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STRONGLY NONLINEAR NONHOMOGENEOUS ELLIPTIC UNILATERAL PROBLEMS WITH L^1 DATA AND NO SIGN CONDITIONS

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ABSTRACT. In this article, we prove the existence of solutions to unilateral problems involving nonlinear operators of the form:

$$Au + H(x, u, \nabla u) = f$$

where A is a Leray Lions operator from $W_0^{1,p(x)}(\Omega)$ into its dual $W^{-1,p'(x)}(\Omega)$ and $H(x, s, \xi)$ is the nonlinear term satisfying some growth condition but no sign condition. The right hand side f belong to $L^1(\Omega)$.

1. INTRODUCTION

Partial differential equations with nonlinearities involving non constant exponents have attracted an increasing amount of attention in recent years. The development, mainly by Rů̃žicka [23], of a theory modeling the behavior of electrorheological fluids, an important class of non-Newtonian fluids, seems to have boosted a still far from completed effort to study and understand nonlinear PDE's involving variable exponents. Other applications relate to image processing [18], elasticity [5], the flow in porous media [16] and problems in the calculus of variations involving variational integrals with nonstandard growth [26].

This in turn, gave rise to a revival of the interest in Lebesgue and Sobolev spaces with variable exponent,where many of the basic properties of these spaces are established by the work of Kovàcik and Rakosnik [20].

Many models of the obstacle problem have already been analyzed for constant exponents of nonlinearity. In [4] the authors have proved the existence of solution for quasilinear degenerated elliptic unilateral problems associated to the operator $Au + g(x, u, \nabla u) = f$ in which the nonlinear term satisfies the sign condition. The principal part A is a differential elliptic operator of the second order in divergence form, acting from $W_0^{1,p}(\Omega, \omega)$ into its dual $W^{-1,p'}(\Omega, \omega)$ and g having natural growth with respect to ∇u and u not assuming any growth restrictions, but assuming the sign-condition.

Porretta [22] studied the same problem in the classical Sobolev space that is ($p(.) = p$ constant) where the right-hand side is a bounded Radon measure on Ω

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and where the sign condition is violated, more precisely the problem treated in [22] is of the form

$$\begin{gathered} Au + g(u)|\nabla u|^p = \mu \quad \text{in} \quad \Omega \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

The work by Aharouch et al [2, 3] can be seen as generalization of [22] in the sense that in [2] the nonlinearity have taken as $H(x, u, \nabla u)$ and in [3] the degenerated case for the same problem. Recently, Rodriguez et al in [24] have proved the existence and uniqueness of an entropy solution to obstacle problem with variable growth and L^1 data, of the form

$$\begin{gathered} -\Delta_{p(.)} u + \beta(., u) = f \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

where β is some function related to a maximal monotone graph. Besides, while $f(x, u, \nabla u)$, Benboubker, Azroul and Barbara have proved the existence results in Sobolev spaces with variable exponent by using a classical theorem of Lions operators of the calculus of variations (see [17]).

Recently, while $Au = -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$, $H \equiv 0$, Bendahmane and Wittbold [6] proved the existence and uniqueness of renormalized solution with L^1-data, and Wittbold and Zimmermann [7] extended the results to the case $Au = -\operatorname{div}(a(x, u))$, (see also Bendahmane and Karlsen [9]).

The objective of our article, is to study the non homogenous obstacle problem with L^1 data associated to the general nonlinear operator of the form

$$\begin{gathered} Au + H(x, u, \nabla u) = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.1}$$

The principal part $Au = -\operatorname{div}(a(x, \nabla u))$ is a differential elliptic operator of the second order in divergence form, acting from $W_0^{1,p(x)}(\Omega)$ into its dual $W^{-1,p'(x)}(\Omega)$ and we suppose that the lower order term satisfies the exact natural growth:

$$|H(x, s, \xi)| \le \gamma(x) + g(s)|\xi|^{p(x)}$$

with $\gamma(x) \in L^1(\Omega)$ and $g \in L^1(\mathbb{R})$ and $g \ge 0$ but not satisfying the sign condition. Under these assumptions the above problem does not admit, in general, a weak solution since the terms $a(u, \nabla u)$ and $H(x, u, \nabla u)$ may not belong to $L^1_{\text{loc}}(\Omega)$. In order to overcome this difficulty, we work with the framework of entropy solutions introduced by Bénilan et al [1]. Let us mention that an equivalent notion of solution, called renormalized solution was first introduced by Di-Perna and Lions [12] for the study of Boltzmann equation. It has been used by many authors to study the elliptic equations (see [11]) and the parabolic equations (see [13, 14, 15]).

Note that our paper can be seen as a generalization of [2] and [24], and as a continuation of [17].

The outline of this paper is as follows. In Section 2, we give some preliminaries and notations. In Section 3, the existence of entropy solutions of (1.1) is obtained. In Section 4, we give the proof of Proposition 2.1, Lemma 3.3 and Lemma 4.2 (see appendix).

2. Preliminaries

In what follows, we recall some definitions and basic properties of Lebesgue and Sobolev spaces with variable exponents. For each open bounded subset Ω of $\mathbb{R}^N$ $(N \geq 1)$, we denote

$$C^+(\overline{\Omega}) = \big\{\text{continuous function } p : \overline{\Omega} \to \mathbb{R}^+ \text{ such that } 1 < p_- \leq p_+ < \infty\big\},$$

where $p_- = \inf_{x\in\overline{\Omega}} p(x)$ and $p_+ = \sup_{x\in\overline{\Omega}} p(x)$. We define the variable exponent Lebesgue space for $p \in C^+(\overline{\Omega})$ by:

$$L^{p(x)}(\Omega) = \big\{u : \Omega \to \mathbb{R} \text{ measurable}, \int_\Omega |u(x)|^{p(x)} dx < \infty\big\}.$$

the space $L^{p(x)}(\Omega)$ under the norm

$$\|u\|_{p(x)} = \inf\big\{\lambda > 0,\ \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} \leq 1\big\}$$

is a uniformly convex Banach space, then reflexive. We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

Proposition 2.1 ([19]). *(i) For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have*

$$|\int_\Omega uv dx| \leq \big(\frac{1}{p_-} + \frac{1}{p'_-}\big)\|u\|_{p(x)}\|v\|_{p'(x)}.$$

(ii) For all $p_1, p_2 \in C^+(\overline{\Omega})$ such that $p_1(x) \leq p_2(x)$ and any $x \in \overline{\Omega}$, we have $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ and the embedding is continuous.

Proposition 2.2 ([19]). *Let us denote*

$$\rho(u) = \int_\Omega |u|^{p(x)} dx, \quad \forall u \in L^{p(x)}(\Omega);$$

then the following assertions hold:

(i) $\|u\|_{p(x)} < 1$ *(resp. = 1 or > 1) if and only if* $\rho(u) < 1$ *(resp. = 1 or > 1)*
(ii) $\|u\|_{p(x)} > 1$ *implies* $\|u\|_{p(x)}^{p_-} \leq \rho(u) \leq \|u\|_{p(x)}^{p_+}$*, and* $\|u\|_{p(x)} < 1$ *implies* $\|u\|_{p(x)}^{p_+} \leq \rho(u) \leq \|u\|_{p(x)}^{p_-}$
(iii) $\|u\|_{p(x)} \to 0$ *if and only if* $\rho(u) \to 0$*, and* $\|u\|_{p(x)} \to \infty$ *if and only if* $\rho(u) \to \infty$.

We define the variable exponent Sobolev space by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) \text{ and } |\nabla u| \in L^{p(x)}(\Omega)\}.$$

where the norm is defined by

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)} \quad \forall u \in W^{1,p(x)}(\Omega).$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ and $p*(x) = \frac{Np(x)}{N-p(x)}$ for $p(x) < N$.

Proposition 2.3 ([19]). *(i) Assuming $1 < p_- \leq p_+ < \infty$, the spaces $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*

(ii) if $q \in C^+(\bar{\Omega})$ and $q(x) < p*(x)$ for any $x \in \overline{\Omega}$, then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$ is compact and continuous.

