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BLOW-UP RESULTS FOR SYSTEMS OF NONLINEAR KLEIN-GORDON EQUATIONS WITH ARBITRARY POSITIVE INITIAL ENERGY

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Abstract. The initial boundary value problem for a system of nonlinear Klein-Gordon equations in a bounded domain is considered. We prove the existence of local solutions by using a successive approximation method. Then, we show blow-up results with arbitrary positive initial energy by a concavity method. Also estimates for the lifespan of solutions are given.

1. Introduction

In this article we study the existence and blow-up of local solutions for the system of nonlinear Klein-Gordon equations

$$(u_i)_{tt} - \Delta u_i + m_i^2 u_i + (u_i)_t = f_i(u) \quad \text{in } \Omega \times [0,T),\ i = 1,2, \tag{1.1}$$

with initial conditions

$$u(x,0) = \phi(x), \quad u_t(x,0) = \varphi(x), \quad x \in \Omega, \tag{1.2}$$

and boundary conditions

$$u(x,t) = 0, \quad x \in \partial\Omega \times (0,T), \tag{1.3}$$

where $u = (u_1, u_2)$, $\phi = (\phi_1, \phi_2)$, $\varphi = (\varphi_1, \varphi_2)$, and $\Omega \subset R^N$, $N \geq 1$, is a bounded domain with smooth boundary $\partial\Omega$ so that Divergence theorem can be applied and $T > 0$. Let $\Delta = \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2}$ be the Laplace operator, $m_i \neq 0$ is a real constant and $f_i(u)$ is a nonlinear function of u, $i = 1,2$.

Before stating our results, we first recall the existing results about the initial boundary value problem for a single wave equation

$$u_{tt} - \Delta u + a|u_t|^{m-1}u_t = b|u|^{p-1}u, \tag{1.4}$$

where $a > 0$, $b > 0$, $m \geq 1$, and $p \geq 1$. There are numerous results about the global existence, asymptotic behavior and blow-up of solutions for (1.4). Levine [6] firstly showed that the solutions with negative initial energy blow up in finite time for equation (1.4) with linear damping ($m = 1$). Georgiev and Todorova [4] extended Levine's result to nonlinear case ($m > 1$). They showed that solutions

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with negative initial energy continue to exist globally in time if $m \geq p$ and blow-up in finite time if $p > m$ and the initial energy is sufficiently negative. Later, Levine and Serrin [9] and Levine, Park, and Serrin [8] generalized this result to an abstract setting and to unbounded domains. By combining the arguments in [4] and [9], Vitillaro [19] extended these results to nonlinear damping ($m > 1$) and the solution has positive initial energy. Messaoudi [12] improved the work of [4] without imposing the condition that energy is sufficiently negative. Similar results have also been established by Todorova [16, 18] for different Cauchy problems. For related results on a single wave equation, we refer the reader to [13, 14, 22] and the references therein.

On the other hand, Levine and Todorova [7] proved the local solution blows up in finite time for some initial data with arbitrary high initial energy. Then this result was improved by Todorova and Vitillaro [17]. However, they did not give a sufficient condition for the initial data such that the corresponding solutions blow up in finite time with arbitrary positive initial energy. Recently, Wang [20] discussed the blow-up phenomena for equation (1.4) with $a = 0$. They obtained a sufficient condition of the initial data such that the solution of (1.4) blows up in finite time when the positive initial energy is arbitrarily large.

