n(\tau, u(\tau))\, \mathrm{d}\tau\, \mathrm{d}s| \\
&\le \lambda k_1 \int_0^1 (1-s)^{p-q-1} h(s) f_n(s, u(s))\, \mathrm{d}s \\
&\quad + \frac{M_0 \lambda k_1}{1-M_0} \int_0^1 (1-s)^{p-q-1} h(s) f_n(s, u(s))\, \mathrm{d}s \\
&= \frac{\lambda k_1}{1-M_0} \int_0^1 (1-s)^{p-q-1} h(s) f_n(s, u(s))\, \mathrm{d}s \\
&\le \frac{\lambda k_1 (f^0 + \varepsilon)}{1-M_0} \int_0^1 (1-s)^{p-q-1} h(s) u(s)\, \mathrm{d}s \\
&\le \frac{\lambda k_1 L (f^0 + \varepsilon)}{1-M_0} \|u\| \le \|u\|.
\end{aligned}$$

We can get $\|S_n u\| \le \|u\|$, for each $u \in \partial K_r$.

Let $R = \max\{2r, \frac{k_1 R_0}{k_2(1-M_0)}\}$ and $e(t) \equiv 1$ for $t \in [0,1]$. Then $R > r$ and $e(t) \in K_1 = \{u \in K : \|u\| < 1\}$. Subsequently, we can show $u \ne S_n u + m e$, for any $m > 0$ and $u \in \partial K_R$.

Otherwise, there exists $u_0 \in \partial K_R$ and $m_1 > 0$ such that $u_0 = S_n u_0 + m_1 e$. We notice that for any $s \in [0,1]$,

$$u_0(s) \ge \min_{s\in[0,1]} u_0(s) \ge \frac{k_2}{k_1}(1-M_0) R \ge R_0.$$

From (3.3), it follows that

$$f_n(t,u_0(t)) = f(t,\max\{1/n,u_0(t)\}) = f(t,u_0(t)) > (f_\infty - \varepsilon)u_0(t)\,.$$

Let $\xi = \min_{t\in[0,1]} u_0(t)$. Consequently, for any $t \in [0,1]$, we have

$$\begin{aligned}
u_0(t) &= \lambda \int_0^1 G(t,s)h(s)f_n(s,u_0(s))\,\mathrm{d}s \\
&\quad + \lambda \int_0^1 \rho(t,s) \int_0^1 G(s,\tau)h(\tau)f_n(\tau,u_0(\tau))\,\mathrm{d}\tau\,\mathrm{d}s + m_1 e(t) \\
&\ge \lambda \int_0^1 G(t,s)h(s)f_n(s,u_0(s))\,\mathrm{d}s + m_1 e(t) \\
&\ge \lambda k_2 (f_\infty - \varepsilon) \int_0^1 (1-s)^{p-q-1} h(s)u_0(s)\,\mathrm{d}s + m_1 \\
&\ge \frac{\xi}{L} \int_0^1 (1-s)^{p-q-1} h(s)\,\mathrm{d}s + m_1 \\
&\ge \xi + m_1 > \xi.
\end{aligned}$$

This implies that $\xi > \xi$, which is a contradiction.

It follows that for $n \ge n_0 = [\frac{k_1}{rk_2(1-M_0)}] + 1$, the operator S_n has a fixed point u_n in K with $r < \|u_n\| < R$, from Lemma 2.5. Hence,

$$\begin{aligned}
&u_n(t) \\
&= \lambda \int_0^1 G(t,s)h(s)f_n(s,u_n(s))\,\mathrm{d}s + \lambda \int_0^1 \rho(t,s) \int_0^1 G(s,\tau)h(\tau)f_n(\tau,u_n(\tau))\,\mathrm{d}\tau\,\mathrm{d}s,
\end{aligned}$$

for $t \in [0,1]$. Since $u_n \in K$, we have

$$u_n(t) \ge \frac{k_2(1-M_0)}{k_1}\|u_n\| = \frac{rk_2(1-M_0)}{k_1} > \frac{1}{n} > 0, \quad t \in [0,1],$$