(iii) There is a constant $C > 0$, such that

$$\|u\|_{p(x)} \leq C\|\nabla u\|_{p(x)} \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

Remark 2.4. By Proposition 2.3 (iii), we know that $\|\nabla u\|_{p(x)}$ and $\|u\|_{1,p(x)}$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$.

3. Existence of an entropy solutions

In this section, we study the existence of an entropy solution of the obstacle problem.

3.1. **Basic assumptions and some Lemmas.** Throughout the paper, we assume that the following assumptions hold.

Let Ω be a bounded open set of $\mathbb{R}^N$ $(N \geq 1)$, $p \in C^+(\overline{\Omega})$ and $(1/p(x)) + (1/p'(x)) = 1$.

The function $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying the following conditions: For all $\xi, \eta \in \mathbb{R}^N$ and for almost every $x \in \Omega$,

$$|a(x,\xi)| \leq \beta(k(x) + |\xi|^{p(x)-1}), \tag{3.1}$$

$$[a(x,\xi) - a(x,\eta)](\xi - \eta) > 0 \quad \forall \xi \neq \eta, \tag{3.2}$$

$$a(x,\xi)\xi \geq \alpha|\xi|^{p(x)}, \tag{3.3}$$

where $k(x)$ is a positive function in $L^{p'(x)}(\Omega)$ and α and β are a positive constants.

Let $H(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function such that for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, the growth condition:

$$|H(x,s,\xi)| \leq \gamma(x) + g(s)|\xi|^{p(x)} \tag{3.4}$$

is satisfied, where $g : \mathbb{R} \to \mathbb{R}^+$ is a continuous positive function that belongs to $L^1(\mathbb{R})$, while $\gamma(x)$ belongs to $L^1(\Omega)$.

$$f \in L^1(\Omega). \tag{3.5}$$

Finally, let the convex set

$$K_\psi = \left\{u \in W_0^{1,p(x)}(\Omega),\ u \geq \psi \text{ a.e. in } \Omega\right\}$$

where ψ is a measurable function such that

$$\psi^+ \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega) \tag{3.6}$$

Lemma 3.1 ([17])**.** *Let $g \in L^{r(x)}(\Omega)$ and $g_n \in L^{r(x)}(\Omega)$ with $\|g_n\|_{r(x)} \leq C$ for $1 < r(x) < \infty$. If $g_n(x) \to g(x)$ a.e. on Ω, then $g_n \rightharpoonup g$ in $L^{r(x)}(\Omega)$.*

Lemma 3.2. *Assume that (3.1)–(3.3), and let $(u_n)_n$ be a sequence in $W_0^{1,p(x)}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p(x)}(\Omega)$ and*

$$\int_\Omega [a(x,\nabla u_n) - a(x,\nabla u)]\nabla(u_n - u)dx \to 0. \tag{3.7}$$

Then $u_n \to u$ strongly in $W_0^{1,p(x)}(\Omega)$.

The proof of the above Lemma is a slight modification of the analogues one of [17, Lemma 3.2].

Lemma 3.3. *Let $F:\mathbb{R}\to\mathbb{R}$ be a uniformly Lipschitz function with $F(0)=0$ and $p\in C_+(\bar\Omega)$. If $u\in W_0^{1,p(x)}(\Omega)$, then $F(u)\in W_0^{1,p(x)}(\Omega)$, moreover, if D is the set of discontinuity points of F' is finite, then*

$$\frac{\partial(F\circ u)}{\partial x_i}=\begin{cases}F'(u)\frac{\partial u}{\partial x_i} & \text{a.e. in } \{x\in\Omega: u(x)\notin D\}\\ 0 & \text{a.e. in } \{x\in\Omega: u(x)\in D\}.\end{cases}$$

The proof of the above lemma is presented in the appendix. The following Lemma is a direct deduction from Lemma 3.3.

Lemma 3.4. *Let $u\in W_0^{1,p(x)}(\Omega)$ then $u^+=\max(u,0)$ and $u^-=\max(-u,0)$ lie in $W_0^{1,p(x)}(\Omega)$. Moreover*

$$\frac{\partial u^+}{\partial x_i}=\begin{cases}\frac{\partial u}{\partial x_i} & \text{if } u>0\\ 0 & \text{if } u\le 0,\end{cases}\quad \frac{\partial u^-}{\partial x_i}=\begin{cases}0 & \text{if } u\ge 0\\ -\frac{\partial u}{\partial x_i} & \text{if } u<0.\end{cases}$$

3.2. **Definition and existence result of an entropy solution.** In this article, T_k denotes the truncation function at height $k\ge 0$: $T_k(r)=\min(k,\ \max(r,\ -k))$. Define

$$T_0^{1,p(x)}(\Omega)=\big\{u \text{ measurable in } \Omega: T_k(u)\in W_0^{1,p(x)}(\Omega),\ \forall\ k>0\big\}.$$

We now give the following definition and existence theorem.

Definition 3.5. An entropy solution of the obstacle problem for $\{f,\psi\}$ is a measurable function $u\in T_0^{1,p(x)}(\Omega)$ such that $u\ge\psi$ a.e. in Ω, and

$$\int_\Omega a(x,\nabla u)\nabla T_k(\varphi-u)dx+\int_\Omega H(x,u,\nabla u)T_k(\varphi-u)dx\ge\int_\Omega fT_k(\varphi-u)dx$$

for all $k\ge 0$ for all $\varphi\in K_\psi\cap L^\infty(\Omega)$.

Theorem 3.6. *Under assumptions* (3.1), (3.2), (3.3), (3.4), (3.5) *and* (3.6) *there exists at least an entropy solution.*

3.3. **Approximate problem.** Let Ω_n be a sequence of compact subsets of Ω such that Ω_n is increasing to Ω as $n\to\infty$. We consider the following sequence of approximate problems

$$\begin{gathered}u_n\in K_\psi\\ \int_\Omega a(x,\nabla u_n)\nabla(u_n-v)dx+\int_\Omega H_n(x,u_n,\nabla u_n)(u_n-v)dx\le\int_\Omega f_n(u_n-v)dx\end{gathered}\tag{3.8}$$

for all $v\in K_\psi$, where f_n are regular functions such that $f_n\in L^\infty(\Omega)$, strongly converge to f in $L^1(\Omega)$ and $\|f_n\|_{L^1(\Omega)}\le\|f\|_{L^1(\Omega)}$ and

$$H_n(x,s,\xi)=\frac{H(x,s,\xi)}{1+\frac{1}{n}|H(x,s,\xi)|}\chi_{\Omega_n}$$

where χ_{Ω_n} is the characteristic function of Ω_n. Note that $|H_n(x,s,\xi)|\le|H(x,s,\xi)|$ and $|H_n(x,s,\xi)|\le n$.