Now, we return to the initial boundary problem for the system of nonlinear wave equations as follows

$$
\begin{gathered}
(u_i)_{tt} - \Delta u_i + m_i^2 u_i + |(u_i)_t|^{p_i-1}(u_i)_t = f_i(u) \quad \text{in } \Omega \times [0,T),\ i = 1,2, \\
u(x,0) = \phi(x), \quad u_t(x,0) = \varphi(x), \quad x \in \Omega, \\
u(x,t) = 0, \quad x \in \partial\Omega \times (0,T),
\end{gathered} \tag{1.5}
$$

where p_1, $p_2 \geq 1$ and Ω is a bounded domain with smooth boundary. Reed [15] proposed this interesting problem without imposing damping terms $|(u_i)_t|^{p-1}(u_i)_t$ in (1.5) to describe the interaction of scalar fields u_1, u_2 of mass m_1, m_2 respectively. As in the case of a single wave equation, it is worth noting that when the damping terms $|(u_i)_t|^{p_i-1}(u_i)_t$ is absent, then the force term $f_i(u)$ causes finite blow-up of solution for (1.5). In this direction, Wang [21] studied (1.5) with $f_1(u_1,u_2) = a_1|u_2|^{q_2+1}|u_1|^{q_1-1}u_1$ and $f_2(u_1,u_2) = a_2|u_1|^{q_1+1}|u_2|^{q_2-1}u_2$ and obtained that the solutions blow up in finite time with arbitrary positive initial energy. On the other hand, if the source term $f_i(u)$ is removed from the equation, then the damping terms should assure global existence and decay of solutions. However, when both damping and source terms are present, then the analysis of their interaction and their influence on the behavior of solutions becomes more difficult. Agre and Rammaha [1] considered (1.5) with

$$
\begin{gathered}
f_1(u_1,u_2) = (r+1)\Big[a|u_1+u_2|^{r-1}(u_1+u_2) + b|u_1|^{\frac{r-3}{2}}|u_2|^{\frac{r+1}{2}}u_1\Big], \\
f_2(u_1,u_2) = (r+1)\Big[a|u_1+u_2|^{r-1}(u_1+u_2) + b|u_2|^{\frac{r-3}{2}}|u_1|^{\frac{r+1}{2}}u_2\Big],
\end{gathered} \tag{1.6}
$$

where $r \geq 3$, $a > 1$ and $b > 0$. They showed the existence of global solutions if $r \leq \min\{p_1,p_2\}$ and proved the blow-up of solutions if $r > \min\{p_1,p_2\}$ and initial energy is negative. Later, Alves et al [2] improved these results and they obtained several results on the global, uniform decay rates, and blow up of solutions in finite time when the initial energy is nonnegative by involving the Nehari manifold. Recently, Li and Tsai [10] considered a class of nonlinear terms which includes (1.6) in a bounded domain where the global existence and blow-up behavior of solutions without imposing damping terms were discussed. However, on considering the blow-up properties, the initial energy can not be arbitrarily large in that paper. This motivates us to consider the problem of how to obtain the blow-up of solutions when the initial energy is arbitrarily large.

Inspired by these previous works [10, 20, 21], in this present paper, we would like to investigate the local existence and then establish a sufficient condition of the initial data with arbitrarily high initial energy such that the corresponding local solution of the system for the nonlinear Klein-Gordon equations (1.1)-(1.3) blows up in finite time. The method used here are the successive approximation method and the concavity method. In this way, we can extend the result of [20] to a system with linear damping terms and the result of [10] without setting any restriction on upper bound of the initial energy. The paper is organized as follows. In section 2, we first introduced some notations used throughout this paper and then state the local existence Theorem 2.4. In section 3, we prove the main result Theorem 3.4 which shows blow-up properties of solutions with highly positive initial energy.

2. Exitance of local solutions

In this section we shall discuss the existence of local solutions for (1.1)-(1.3) by the method of successive approximations. First we give the notation which will be used throughout the paper. Let $W^{m,p}(\Omega)$ be the usual Sobolev space. Specially, $W^{m,2}(\Omega)$ and $W^{0,p}(\Omega)$ will be marked by $H^m(\Omega)$ and $L^p(\Omega)$, respectively. And we denote $\|\cdot\|_p$ to be L^p-norm for $1 \leq p \leq \infty$. $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{H_0^1} = \|\nabla u\|_2$.