and

$$f_n(t,u_n(t)) = f(t,\max\{1/n,u_n(t)\}) = f(t,u_n(t)), \quad t \in [0,1].$$

It is easy to see that

$$\begin{aligned}
&u_n(t) \\
&= \lambda \int_0^1 G(t,s)h(s)f(s,u_n(s))\,\mathrm{d}s + \lambda \int_0^1 \rho(t,s) \int_0^1 G(s,\tau)h(\tau)f(\tau,u_n(\tau))\,\mathrm{d}\tau\,\mathrm{d}s,
\end{aligned}$$

for $t \in [0,1]$. By Lemma 2.8, we obtain that u_n is a positive solution of (1.1). $\square$

By proof similar to the one for Theorem 3.1, we can show the following theorem.

Theorem 3.2. *Suppose* $0 \le m_0 \le M_0 < 1$ *and* (H) *holds. If*

$$f^0 = 0 \quad \text{and} \quad f_\infty = +\infty,$$

then (1.1) *has at least one positive solution for* $\lambda \in (0,+\infty)$.

Remark 3.3. In Theorem 3.1, if $f^0 = 0$ or $f_\infty = +\infty$, we can obtain conclusions similar to Theorems 3.1 and 3.2.

Theorem 3.4. *Suppose* $0 \le m_0 \le M_0 < 1$ *and* (H) *holds. If*

$$0 < f^\infty < \frac{1-M_0}{k_1 L} \quad \text{and} \quad 0 < \frac{1}{k_2 L} < f_0 < +\infty, \tag{3.5}$$

then (1.1) *has at least one positive solution for* $\lambda \in \big(\frac{1}{k_2 L f_0}, \frac{1-M_0}{k_1 L f^\infty}\big)$.

Proof. For $\lambda \in \big(\frac{1}{k_2 L f_0}, \frac{1-M_0}{k_1 L f^\infty}\big)$, there exists $\varepsilon > 0$ such that

$$f_0 - \varepsilon > 0, \quad \frac{1}{(f_0-\varepsilon)k_2 L} \le \lambda \le \frac{1-M_0}{k_1 L(f^\infty + \varepsilon)}.$$

By (3.5), there exist $r > 0$ and $R_0 > 1$, such that

$$f(t,u) \ge (f_0 - \varepsilon)u, \quad \text{for } t \in [0,1],\ 0 < u \le r. \tag{3.6}$$

$$f(t,u) \le (f^\infty + \varepsilon)u, \quad \text{for } t \in [0,1],\ u \ge R_0. \tag{3.7}$$

Take $R \ge \max\{r, R_0, \frac{k_1 R_0}{k_2(1-M_0)}\}$ For $u \in \partial K_R$ and $n > [\frac{k_1}{Rk_2(1-M_0)}] + 1 =: n_0$, we have

$$u(t) \ge \frac{k_2(1-M_0)}{k_1}\|u\| = \frac{Rk_2(1-M_0)}{k_1} \ge R_0.$$

From (3.7), we have

$$f_n(t,u(t)) = f(t, \max\{1/n, u(t)\}) = f(t,u(t)) \le (f^\infty + \varepsilon)u.$$

Hence,

$$\begin{aligned}
\|S_n u\| &= \max_{t\in[0,1]} |\lambda \int_0^1 G(t,s)h(s)f_n(s,u(s))\,\mathrm{d}s \\
&\quad + \lambda \int_0^1 \rho(t,s)\int_0^1 G(s,\tau)h(\tau)f_n(\tau,u(\tau))\,\mathrm{d}\tau\,\mathrm{d}s| \\
&\le \lambda k_1 \int_0^1 (1-s)^{p-q-1}h(s)f_n(s,u(s))\,\mathrm{d}s \\
&\quad + \frac{M_0\lambda k_1}{1-M_0}\int_0^1 (1-s)^{p-q-1}h(s)f_n(s,u(s))\,\mathrm{d}s \\
&= \frac{\lambda k_1}{1-M_0}\int_0^1 (1-s)^{p-q-1}h(s)f_n(s,u(s))\,\mathrm{d}s \\
&\le \frac{\lambda k_1(f^\infty+\varepsilon)}{1-M_0}\int_0^1 (1-s)^{p-q-1}h(s)u(s)\,\mathrm{d}s \\
&\le \frac{\lambda k_1 L(f^\infty+\varepsilon)}{1-M_0}\|u\| \le \|u\|.
\end{aligned}$$

We can get $\|S_n u\| \le \|u\|$, for each $u \in \partial K_R$.