Theorem 3.7. *For fixed n, the approximate problem* (3.8) *has at least one solution.*

Proof. Let $X = K_\psi$, we define the operator $G_n : X \to X^*$ by

$$\langle G_n u, v\rangle = \int_\Omega H_n(x, u, \nabla u) v dx$$

Thanks to Hölder's inequality, for all $u, v \in X$,

$$\begin{aligned}\Big|\int_\Omega H_n(x,u,\nabla u)vdx\Big| &\le (\frac{1}{p_-}+\frac{1}{p'_-})(\int_\Omega |H_n(x,u,\nabla u)|^{p'(x)}dx)^\theta \|v\|_{L^{p(x)}(\Omega)}\\ &\le (\frac{1}{p_-}+\frac{1}{p'_-})n^{\theta p'_+}(\operatorname{meas}(\Omega))^\theta \|v\|_{L^{p(x)}(\Omega)}\end{aligned}$$

with

$$\theta = \begin{cases} 1/p'_- & \text{if } \|H_n(x,u,\nabla u)\|_{L^{p'(x)}(\Omega)} \ge 1\\ 1/p'_+ & \text{if } \|H_n(x,u,\nabla u)\|_{L^{p'(x)}(\Omega)} \le 1\end{cases} \tag{3.9}$$

We deduce that the operator $B_n = A + G_n$ is pseudomonotone (see appendix, Lemma 4.2). On the other hand, we show that B_n is coercive in the following sense: there exists $v_0 \in K_\psi$ such that

$$\frac{\langle B_n v, v - v_0\rangle}{\|v\|_{1,p(x)}} \to +\infty \quad \text{if } \|v\|_{1,p(x)} \to \infty \text{ and } v \in K_\psi.$$

Let $v_0 \in K_\psi$, we use Hölder inequality and the growth condition to have

$$\begin{aligned}\langle Av, v_0\rangle &= \int_\Omega a(x,\nabla v)\nabla v_0 dx\\ &\le C(\frac{1}{p^-}+\frac{1}{p'^-})\Big(\int_\Omega |a(x,\nabla v)|^{p'(x)}\Big)^{\theta'}\|v_0\|_{W_0^{1,p(x)}(\Omega)}\\ &\le C(\frac{1}{p^-}+\frac{1}{p'^-})\|v_0\|_{W_0^{1,p(x)}(\Omega)}\Big(\int_\Omega \beta(K(x)^{p'(x)}+|\nabla v|^{p(x)})\Big)^{\theta'}\\ &\le C_0(C_1+\rho(\nabla v))^{\theta'}\end{aligned}$$

where

$$\theta' = \begin{cases} \frac{1}{p'^-} & \text{if } \|a(x,\nabla v)\|_{L^{p'(x)}(\Omega)} \ge 1\\ \frac{1}{p'^+} & \text{if } \|a(x,\nabla v)\|_{L^{p'(x)}(\Omega)} \le 1\end{cases} \tag{3.10}$$

From (3.3), we have

$$\frac{\langle Av, v\rangle}{\|v\|_{1,p(x)}} - \frac{\langle Av, v_0\rangle}{\|v\|_{1,p(x)}} \ge \frac{1}{\|v\|_{1,p(x)}}(\alpha\rho(\nabla v) - C_0(C_1+\rho(\nabla v))^{\theta'}) \tag{3.11}$$

hence $\frac{\rho(\nabla v)}{\|v\|_{1,p(x)}} \to \infty$ as $\|v\|_{1,p(x)} \to \infty$. Since $\frac{<G_n v, v>}{\|v\|_{1,p(x)}}$ and $\frac{<G_n v, v_0>}{\|v\|_{1,p(x)}}$ are bounded, then we have

$$\frac{\langle B_n v, v - v_0\rangle}{\|v\|_{1,p(x)}} = \frac{\langle Av, v - v_0\rangle}{\|v\|_{1,p(x)}} + \frac{\langle G_n v, v\rangle}{\|v\|_{1,p(x)}} - \frac{\langle G_n v, v_0\rangle}{\|v\|_{1,p(x)}} \to \infty$$

as $\|v\|_{1,p(x)} \to \infty$. Finally B_n is pseudomonotone and coercive. Hence by virtue of [21, Theorem 8.2, chapter 2], the approximate problem (3.8) has at least one solution. $\square$

3.3.1. *A priori estimate.*

Proposition 3.8. *Assume that* (3.1)–(3.6) *hold, and let u_n is a solution of the approximate problem* (3.8)*. Then, there exists a constant C (which does not depend on the n and k) such that*

$$\int_\Omega |\nabla T_k(u_n)|^{p(x)} dx \le Ck \quad \forall\ k > 0.$$

Proof. Let $v = u_n - \eta \exp(G(u_n)) T_k(u_n^+ - \psi^+)$ where $G(s) = \int_0^s \frac{g(t)}{\alpha} dt$ and $\eta \ge 0$, we have $v \in W_0^{1,p(x)}(\Omega)$, and for η small enough we deduce that $v \ge \psi$, and thus v is an admissible test function in (3.8). Then

$$\begin{aligned}
&\int_\Omega a(x, \nabla u_n) \nabla\Big(\exp(G(u_n)) T_k(u_n^+ - \psi^+)\Big) dx \\
&+ \int_\Omega H_n(x, u_n, \nabla u_n) \exp(G(u_n)) T_k(u_n^+ - \psi^+) dx \\
&\le \int_\Omega f_n \exp(G(u_n)) T_k(u_n^+ - \psi^+) dx
\end{aligned}$$

which implies

$$\begin{aligned}
&\int_\Omega a(x, \nabla u_n) \nabla u_n \frac{g(u_n)}{\alpha} \exp(G(u_n)) T_k(u_n^+ - \psi^+) dx \\
&+ \int_\Omega a(x, \nabla u_n) \nabla T_k(u_n^+ - \psi^+) \exp(G(u_n)) dx \\
&\le - \int_\Omega H_n(x, u_n, \nabla u_n) \exp(G(u_n)) T_k(u_n^+ - \psi^+) dx \\
&\quad + \int_\Omega f_n \exp(G(u_n)) T_k(u_n^+ - \psi^+) dx \\
&\le \int_\Omega (f_n + \gamma(x)) \exp(G(u_n)) T_k(u_n^+ - \psi^+) dx \\
&\quad + \int_\Omega g(u_n) |\nabla u_n|^{p(x)} \exp(G(u_n)) T_k(u_n^+ - \psi^+) dx.
\end{aligned}$$

In view of (3.3) and since $\|f_n\|_{L^1(\Omega)} \le \|f\|_{L^1(\Omega)}$, $\gamma \in L^1(\Omega)$ we deduce that

$$\begin{aligned}
&\int_\Omega a(x, \nabla u_n) \nabla T_k(u_n^+ - \psi^+) \exp(G(u_n) dx \\
&\le \int_\Omega f_n \exp(G(u_n)) T_k(u_n^+ - \psi^+)\, dx + \int_\Omega \gamma(x) \exp(G(u_n)) T_k(u_n^+ - \psi^+)\, dx \\
&\le (\|f\|_{L^1(\Omega)} + \|\gamma\|_{L^1(\Omega)}) \exp(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}) k \le C_1 k
\end{aligned}$$

where C_1 is a positive constant. Consequently,

$$\begin{aligned}
&\int_{\{|u_n^+ - \psi^+| \le k\}} a(x, \nabla u_n) \nabla u_n^+ \exp(G(u_n)) dx \\
&\le \int_{\{|u_n^+ - \psi^+| \le k\}} a(x, \nabla u_n) \nabla \psi^+ \exp(G(u_n)) dx + C_1 k
\end{aligned}$$

Thanks to (3.3) and Young's inequality, we deduce that

$$\int_{\{|u_n^+-\psi^+|\le k\}} |\nabla u_n^+|^{p(x)}dx \le C_2 k. \tag{3.12}$$

Since $\{x\in\Omega, |u_n^+|\le k\}\subset\{x\in\Omega, |u_n^+-\psi^+|\le k+\|\psi^+\|_\infty\}$, it follows that

$$\int_\Omega |\nabla T_k(u_n^+)|^{p(x)}dx = \int_{\{|u_n^+|\le k\}} |\nabla u_n^+|^{p(x)} \le \int_{\{|u_n^+-\psi^+|\le k+\|\psi^+\|_\infty\}} |\nabla u_n^+|^{p(x)}dx$$

Moreover, (3.12) implies

$$\int_\Omega |\nabla T_k(u_n^+)|^{p(x)}dx \le C_3 k, \quad \forall k>0, \tag{3.13}$$

where C_3 is a positive constant.