Define

$$H1 = C^1([0,T]; L^2(\Omega)) \cap C^0([0,T]; H_0^1(\Omega)),$$
$$H2 = C^2([0,T]; L^2(\Omega)) \cap C^1([0,T]; H_0^1(\Omega)), \text{ for } T > 0.$$

Now, we make the following assumptions:

(A1) $f_i : R^2 \to \mathbb{R}$ is continuously differentiable such that for each $u = (u_1, u_2) \in H_0^1(\Omega) \times H_0^1(\Omega)$, we have $u_i f_i \in L^1(\Omega)$, $i = 1, 2$ and $F(u) \in L^1(\Omega)$, where

$$F(u) = \int_0^{u_1} f_1(s, u_2)ds + \int_0^{u_2} f_2(0, s)ds.$$

(A2) $f_i(0) = 0$ and for any $\rho > 0$ there exists a constant $k(\rho) > 0$ such that

$$\|f_i(u) - f_i(v)\|_2 \leq k(\rho)\|u - v\|_{H_0^1 \times H_0^1}, \quad i = 1, 2,$$

where $u, v \in H_0^1(\Omega) \times H_0^1(\Omega)$ with $\|u\|_{H_0^1 \times H_0^1}, \|v\|_{H_0^1 \times H_0^1}$.

(A3)
$$\frac{\partial f_1}{\partial u_2} = \frac{\partial f_2}{\partial u_1}.$$

Note that the function of the form $f_1(u_1, u_2) = u_1^{s-1} u_2^s + u_1^p$, $f_2(u_1, u_2) = u_2^{s-1} u_1^s + u_2^q$ satisfy the assumptions (A1)-(A3) where $1 < s, p, q \leq \frac{N}{N-2}$ for $N \geq 3$ or $s, p, q > 1$ for $N = 1, 2$.

Lemma 2.1 (Sobolev-Poincaré [11])**.** *Let $2 \leq p \leq \frac{2N}{N-2}$. then the inequality*

$$\|u\|_p \leq c_s \|\nabla u\|_2, \text{ for } u \in H_0^1(\Omega),$$

holds for some positive constant c_s.

Lemma 2.2 ([3]). *Let $\delta \geq 0$, $T > 0$ and h be a Lipschitizan function over $[0,T)$. Assume that $h(0) \geq 0$ and $h'(t) + \delta h(t) > 0$ for a.e. $t \in (0,T)$. Then $h(t) > 0$ for all $t \in (0,T)$.*

Before proving the existence theorem for nonlinear equations (1.1)-(1.3), we need the existence result for a linear wave equation which is given in [5].

Lemma 2.3. *Assume that $f \in W^{1,1}([0,T]; L^2(\Omega))$ and that $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $u_1 \in H_0^1(\Omega)$, then the linear problem with damping*

$$\begin{gathered} u_{tt} - \Delta u + u_t = f(t,x), \\ u(0) = u_0, \quad u_t(0) = u_1, \quad x \in \Omega, \\ u(x,t) = 0, \quad x \in \partial\Omega \times (0,T), \end{gathered}$$

has a unique solution $u \in H2$.

Theorem 2.4. *Assume that the assumptions* (A1)–(A3) *hold and let $(\phi_1, \phi_2) \in H_0^1(\Omega) \times H_0^1(\Omega)$ and $(\varphi_1, \varphi_2) \in L^2(\Omega) \times L^2(\Omega)$. Then problem* (1.1)–(1.3) *admits a unique solution $(u_{1,}u_2)$ in $H1 \times H1$.*