Let $e(t) \equiv 1$, $t \in [0,1]$. Then $e(t) \in \partial K_1$, and we can prove $u \ne S_n u + me$, for any $m > 0$, *and* $u \in K_r$. Otherwise there exists $u_0 \in K_r$ and $m_1 > 0$ such that $u_0 = S_n u_0 + m_1 e$. Let $\eta = \min\{u_0(t) : t \in [0,1]\}$, for $t \in [0,1]$, by (3.6), we have

$$\begin{aligned}
u_0(t) &= \lambda \int_0^1 G(t,s)h(s)f_n(s,u_0(s))\,\mathrm{d}s \\
&\quad + \lambda \int_0^1 \rho(t,s)\int_0^1 G(s,\tau)h(\tau)f_n(\tau,u_0(\tau))\,\mathrm{d}\tau\,\mathrm{d}s + m_1 e(t)
\end{aligned}$$

$$
\begin{aligned}
&\ge \lambda \int_0^1 G(t,s)h(s)f_n(s,u_0(s))\,\mathrm{d}s + m_1\\
&\ge \lambda \int_0^1 k_2(1-s)^{p-q-1}h(s)(f_0-\varepsilon)u_0(s)\,\mathrm{d}s + m_1\\
&\ge \frac{\eta}{L}\int_0^1 (1-s)^{p-q-1}h(s)\,\mathrm{d}s + m_1 = \eta + m_1.
\end{aligned}
$$

This is a contradiction. It follows from Lemma 2.5 that S_n has a fixed point u_n in K with $r < \|u_n\| < R$. Hence,

$$
\begin{aligned}
u_n(t) = \lambda &\int_0^1 G(t,s)h(s)f_n(s,u_n(s))\,\mathrm{d}s\\
&+ \lambda \int_0^1 \rho(t,s)\int_0^1 G(s,\tau)h(\tau)f_n(\tau,u_n(\tau))\,\mathrm{d}\tau\,\mathrm{d}s,
\end{aligned}
$$

for $t \in [0,1]$. Since $u_n \in K$, for $n > \frac{k_1}{rk_2(1-M_0)}$, we have

$$
u_n(t) \ge \frac{k_2(1-M_0)}{k_1}\|u_n\| = \frac{rk_2(1-M_0)}{k_1} > \frac{1}{n} > 0, \quad t \in [0,1],
$$

and

$$
f_n(t,u_n(t)) = f(t, \max\{1/n, u_n(t)\}) = f(t,u_n(t)), \quad t \in [0,1].
$$

It is easy to see that

$$
\begin{aligned}
u_n(t) = \lambda &\int_0^1 G(t,s)h(s)f(s,u_n(s))\,\mathrm{d}s\\
&+ \lambda \int_0^1 \rho(t,s)\int_0^1 G(s,\tau)h(\tau)f(\tau,u_n(\tau))\,\mathrm{d}\tau\,\mathrm{d}s,
\end{aligned}
$$

for $t \in [0,1]$. By Lemma 2.8, we can get u_n is a positive solution of (1.1). □

Similarly to the proof of Theorem 3.1, we can obtain the following theorem.

Theorem 3.5. *Suppose* $0 \le m_0 \le M_0 < 1$ *and* (H) *holds. If*

$$
f^\infty = 0 \quad \text{and} \quad f_0 = +\infty,
$$

then (1.1) *has at least one positive solution for* $\lambda \in (0,+\infty)$.

Remark 3.6. In Theorem 3.4, if $f^\infty = 0$ or $f_0 = +\infty$, we can obtain similar conclusions as those in Theorems 3.4 and 3.5.

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