On the other hand, taking $v=u_n+\exp(-G(u_n)T_k(u_n^-)$ as test function in (3.8), we obtain

$$\begin{aligned}
&-\int_\Omega a(x,\nabla u_n)\nabla(\exp(-G(u_n))T_k(u_n^-))dx\\
&-\int_\Omega H_n(x,u_n,\nabla u_n)\exp(-G(u_n))T_k(u_n^-)dx\\
&\le -\int_\Omega f_n\exp(-G(u_n))T_k(u_n^-)dx
\end{aligned}$$

Using (3.4), we have

$$\begin{aligned}
&\int_\Omega a(x,\nabla u_n)\nabla u_n\frac{g(u_n)}{\alpha}\exp(-G(u_n))T_k(u_n^-)dx\\
&-\int_\Omega a(x,\nabla u_n)\nabla T_k(u_n^-)\exp(-G(u_n))dx\\
&\le \int_\Omega \gamma(x)\exp(-G(u_n))T_k(u_n^-)dx + \int_\Omega g(u_n)|\nabla u_n|^{p(x)}\exp(-G(u_n))T_k(u_n^-)dx\\
&\quad -\int_\Omega f_n\exp(-G(u_n))T_k(u_n^-)dx
\end{aligned}$$

By (3.3) and since $\gamma\in L^1(\Omega), \|f_n\|_{L^1(\Omega)}\le\|f\|_{L^1(\Omega)}$ we have

$$\begin{aligned}
&-\int_\Omega a(x,\nabla u_n)\nabla T_k(u_n^-)\exp(-G(u_n))dx\\
&=\int_{\{u_n\le 0\}} a(x,\nabla u_n)\nabla T_k(u_n)\exp(-G(u_n))dx \le C_3 k
\end{aligned}$$

By using again (3.3) we deduce that

$$\int_{\{u_n\le 0\}} |\nabla T_k(u_n)|^{p(x)}dx \le C_4 k, \tag{3.14}$$

where C_4 is a constant positive. Combining (3.13) and (3.14), we conclude

$$\int_\Omega |\nabla T_k(u_n)|^{p(x)}dx \le Ck \quad \textit{with} \quad C>0, \tag{3.15}$$

$$\|\nabla T_k(u_n)\|_{L^{p(x)}(\Omega)} \le (Ck)^{\theta''}, \tag{3.16}$$

with

$$\theta'' = \begin{cases} 1/p^- & \text{if } \|\nabla T_k(u_n)\|_{L^{p(x)}(\Omega)} \geq 1 \\ 1/p^+ & \text{if } \|\nabla T_k(u_n)\|_{L^{p(x)}(\Omega)} \leq 1. \end{cases} \tag{3.17}$$

□

3.3.2. *Strong convergence of truncations.*

Proposition 3.9. *There exist a measurable function u and a subsequence of u_n such that*

$$T_k(u_n) \to T_k(u) \quad \textit{strongly in } W_0^{1,p(x)}(\Omega).$$

The proof of the above proposition is done in two steps.

Step 1. We will show that $(u_n)_n$ is a Cauchy sequence in measure in Ω. According to the Poincaré inequality and (3.16),

$$\begin{aligned} k \text{ meas}\{|u_n| > k\} &= \int_{\{|u_n|>k\}} |T_k(u_n)|dx \leq \int_\Omega |T_k(u_n)|dx \\ &\leq \Big(\frac{1}{p_-} + \frac{1}{p'_-}\Big)\|1\|_{p'(x)}\|T_k(u_n)\|_{p(x)} \\ &\leq \Big(\frac{1}{p_-} + \frac{1}{p'_-}\Big)(\text{meas}(\Omega) + 1)^{1/p'_-}\|T_k(u_n)\|_{p(x)} \leq Ck^{1/\gamma} \end{aligned} \tag{3.18}$$

Thus

$$\text{meas}\{|u_n| > k\} \leq C\frac{1}{k^{1-\frac{1}{\gamma}}} \to 0 \quad \text{as } k \to \infty. \tag{3.19}$$

For all $\delta > 0$, we obtain

$$\begin{aligned} \text{meas}\{|u_n - u_m| > \delta\} &\leq \text{meas}\{|u_n| > k\} + \text{meas}\{|u_m| > k\} \\ &\quad + \text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\}. \end{aligned}$$

In view of(3.19), we deduce that for all $\varepsilon > 0$, there exists $k_0 > 0$ such that

$$\text{meas}\{|u_n| > k\} \leq \frac{\varepsilon}{3} \quad \text{and} \quad \text{meas}\{|u_n| > k\} \leq \frac{\varepsilon}{3} \quad \forall k \geq k_0. \tag{3.20}$$

and by (3.15), we have $(T_k(u_n))_n$ bounded in $W_0^{1,p(x)}(\Omega)$, then there exists a subsequence $(T_k(u_n))_n$ such that $T_k(u_n)$ converges to η_k a.e. in Ω, strongly in $L^{p(x)}(\Omega)$ and weakly in $W_0^{1,p(x)}(\Omega)$ as n tends to ∞. Thus, we can assume that $(T_k(u_n))_n$ is a Cauchy sequence in measure in Ω, then there exists n_0 which depend on δ and ε such that

$$\text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\} \leq \frac{\varepsilon}{3} \quad \forall m, n \geq n_0 \text{ and } k \geq k_0. \tag{3.21}$$

by combining(3.20)*and*(3.21), we obtain for all $\delta > 0$, there exists $\varepsilon > 0$ such that

$$\text{meas}\{|u_n - u_m| > \delta\} \leq \varepsilon \quad \forall n,\ m \geq n_0(k_0, \delta).$$

Then $(u_n)_n$ is a Cauchy sequence in measure in Ω, thus, there exists a subsequence still denoted u_n which converges almost everywhere to some measurable function u, and by Lemma 3.1, we obtain

$$T_k(u_n) \to T_k(u) \text{ strongly in } L^{p(x)}(\Omega) \text{ and weakly in } W_0^{1,p(x)}(\Omega). \tag{3.22}$$

Step 2. We will use the following function of one real variable, which is defined as follows

$$h_j(s) = \begin{cases} 1 & \text{if } |s| \leq j \\ 0 & \text{if } |s| \geq j+1 \\ j+1-|s| & \text{if } j \leq |s| \leq j+1 \end{cases} \tag{3.23}$$

where j is a nonnegative real parameter.

To prove the strong convergence of truncation $T_k(u_n)$, we have to prove the following assertions:

Proposition 3.10. *The subsequence of u_n solution of problem* (3.8) *satisfies, for any $k \geq 0$, Assertion (i):*

$$\lim_{j\to\infty}\lim_{n\to\infty}\int_{\{j\leq|u_n|\leq j+1\}} a(x,\nabla u_n)\nabla u_n dx = 0. \tag{3.24}$$

Assertion(ii):

$$\lim_{j\to\infty}\lim_{n\to\infty}\int_\Omega a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u))(\nabla T_k(u_n) - \nabla T_k(u))h_j(u_n)dx = 0. \tag{3.25}$$

Assertion(iii):

$$\lim_{j\to\infty}\lim_{n\to\infty}\int_\Omega a(x,\nabla T_k(u_n))\nabla T_k(u_n)(1-h_j(u_n))dx = 0. \tag{3.26}$$

Assertion(iv):

$$\lim_{j\to\infty}\lim_{n\to\infty}\int_\Omega \Big(a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u))\Big)(\nabla T_k(u_n) - \nabla T_k(u))dx = 0. \tag{3.27}$$

The proof of the above proposition is shown in the appendix. Thanks to (3.27) and lemma 3.2, we have

$$T_k(u_n) \to T_k(u) \quad \text{strongly in } W_0^{1,p(x)}(\Omega) \text{ as n tends to } +\infty, \tag{3.28}$$

$$\nabla u_n \to \nabla u \quad \text{a.e. in } \Omega. \tag{3.29}$$

3.3.3. *Passing to the limit.*

$$H_n(x,u_n,\nabla u_n) \to H(x,u,\nabla u) \quad \text{strongly in } L^1(\Omega). \tag{3.30}$$