Proof. Since $H^2(\Omega) \cap H_0^1(\Omega)$ is dense in $H_0^1(\Omega)$ and $H_0^1(\Omega)$ is dense in $L^2(\Omega)$, it suffices to consider problem (1.1)–(1.3) for $\phi_i \in H^2(\Omega) \cap H_0^1(\Omega)$ and $\varphi_i \in H_0^1(\Omega)$, $i = 1, 2$. Let $\{u^m = (u_1^m, u_2^m)\}_{m \geq 1}$ be a sequence of solutions obtained by considering the approximation problem

$$\begin{gathered} (u_i^{m+1})_{tt} - \Delta u_i^{m+1} + (u_i^{m+1})_t = -m_i^2 u_i^m + f_i(u^m), \quad i = 1, \quad 2, \\ u^{m+1}(x,0) = \phi(x), \quad u_t^{m+1}(x,0) = \varphi(x), \quad x \in \Omega, \\ u^{m+1}(x,t) = 0, \quad x \in \partial\Omega \times (0,T), \end{gathered} \tag{2.1}$$

with the initial function $u^1(x,0) = \phi(x)$.

Using Lemma 2.3 and (A1)–(A2), we see that (2.1) has a unique solution $u^m \in H2 \times H2$. In the following, we would like to estimate the solution obtained above. Multiplying by $(u_i^{m+1})_t$ on both sides of (2.1) and then integrating it over Ω, we have

$$\begin{aligned} &\int_\Omega (u_i^{m+1})_t [(u_i^{m+1})_{tt} - \Delta u_i^{m+1} + (u_i^{m+1})_t] dx \\ &= \int_\Omega (u_i^{m+1})_t [-m_i^2 u_i^m + f_i(u^m)] dx. \end{aligned}$$

Using the Divergence theorem and Hölder inequality, we obtain

$$\frac{d}{dt} \|D u_i^{m+1}\|_2 \leq \|m_i^2 u_i^m + f_i(u^m)\|_2, \tag{2.2}$$

where $D \equiv (\partial_t, \nabla_x)$ and $\|Du_i\|_2^2 = \int_\Omega (|(u_i)_t|^2 + |\nabla u_i|^2) dx$. Integrating (2.2) from 0 to t, we obtain

$$\|Du_i^{m+1}\|_2(t) \leq \|Du_i^{m+1}\|_2(0) + \int_0^t \|m_i^2 u_i^m + f_i(u_1^m, u_2^m)\|_2(r) dr. \tag{2.3}$$

For simplicity, we denote

$$\begin{gathered} \beta_i = \|Du_i^{m+1}\|(0) = (\|\varphi_i\|_2^2 + \|\phi_i\|_2^2)^{1/2}, \quad i = 1, 2, \\ \beta = \beta_1 + \beta_2, \end{gathered} \tag{2.4}$$

$$G_{m,i} = m_i^2 \|u_i^m\|_2 + \|f_i(u_1^m, u_2^m)\|_2, \quad i = 1, 2, \ m \geq 1, \tag{2.5}$$

$$H^k(t) = \|Du^k\|_2(t) = (\|Du_1^k\|_2 + \|Du_2^k\|_2)(t), \quad k \geq 1, \tag{2.6}$$

where $Du^k = (Du_1^k, Du_2^k)$. Then using Lemma 2.1 and (A2), we have

$$G_{m,1} + G_{m,2} \leq c\|Du^m\|_2(t), \tag{2.7}$$

here c is some positive constant. It follows from (2.3)-(2.5) that

$$\|Du_i^2\|_2(t) \leq \beta_i + \int_0^t m_i^2\|\phi_i\|_2 + \|f_i(\phi)\|_2 dt \leq \beta_i + G_{1,i}t. \tag{2.8}$$