Let $v = u_n + \exp(-G(u_n))\int_{u_n}^0 g(s)\chi_{\{s<-h\}}ds$. Since $v \in W_0^{1,p(x)}(\Omega)$ and $v \geq \psi$ is an admissible test function in (3.8),

$$\begin{aligned}
&\int_\Omega a(x,\nabla u_n)\nabla\Big(-\exp(-G(u_n))\int_{u_n}^0 g(s)\chi_{\{s<-h\}}\Big)\,ds\,dx \\
&+\int_\Omega H(x,u_n,\nabla u_n)(-\exp(-G(u_n))\int_{u_n}^0 g(s)\chi_{\{s<-h\}}ds)dx \\
&\leq \int_\Omega f_n(-\exp(-G(u_n))\int_{u_n}^0 g(s)\chi_{\{s<-h\}}\,ds\,dx.
\end{aligned}$$

This implies

$$\begin{aligned}
&\int_\Omega a(x,\nabla u_n)\nabla u_n\frac{g(u_n)}{\alpha}\exp(-G(u_n))(\int_{u_n}^0 g(s)\chi_{\{s<-h\}}ds)dx \\
&+\int_\Omega a(x,\nabla u_n)\nabla u_n\exp(-G(u_n))g(u_n)\chi_{\{u_n<-h\}}dx
\end{aligned}$$

$$
\begin{aligned}
&\le \int_\Omega \gamma(x)\exp(-G(u_n))\int_{u_n}^0 g(s)\chi_{\{s<-h\}}\,ds\,dx\\
&\quad+\int_\Omega g(u_n)|\nabla u_n|^{p(x)}\exp(-G(u_n))\int_{u_n}^0 g(s)\chi_{\{s<-h\}}\,ds\,dx\\
&\quad-\int_\Omega f_n\exp(-G(u_n))\int_{u_n}^0 g(s)\chi_{\{s<-h\}}\,ds\,dx,
\end{aligned}
$$

using (3.3) and since $\int_{u_n}^0 g(s)\chi_{\{s<-h\}}ds \le \int_{-\infty}^{-h} g(s)ds$, we obtain

$$
\begin{aligned}
&\int_\Omega a(x,\nabla u_n)\nabla u_n\exp(-G(u_n))g(u_n)\chi_{\{u_n<-h\}}dx\\
&\le \exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\int_{-\infty}^{-h} g(s)ds(\|\gamma\|_{L^1(\Omega)}+\|f_n\|_{L^1(\Omega)})\\
&\le \exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\int_{-\infty}^{-h} g(s)ds(\|\gamma\|_{L^1(\Omega)}+\|f\|_{L^1(\Omega)})
\end{aligned}
$$

using again (3.3), we obtain

$$
\int_{\{u_n<-h\}} g(u_n)|\nabla u_n|^{p(x)}dx \le c\int_{-\infty}^{-h} g(s)ds \tag{3.31}
$$

and since $g\in L^1(\mathbb{R})$, we deduce that

$$
\lim_{h\to+\infty}\sup_n\int_{\{u_n<-h\}} g(u_n)|\nabla u_n|^{p(x)}dx=0. \tag{3.32}
$$

On the other hand, let

$$
M=\exp\Big(\frac{\|g\|_{L^1(R)}}{\alpha}\Big)\int_0^{+\infty} g(s)ds
$$

and $h\ge M+\|\psi^+\|_{L^\infty(\Omega)}$. Consider

$$
v=u_n-\exp(G(u_n))\int_0^{u_n} g(s)\chi_{\{s>h\}}ds.
$$

Since $v\in W_0^{1,p(x)}(\Omega)$ and $v\ge\psi$, v is an admissible test function in (3.8). Then, similarly to (3.32), we obtain

$$
\lim_{h\to+\infty}\sup_{n\in N}\int_{\{u_n>h\}} g(u_n)|\nabla u_n|^{p(x)}dx=0. \tag{3.33}
$$

Combining (3.28), (3.32), (3.33) and Vitali's theorem, we conclude (3.30). Now, let $\varphi\in K_\psi\cap L^\infty(\Omega)$ and take $v=u_n-T_k(u_n-\varphi)$ as a test function in (3.8). We obtain

$$
\begin{gathered}
u_n\in K_\psi\\
\int_\Omega a(x,\nabla u_n)\nabla T_k(u_n-\varphi)dx+\int_\Omega H_n(x,u_n,\nabla u_n)T_k(u_n-\varphi)dx\\
\le\int_\Omega f_nT_k(u_n-\varphi)dx\quad\forall\varphi\in K_\psi\cap L^\infty(\Omega),\ \forall k>0.
\end{gathered} \tag{3.34}
$$

Finally, from (3.28) and (3.30), we can pass to the limit in (3.34). This completes the proof of Theorem 3.6.

4. Appendix

Proof of Proposition 2.1. **Assertion (i):** Consider the function

$$v = u_n - \eta \exp(G(u_n))T_1(u_n - T_j(u_n))^+.$$

For j large enough and η small enough, we can deduce that $v \geq \psi$ and since $v \in W_0^{1,p(x)}(\Omega)$, v is a admissible test function in (3.8). Then, we obtain

$$\begin{aligned}
&\int_\Omega a(x, \nabla u_n)\nabla\Big(\exp(G(u_n))T_1(u_n - T_j(u_n))^+\Big)dx \\
&+ \int_\Omega H_n(x, u_n, \nabla u_n) \exp(G(u_n))T_1(u_n - T_j(u_n))^+ dx \\
&\leq \int_\Omega f_n \exp(G(u_n))T_1(u_n - T_j(u_n))^+ dx.
\end{aligned}$$

From the growth conditions (3.3) and (3.4), we have

$$\begin{aligned}
&\int_\Omega a(x, \nabla u_n)\nabla(T_1(u_n - T_j(u_n))^+) \exp(G(u_n))dx \\
&\leq \int_\Omega \gamma(x) \exp(G(u_n))T_1(u_n - T_j(u_n))^+ dx \\
&\quad + \int_\Omega f_n \exp(G(u_n))T_1(u_n - T_j(u_n))^+ dx.
\end{aligned} \tag{4.1}$$

Since f_n converges to f strongly in $L^1(\Omega)$ and $\gamma \in L^1(\Omega)$, by Lebesgue's theorem, the right-hand side approaches zero as $n, j \to \infty$. Therefore, passing to the limit first in n, then in j, we obtain from (4.1)

$$\lim_{j\to\infty} \lim_{n\to\infty} \int_{\{j\leq u_n\leq j+1\}} a(x, \nabla u_n)\nabla u_n dx = 0. \tag{4.2}$$

On the other hand, consider the test function $v = u_n + \exp(-G(u_n))T_1(u_n - T_j(u_n))^-$ in (3.8). Similarly to (4.2), it is easy to see that

$$\lim_{j\to\infty} \lim_{n\to\infty} \int_{\{-j-1\leq u_n\leq -j\}} a(x, \nabla u_n)\nabla u_n dx = 0 \tag{4.3}$$

Finally, by (4.2) and (4.3) we obtain assertion (i).

Assertion (ii): On one hand, let $v = u_n - \eta \exp(G(u_n))(T_k(u_n) - T_k(u))^+ h_j(u_n)$ with h_j is defined in (3.23) and η small enough such that $v \in K_\psi$, then we take v as test function in (3.8), we obtain

$$\begin{aligned}
&\int_\Omega a(x, \nabla u_n)\nabla\Big(\eta \exp(G(u_n))(T_k(u_n) - T_k(u))^+ h_j(u_n)\Big)dx \\
&+ \int_\Omega H_n(x, u_n, \nabla u_n)\Big(\eta \exp(G(u_n))(T_k(u_n) - T_k(u))^+ h_j(u_n)\Big)dx \\
&\leq \int_\Omega f_n \eta \exp(G(u_n))(T_k(u_n) - T_k(u))^+ h_j(u_n)dx.
\end{aligned}$$

Similarly, using (3.3))and(3.4)), we deduce

$$\begin{aligned}
&\int_\Omega a(x, \nabla u_n)\nabla(T_k(u_n) - T_k(u))^+ \exp(G(u_n))h_j(u_n)dx \\
&\leq \int_\Omega \gamma(x) \exp(G(u_n))(T_k(u_n) - T_k(u))^+ h_j(u_n)dx
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\{j\le u_n\le j+1\}} a(x,\nabla u_n)\nabla u_n \exp(G(u_n))(T_k(u_n)-T_k(u))^+dx\\
&+\int_\Omega f_n \exp(G(u_n))(T_k(u_n)-T_k(u))^+h_j(u_n)dx
\end{aligned}
$$