Thus by (2.6) and (2.8), we obtain

$$H^2(t) \leq \beta + ct\|Du^1\|_2(t). \tag{2.9}$$

Define

$$K_{\infty,\tau}(u^i) = \sup\{\|Du^i\|_2(t) \mid 0 \leq t \leq \tau\}, \tag{2.10}$$

and take a constant $M > \beta$. Then $H^1(t) \leq M$, and hence $K_{\infty,\tau}(u^1) \leq M$. Therefore, from (2.9), we see that

$$H^2(t) \leq \beta + ctM \leq M,$$

provided that $\tau = (M - \beta)/(cM)$. That is, $K_{\infty,\tau}(u^2) \leq M$. Suppose that $K_{\infty,\tau}(u^m) \leq M$, then, using (2.3), (2.5), (2.7) and (2.10), we obtain

$$\begin{aligned} H^{m+1}(t) &\leq \beta + \int_0^t (G_{m,1} + G_{m,2})(r)dr \\ &\leq \beta + \int_0^t c\|Du^m\|_2(r)dr \\ &\leq \beta + cK_{\infty,\tau}(u^m)t \leq M, \quad 0 \leq t \leq \tau. \end{aligned} \tag{2.11}$$

Thus $K_{\infty,\tau}(u^{m+1}) \leq M$. Hence, we have

$$K_{\infty,\tau}(u^m) \leq M, \text{ for all } m \geq 1. \tag{2.12}$$

Below we shall show that $\{u^m\}_{m\geq 1}$ is a Cauchy sequence in $H1 \times H1$. Let $z^m = u^{m+1} - u^m$. From (2.1), for $i = 1, 2$, we have

$$\begin{gathered} (z_i^m)_{tt} - \Delta z_i^m + (z_i^m)_t = -m_i^2 z_i^{m-1} + f_i(u^m) - f_i(u^{m-1}), \\ z^m(x,0) = 0, \quad z_t^m(x,0) = 0, \quad x \in \Omega, \\ z^m(x,t) = 0, \quad x \in \partial\Omega \times (0,T). \end{gathered} \tag{2.13}$$

As in the previous arguments, we obtain

$$\begin{aligned} &\|Dz^m\|_2(t) \\ &\leq \|Dz^m\|_2(0) + \sum_{i=1}^2 \int_0^t (m_i^2\|z_i^{m-1}\|_2 + \|f_i(u^m) - f_i(u^{m-1})\|_2)dr. \end{aligned} \tag{2.14}$$

From (2.13), we obtain $\|Dz^m\|_2(0) = 0$. Then, by (2.12), Lemma 2.1 and (A2), we have

$$\|Dz^m\|_2(t) \leq L\int_0^t \|Dz^{m-1}\|_2(r)dr, \quad 0 \leq t \leq \tau,$$

where L is a constant depending on m_1, m_2 and Sobolev constant. Thus by induction, we obtain

$$K_{\infty,\tau}(z^m) \leq L\tau K_{\infty,\tau}(z^{m-1}) \leq \dots \leq (L\tau)^{m-1}K_{\infty,\tau}(z^1). \tag{2.15}$$

Therefore, for any positive integer p and $L\tau \in (0,1)$, we see that

$$\begin{aligned} K_{\infty,\tau}(u^{m+p}-u^m) &\le ((L\tau)^{m+p-2}+\dots+(L\tau)^{m-1})K_{\infty,\tau}(u^2-u^1)\\ &\le \frac{(L\tau)^{m-1}}{1-L\tau}K_{\infty,\tau}(u^2-u^1)\to 0 \quad \text{as } m\to\infty. \end{aligned}$$

Hence, the Cauchy sequence $\{u^m\}_{m\ge 1}$ converges in $H1\times H1$ and the limit function $u=\lim_{m\to\infty}u^m$ in $H1\times H1$ is a solution defined on $[0,\tau)$ for problem (1.1)–(1.3).

Uniqueness. Let u and $\widehat{u}$ be two solutions defined on $[0,T)$ of problem (1.1)-(1.3). Set $w=u-\widehat{u}$. From (1.1), we have

$$\begin{gathered} (w_i)_{tt}-\Delta w_i+(w_i)_t=-m_i^2w_i+f_i(u)-f_i(\widehat{u}),\quad i=1,2\\ w(x,0)=0,\quad w_t(x,0)=0,\quad x\in\Omega,\\ w(x,t)=0,\quad x\in\partial\Omega\times(0,T). \end{gathered}$$

Similar to (2.14), we obtain

$$\|Dw\|_2^2(t)\le \|Dw\|_2^2(0)+c\int_0^t\|Dw\|_2^2(r)dr.$$

The Gronwall's inequality implies

$$\|Dw\|_2^2(t)=0,\ \text{for } 0\le t<T.$$

Therefore, we have $u=\widehat{u}$. □

3. Blow-up property

In this section, we shall investigate blow-up phenomena of solutions of system (1.1)-(1.3) with $m_1=m_2=1$. For this purpose, we further make the following assumption:

(A4) there exists a positive constant $\delta>0$ such that

$$u_1f_1(u)+u_2f_2(u)\ge(2+4\delta)F(u),\quad \text{for all } u_1,u_2\in\mathbb{R},$$

where $F(u)$ is given in (A1).

Definition. A solution $(u_1(t),u_2(t))$ of (1.1)-(1.3) is said to blow up if there exists a finite time T such that

$$\lim_{t\to T^-}(\|u_1(t)\|_2^2+\|u_2(t)\|_2^2)=\infty. \tag{3.1}$$

Let $(u_1(t),u_2(t))$ be the solution of (1.1)-(1.3), we define the energy function

$$E(t)=\frac12\sum_{i=1}^2[\|(u_i)_t\|_2^2+\|\nabla u_i\|_2^2+\|u_i\|_2^2]-\int_\Omega F(u)dx,\quad t\ge 0 \tag{3.2}$$

and

$$I(u(t))\equiv I(t)=\sum_{i=1}^2[\|\nabla u_i\|_2^2+\|u_i\|_2^2]-\int_\Omega\sum_{i=1}^2u_if_i(u)dx. \tag{3.3}$$

Lemma 3.1. *Let u be a solution of (1.1)-(1.3). Then $E(t)$ is a nonincreasing function and*

$$E(t)=E(0)-\int_0^t\sum_{i=1}^2\|(u_i)_t\|_2^2dt. \tag{3.4}$$

Proof. By differentiating (3.2) and using (1.1)-(1.3), (A1) and (A3), we obtain

$$\frac{dE(t)}{dt} = -\sum_{i=1}^{2} \|(u_i)_t\|_2^2.$$

Thus, the result of Lemma 3.1 follows. $\square$

Lemma 3.2. *Assume* (A4) *and that* $(\phi_1, \varphi_1), (\phi_2, \varphi_2) \in H_0^1(\Omega) \times L^2(\Omega)$ *satisfy* $E(0) > 0$, $I(0) < 0$,

$$\|\phi_1\|_2^2 + \|\phi_2\|_2^2 > \frac{1+2\delta}{\delta} E(0), \tag{3.5}$$

$$\int_\Omega (\phi_1\varphi_1 + \phi_2\varphi_2) dx > 0. \tag{3.6}$$

Then

$$\|u_1(t)\|_2^2 + \|u_2(t)\|_2^2 > \frac{1+2\delta}{\delta} E(0) \quad \textit{and} \quad I(t) < 0,$$

for all $t \in [0, T)$.

Proof. First, we prove that $I(t) < 0$, for all $t \in [0, T)$. Suppose not, then there exists $T^* > 0$ such that $T^* = \min\{t \in [0, T); I(t) = 0\}$. We define

$$G(t) = \int_\Omega (u_1^2(x,t) + u_2^2(x,t)) dx.$$

Using (1.1), we have

$$G'(t) = 2\int_\Omega \sum_{i=1}^{2} u_i(u_i)_t dx,$$

$$G''(t) = 2\int_\Omega \sum_{i=1}^{2} ((u_i)_t^2 - |\nabla u_i|^2 - u_i^2 + u_i f_i(u)) dx - 2\int_\Omega \sum_{i=1}^{2} u_i(u_i)_t dx.$$