In view of (4.2), the convergence f_n to f in $L^1(\Omega)$ and $\gamma\in L^1(\Omega)$, it is easy to see that

$$
\begin{aligned}
&\lim_{j\to+\infty}\lim_{n\to+\infty}\int_{\{T_k(u_n)-T_k(u)\ge 0\}} a(x,\nabla u_n)\nabla(T_k(u_n)-T_k(u))^+\\
&\quad\times\exp(G(u_n))h_j(u_n)dx\le 0.
\end{aligned}
\tag{4.4}
$$

Moreover, (4.4) becomes

$$
\begin{aligned}
&\lim_{j\to+\infty}\lim_{n\to+\infty}\int_{\{T_k(u_n)-T_k(u)\ge 0,\ |u_n|\le k\}} a(x,\nabla u_n)\nabla(T_k(u_n)-T_k(u))\\
&\quad\times\exp(G(u_n))h_j(u_n)dx\\
&-\lim_{j\to+\infty}\lim_{n\to+\infty}\int_{\{T_k(u_n)-T_k(u)\ge 0,\ |u_n|> k\}} a(x,\nabla u_n)\nabla T_k(u)\\
&\quad\times\exp(G(u_n))h_j(u_n)dx\le 0
\end{aligned}
$$

Since $h_j(u_n)=0$ if $|u_n|>j+1$, we obtain

$$
\begin{aligned}
&\lim_{j\to+\infty}\lim_{n\to+\infty}\int_{\{T_k(u_n)-T_k(u)\ge 0,\ |u_n|> k\}} a(x,\nabla u_n)\nabla T_k(u)\exp(G(u_n))h_j(u_n)dx\\
&=\lim_{j\to+\infty}\lim_{n\to+\infty}\int_{\{T_k(u_n)-T_k(u)\ge 0,\ |u_n|> k\}} a(x,\nabla T_{j+1}(u_n))\nabla T_k(u)\\
&\quad\times\exp(G(u_n))h_j(u_n)dx\\
&=\lim_{j\to+\infty}\int_{\{|u|>k\}} X_j\nabla T_k(u)\exp(G(u))h_j(u)dx=0,
\end{aligned}
$$

where X_j is the limit of $a(x,\nabla T_{j+1}(u_n))$ in $(L^{p'(x)}(\Omega))^N$ as n goes to infinity and $\nabla T_k(u)\chi_{\{|u|>k\}}=0$ a.e. in Ω. Consequently,

$$
\begin{aligned}
&\lim_{j,n\to\infty}\int_{\{T_k(u_n)-T_k(u)\ge 0\}}\Big(a(x,\nabla T_k(u_n))-a(x,\nabla T_k(u))\Big)\\
&\times(\nabla T_k(u_n)-\nabla T_k(u))h_j(u_n)=0.
\end{aligned}
\tag{4.5}
$$

On the other hand, taking $v=u_n+\exp(-G(u_n))(T_k(u_n)-T_k(u))^-h_j(u_n)$ as test function in (3.8) and reasoning as in (4.5) we have

$$
\begin{aligned}
&\int_\Omega a(x,\nabla u_n)\nabla(-\exp(-G(u_n))(T_k(u_n)-T_k(u))^-h_j(u_n))dx\\
&+\int_\Omega H_n(x,u_n,\nabla u_n)(-\exp(-G(u_n))(T_k(u_n)-T_k(u))^-h_j(u_n))dx\\
&\le-\int_\Omega f_n(\exp(-G(u_n))(T_k(u_n)-T_k(u))^-h_j(u_n))dx
\end{aligned}
$$

Similarly to (4.5), it is easy to see that

$$
\lim_{j,n\to\infty}\int_{\{T_k(u_n)-T_k(u)\le 0\}} a(x,\nabla u_n)\nabla(T_k(u_n)-T_k(u))\exp(-G(u_n))h_j(u_n)dx=0.
\tag{4.6}
$$

Combing (4.5) and (4.6) we obtain the desired assertion (ii).

Assertion (iii): Let $v = u_n + \exp(-G(u_n))T_k(u_n)^-(1-h_j(u_n))$ as test function in(3.8). Then we have

$$\begin{aligned}
&\int_\Omega a(x,\nabla u_n)\nabla\Big(-\exp(-G(u_n))T_k(u_n)^-(1-h_j(u_n))\Big)dx\\
&+\int_\Omega H_n(x,u_n,\nabla u_n)\Big(-\exp(-G(u_n))T_k(u_n)^-(1-h_j(u_n))\Big)dx\\
&\le -\int_\Omega f_n\exp(-G(u_n))T_k(u_n)^-(1-h_j(u_n))dx
\end{aligned}$$

Using(3.4) and (3.3), we deduce that

$$\begin{aligned}
&\int_{\{u_n\le 0\}} a(x,\nabla u_n)\nabla T_k(u_n)\exp(-G(u_n))(1-h_j(u_n))dx\\
&\le -\int_{\{-1-j\le u_n\le -j\}} a(x,\nabla u_n)\nabla u_n\exp(-G(u_n))T_k(u_n)^-dx\\
&\quad+\int_\Omega \gamma(x)\exp(-G(u_n))T_k(u_n)^-(1-h_j(u_n))dx\\
&\quad-\int_\Omega f_n\exp(-G(u_n))T_k(u_n)^-(1-h_j(u_n))dx
\end{aligned}$$

In view of (3.24), the second integral tends to zero as n and j approach infinity. By Lebesgue's theorem, it is possible to conclude that the third and the fourth integrals converge to zero as n and j approach infinity. Then

$$\lim_{j,n\to\infty}\int_{\{u_n\le 0\}} a(x,\nabla T_k(u_n))\nabla T_k(u_n)(1-h_j(u_n))dx = 0. \tag{4.7}$$

On the other hand, we take $v = u_n - \eta\exp(G(u_n))T_k(u_n^+-\psi^+)(1-h_j(u_n))$ which is an admissible test function in (3.8), we have

$$\begin{aligned}
&\int_\Omega a(x,\nabla u_n)\nabla\Big(\eta\exp(G(u_n))T_k(u_n^+-\psi^+)(1-h_j(u_n))\Big)dx\\
&+\int_\Omega H_n(x,u_n,\nabla u_n)\Big(\eta\exp(G(u_n))T_k(u_n^+-\psi^+)(1-h_j(u_n))\Big)dx\\
&\le\int_\Omega f_n\Big(\eta\exp(G(u_n))T_k(u_n^+-\psi^+)(1-h_j(u_n))\Big)dx
\end{aligned}$$

Which takes, by using (3.4) and (3.3), the from

$$\begin{aligned}
&\int_\Omega a(x,\nabla u_n)\nabla T_k(u_n^+-\psi^+)\exp(G(u_n))(1-h_j(u_n))dx\\
&\le -\int_{\{j\le u_n\le j+1\}} a(x,\nabla u_n)\nabla u_n\exp(G(u_n))T_k(u_n^+-\psi^+)dx\\
&\quad+\int_{\{-j-1\le u_n\le -j\}} a(x,\nabla u_n)\nabla u_n\exp(G(u_n))T_k(u_n^+-\psi^+)dx \qquad (4.8)\\
&\quad+\int_\Omega \gamma(x)\exp(G(u_n))T_k(u_n^+-\psi^+)(1-h_j(u_n))dx\\
&\quad+\int_\Omega f_n\exp(G(u_n))T_k(u_n^+-\psi^+)(1-h_j(u_n))dx = \varepsilon_1(j,n)
\end{aligned}$$

By (3.24) and Lebesgue's theorem, we conclude that $\varepsilon_1(j,n)$ converges to zero as n and j appraoch infinity. From (4.8), we have

$$\begin{aligned}&\int_{\{|u_n^+-\psi^+|\le k\}} a(x,\nabla u_n)\nabla u_n^+\exp(G(u_n))(1-h_j(u_n))dx\\&\le \int_{\{|u_n^+-\psi^+|\le k\}} a(x,\nabla u_n)\nabla\psi^+\exp(G(u_n)(1-h_j(u_n)))dx+\varepsilon_1(j,n)\end{aligned}$$

Thanks to (3.1) and Young's inequality, it is possible to conclude that

$$\int_{\{|u_n^+-\psi^+|\le k\}} a(x,\nabla u_n)\nabla\psi^+\exp(G(u_n)(1-h_j(u_n)))dx\le\varepsilon_2(j,n),$$

where $\varepsilon_2(j,n)$ converges to zero as n and j go to infinity. Since $\exp(G(u_n))$ is bounded,

$$\int_{\{|u_n^+-\psi^+|\le k\}} a(x,\nabla u_n)\nabla u_n^+(1-h_j(u_n)))dx\le\varepsilon_3(j,n).$$

Since $\{x\in\Omega,\quad |u_n^+|\le k\}\subset\{x\in\Omega,\quad |u_n^+-\psi^+|\le k+\|\psi^+\|_\infty\}$, hence

$$\begin{aligned}&\int_{\{|u_n^+|\le k\}} a(x,\nabla u_n)\nabla u_n(1-h_j(u_n)))dx\\&\le\int_{\{|u_n^+-\psi^+|\le k+\|\psi^+\|_\infty\}} a(x,\nabla u_n)\nabla u_n(1-h_j(u_n)))dx\le\varepsilon_3(j,n)\end{aligned}$$

Which, for all $k\ge 0$, yields

$$\lim_{j,n\to\infty}\int_{\{u_n\ge 0\}} a(x,\nabla T_k(u_n))\nabla T_k(u_n)(1-h_j(u_n))dx=0, \tag{4.9}$$

using (4.7) and (4.9), we conclude (3.26) of assertion (iii).

Assertion(iv): First we have

$$\begin{aligned}&\int_\Omega(a(x,\nabla T_k(u_n))-a(x,\nabla T_k(u)))(\nabla T_k(u_n)-\nabla T_k(u))dx\\&=\int_\Omega(a(x,\nabla T_k(u_n))-a(x,\nabla T_k(u)))(\nabla T_k(u_n)-\nabla T_k(u))h_j(u_n)dx\\&\quad+\int_\Omega(a(x,\nabla T_k(u_n))-a(x,\nabla T_k(u)))(\nabla T_k(u_n)-\nabla T_k(u))(1-h_j(u_n))dx\end{aligned}$$

Thanks to (3.25), the first integral of the right hand side converges to zero as n and j tend to infinity. For the second term, we have

$$\begin{aligned}&\int_\Omega a(x,\nabla T_k(u_n))-a(x,\nabla T_k(u))(\nabla T_k(u_n)-\nabla T_k(u))(1-h_j(u_n))dx\\&=\int_\Omega a(x,\nabla T_k(u_n))\nabla T_k(u_n)(1-h_j(u_n))\,dx\\&\quad-\int_\Omega a(x,\nabla T_k(u_n))\nabla T_k(u)(1-h_j(u_n))\,dx\\&\quad-\int_\Omega a(x,\nabla T_k(u))(\nabla T_k(u_n)-\nabla T_k(u))(1-h_j(u_n))\,dx\end{aligned}$$

By (3.26), the first integral of the right-hand side approaches zero as n and j tend to infinity, and since $a(x,\nabla T_k(u_n))$ in $(L^{p'(x)}(\Omega))^N$ and $\nabla T_k(u)(1-h_j(u_n))$ converges to zero, hence the second integral converges to zero. For the third integral, it converges to zero because $\nabla T_k(u_n) \to \nabla T_k(u)$ weakly in $(L^{p(x)}(\Omega))^N$. Finally we conclude that,

$$\lim_{n\to\infty} \int_\Omega \Big(a(x, \nabla T_k(u_n)) - a(x, \nabla T_k(u))\Big)(\nabla T_k(u_n) - \nabla T_k(u))dx = 0.$$

The proof of Proposition 2.1 is complete. □

Proof of Lemma 3.3. Take at first the case of $F \in C^1(\mathbb{R})$ and $F' \in L^\infty(\mathbb{R})$. Let $u \in W_0^{1,p(x)}(\Omega)$. Since $\overline{C_0^\infty(\Omega)}^{W^{1,p(x)}(\Omega)} = W_0^{1,p(x)}(\Omega)$, there exists $u_n \in C_0^\infty(\Omega)$ such that $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$, then $u_n \to u$ a.e, in Ω and $\nabla u_n \to \nabla u$ a.e. in Ω, then $F(u_n) \to F(u)$ a.e. in Ω. In the the other hand, we have $|F(u_n)| = |F(u_n) - F(0)| \le \|F'\|_\infty |u_n|$, then

$$|F(u_n)|^{p(x)} \le (\|F'\|_\infty + 1)^{p_+} |u_n|^{p(x)},$$
$$|\frac{\partial F(u_n)}{\partial x_i}|^{p(x)} = |F'(u_n)\frac{\partial u_n}{\partial x_i}|^{p(x)} \le M|\frac{\partial u_n}{\partial x_i}|^{p(x)},$$

where $M = (\|F'\|_\infty + 1)^{p_+}$. Then $F(u_n)$ is bounded in $W_0^{1,p(x)}(\Omega)$ and we obtain $F(u_n) \rightharpoonup \nu$ in $W_0^{1,p(x)}(\Omega)$, then $F(u_n) \to \nu$ strongly in $L^{q(x)}(\Omega)$ with $1 < q(x) < p^*(x)$ and $p^*(x) = \frac{N.p(x)}{N-p(x)}$. Since $F(u_n) \to \nu$ a.e. in Ω, we obtain $\nu = F(u) \in W_0^{1,p(x)}(\Omega)$.

Let $F : \mathbb{R} \to \mathbb{R}$ a uniformly Lipschitz function, then $F_n = F * \varphi_n \to F$ uniformly on each compact, where φ_n is a regularizing sequence, then $F_n \in C^1(\mathbb{R})$ and $F_n' \in L^\infty(\mathbb{R})$, and from the first part, we have $F_n(u) \in W_0^{1,p(x)}(\Omega)$ and $F_n(u) \to F(u)$ a.e. in Ω. Since $(F_n(u))_n$ is bounded in $W_0^{1,p(x)}(\Omega)$, then $F_n(u) \rightharpoonup \overline{\nu}$ weakly in $W_0^{1,p(x)}(\Omega)$ a.e. in Ω, then $\overline{\nu} = F(u) \in W_0^{1,p(x)}(\Omega)$. The following Lemma is a direct deduction of the Lemma 3.3. □

Definition 4.1. Let Y be a separable reflexive Banach space. The operator B from Y to its dual Y^* is called of the calculus of variations type, if B is bounded and is of the form

$$B(u) = B(u, u) \tag{4.10}$$

where $(u, v) \to B(u, v)$ is an operator from $Y \times Y$ into Y^* satisfying the following properties:

$$\begin{gathered}\forall u \in Y,\ v \longmapsto B(u, v) \text{ is bounded hemicontinuous from } Y \text{ to } Y^* \\ \text{and } (B(u, u) - B(u, v), u - v) \ge 0.\end{gathered} \tag{4.11}$$

$$\forall v \in Y,\ u \longmapsto B(u, v) \text{ is bounded hemicontinuous from } Y \text{ to } Y^*, \tag{4.12}$$

$$\begin{gathered}\text{if } u_n \rightharpoonup u \text{ weakly in } Y \text{ and if } (B(u_n, u_n) - B(u_n, u), u_n - u) \to 0 \\ \text{then } (B(u_n, v), u_n) \to B(u, v) \text{ weakly in } Y^*,\ \forall v \in Y.\end{gathered} \tag{4.13}$$

$$\begin{gathered}\text{if } u_n \rightharpoonup u \text{ weakly in } Y \text{ and if } B(u_n, v) \rightharpoonup \psi \text{ weakly in } Y^* \\ \text{then } \langle B(u_n, v), u_n\rangle \to \langle \psi, u\rangle.\end{gathered} \tag{4.14}$$

Lemma 4.2. *The operator B_ε is of the calculus of variations type.*

Proof. We put

$$b_1(v,\tilde{w}) = \int_\Omega a(x,\nabla v)\nabla\tilde{w}dx, \quad b_2(u,\tilde{w}) = \int_\Omega H_\varepsilon(x,u,\nabla u)\tilde{w}dx,$$

where

$$H_\varepsilon(x,s,\xi) = \frac{H(x,s,\xi)}{1+\varepsilon|H(x,s,\xi)|}$$

The function $\tilde{w} \mapsto b_1(v,\tilde{w}) + b_2(u,\tilde{w})$ is continuous in $W_0^{1,p(x)}(\Omega)$. Then

$$b_1(v,\tilde{w}) + b_2(u,\tilde{w}) = b(u,v,\tilde{w}) = \langle B_\varepsilon(u,v),\tilde{w}\rangle$$

and $B_\varepsilon(u,v) \in W^{-1,p'(x)}(\Omega)$. We have $B_\varepsilon(u,u) = B_\varepsilon u$ and B_ε is bounded. Then, it is sufficient to check (4.11)-(4.14).

Next we show that (4.11) and (4.12) are true. By (3.3), we have

$$\begin{aligned}\langle B_\varepsilon(u,u) - B_\varepsilon(u,v), u-v\rangle &= b_1(u,u-v) - b_1(v,u-v)\\ &= \int_\Omega (a(x,\nabla u) - a(x,\nabla v))(\nabla u - \nabla v)dx \geq 0.\end{aligned}$$

The operator $v \to B_\varepsilon(u,v)$ is bounded hemi-continuous. We have: $a(x,\nabla(v_1 + \lambda v_2)) \to a(x,\nabla v_1)$ strongly in $L^{p'(x)}(\Omega)$ as $\lambda \to 0$. On the other hand, $(H_\varepsilon(x,u_1 + \lambda u_2, \nabla(u_1+\lambda u_2)))_\lambda$ is bounded in $L^{p'(x)}(\Omega)$ and $H_\varepsilon(x,u_1+\lambda u_2,\nabla(u_1+\lambda u_2)) \to H_\varepsilon(x,u_1,\nabla u_1)$ a.e. in Ω hence Lemma 3.1 gives

$$H_\varepsilon(x,u_1+\lambda u_2,\nabla(u_1+\lambda u_2)) \rightharpoonup H_\varepsilon(x,u_1,\nabla u_1) \quad \text{weakly in } L^{p'(x)}(\Omega) \text{ as } \lambda \to 0.$$

It is easy to see that $b(u,v_1+\lambda v_2,\tilde{w})$ converges to $b(u,v_1,\tilde{w})$ as λ tends to 0, for all $u,v,\tilde{w} \in W_0^{1,p(x)}(\Omega)$ and $b(u_1+\lambda u_2,v,\tilde{w})$ converges to $b(u_1,v,\tilde{w})$ as λ tends to 0, for all $u,v,\tilde{w} \in W_0^{1,p(x)}(\Omega)$, then we deduce (4.12).

Now we prove (4.13). Assume $u_n \to u$ weakly in $W_0^{1,p(x)}(\Omega)$ and $(B(u_n,u_n) - B(u_n,u),u_n-u) \to 0$. Then

$$(B(u_n,u_n) - B(u_n,u),u_n-u) = \int_\Omega (a(x,\nabla u_n) - a(x,\nabla u))\nabla(u_n-u)dx \to 0$$

then, by Lemma 3.2 we have, $u_n \to u$ strongly in $W_0^{1,p(x)}(\Omega)$, which gives $b(u_n,v,\tilde{w})$ converges to $b(u,v,\tilde{w})$ $\forall \tilde{w} \in W_0^{1,p(x)}(\Omega)$ and then $B_\varepsilon(u_n,v)$ converges to $B_\varepsilon(u,v)$ weakly to $W^{-1,p'(x)}(\Omega)$. It remains to prove (4.14), we assume that, u_n converges to u weakly in $W_0^{1,p(x)}(\Omega)$ and that

$$B(u_n,v) \rightharpoonup \psi \quad \textit{weakly} \quad \textit{in} \quad W_0^{1,p(x)}(\Omega). \tag{4.15}$$

Thanks to (3.1), we obtain $a(x,\nabla v) \in (L^{p'(x)}(\Omega))^N$ then,

$$b_1(v,u_n) \to b_1(v,u). \tag{4.16}$$

On other hand, by Hölder inequality,

$$\begin{aligned}|b_2(u_n,u_n-v)| &\leq r_p\Big(\int_\Omega |H_\varepsilon(x,u_n,\nabla u_n)|^{p'(x)}dx\Big)^{\gamma'} \|u_n-u\|_{L^{p(x)}(\Omega)}\\ &\leq C_\varepsilon \|u_n-u\|_{L^{p(x)}(\Omega)} \to 0 \quad \text{as } n\to\infty\,.\end{aligned}$$

Then

$$b_2(u_n,u_n-v) \to 0 \quad \text{as } n \to \infty. \tag{4.17}$$

In view of (4.15) and (4.16), we obtain

$$b_2(u_n,u)=(B_\varepsilon(u_n,v),u)-b_1(u_n,v,u)\to(\psi-u)-b_1(u,v,u)$$

and from (4.17) we obtain $b_2(u_n,u_n)\to(\psi-u)-b_1(v,u)$, then

$$(B_\varepsilon(u_n,v),u_n)=b_1(v,u_n)+b_2(u_n,u_n)\to(\psi,u).$$

Thus, the proof is complete. □

Remark 4.3. Our approach can be applied for a function $p(x)$ satisfying the log-continuity

$$\forall\ x,\ y\in\ \bar{\Omega}\ |x-y|<1\ \Rightarrow\ |p(x)-p(y)|<w(|x-y|), \tag{4.18}$$

where $w:(0,\infty)\mapsto\mathbb{R}$ is a nondecreasing function with $\lim_{\alpha\to0^+}w(\alpha)\ln(\frac{1}{\alpha})<\infty$.

Remark 4.4. Note that in general there is no uniqueness of the entropy solution of (1.1), but if we assume that the condition

$$\Big(H(x,s,\xi)-H(x,r,\eta)\Big)(s-r)>0$$

holds for almost all $x\in\Omega$, for $r,s\ge0$, and for $\xi\neq\eta$, then we are able to prove the following result.

Proposition 4.5. *Let u and v be two entropy solutions of* (1.1)*, where $f\in L^1(\Omega)$ and $f\ge0$, then one has*

$$\lim_{k\to+\infty}k\int_{\{|u-v|\ge k\}}[H(x,u,Du)-H(x,v,Dv)]\operatorname{sign}(u-v)\,dx\le0,$$

and the condition

$$\lim_{k\to+\infty}k\int_{\{|u-v|\ge k\}}[H(x,u,Du)-H(x,v,Dv)]\operatorname{sign}(u-v)\,dx\ge0$$

implies $u=v$.

For a proof of the above propositions, see [10, Proposition 2.2] for $p(.)=p$ constant.

The existence result of an entropy solution (similar to those of the present paper) for a class of nonlinear parabolic unilateral of the type

$$\begin{gathered}
u\ge\psi\quad\text{a.e. in }\Omega\times(0,T),\\
\frac{\partial b(u)}{\partial t}-\operatorname{div}(a(x,Du))+H(x,u,Du)=f\quad\text{in }\Omega\times(0,T),\\
u=0\quad\text{on }\partial\Omega\times(0,T),\\
b(u)(t=0)=b(u_0)\quad\text{in }\Omega,
\end{gathered}\tag{4.19}$$

(where b is a strictly increasing function of u) will be treated by the authors in a forthcoming paper.

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