Then, from (3.3) it follows that

$$G''(t) + G'(t) = 2[\sum_{i=1}^{2} \int_\Omega (u_i)_t^2 dx - I(t)] > 0, \tag{3.7}$$

for all $t \in [0, T^*)$. By Lemma 2.2 and (3.6), we obtain $G'(t) > 0$, for all $t \in [0, T^*)$. This implies $G(t)$ is strictly increasing on $[0, T^*)$. Thus, from (3.5), we have

$$G(t) > G(0) > \frac{1+2\delta}{\delta} E(0),$$

for all $t \in (0, T^*)$. From the continuity of $u(t)$ at $t = T^*$, we see that

$$G(T^*) = \sum_{i=1}^{2} \|u_i(T^*)\|_2^2 > \frac{1+2\delta}{\delta} E(0). \tag{3.8}$$

On the other hand, from (3.2) and Lemma 3.1, we have

$$\begin{aligned} &\sum_{i=1}^{2} \big(\|\nabla u_i(T^*)\|_2^2 + \|u_i(T^*)\|_2^2\big) - 2\int_\Omega F(u_1(T^*), u_2(T^*)) dx \\ &\le 2E(T^*) \le 2E(0). \end{aligned} \tag{3.9}$$

Noting that from the assumption $I(T^*) = 0$ and (A4) give us

$$\sum_{i=1}^{2}\big(\|\nabla u_i(T^*)\|_2^2 + \|u_i(T^*)\|_2^2\big) \geq (2+4\delta)\int_\Omega F(u_1(T^*), u_2(T^*))dx, \tag{3.10}$$

which together with (3.9) implies

$$\sum_{i=1}^{2}\big(\|\nabla u_i(T^*)\|_2^2 + \|u_i(T^*)\|_2^2\big) \leq \frac{1+2\delta}{\delta}E(0).$$

It is a contradiction to (3.8). Hence, $I(t) < 0$, for all $t \in [0,T)$. Therefore, following the same arguments as above, we deduce that $G(t)$ is strictly increasing on $[0,T)$ and

$$\|u_1(t)\|_2^2 + \|u_2(t)\|_2^2 > \frac{1+2\delta}{\delta}E(0),$$

for all $t \in [0,T)$. □

Now, let

$$a(t) = \sum_{i=1}^{2}\Big(\int_\Omega u_i^2 dx + \int_0^t \|u_i\|_2^2 dt\Big), \quad t \geq 0. \tag{3.11}$$

We need the following lemma to derive our result.

Lemma 3.3. . *Assume that* (A1), (A3) (A4) *hold. Then*

$$a''(t) \geq 4(\delta+1)\int_\Omega \sum_{i=1}^{2}(u_i)_t^2 dx + (4+8\delta)\int_0^t \sum_{i=1}^{2}\|(u_i)_t\|_2^2 dt. \tag{3.12}$$

Proof. Form (3.11) and using (1.1), we have

$$a'(t) = \sum_{i=1}^{2}\Big(\int_\Omega 2u_i(u_i)_t dx + \|u_i\|_2^2\Big), \tag{3.13}$$

and

$$a''(t) = 2\sum_{i=1}^{2}\Big(\int_\Omega (u_i)_t^2 dx - \|\nabla u_i\|_2^2 - \|u_i\|_2^2\Big) + 2\int_\Omega \sum_{i=1}^{2} u_i f_i(u) dx. \tag{3.14}$$

Employing (3.2) , (3.4) and (A4), we obtain

$$\begin{aligned}
a''(t) &= 4\int_\Omega \sum_{i=1}^{2}(u_i)_t^2 dx - 4E(t) + 2\int_\Omega (u_1 f_1(u) + u_2 f_2(u) - 2F(u))dx \\
&\geq 4\int_\Omega \sum_{i=1}^{2}(u_i)_t^2 dx - 4E(0) + 4\int_0^t \sum_{i=1}^{2}\|(u_i)_t\|_2^2 dt + 8\delta\int_\Omega F(u)dx.
\end{aligned}$$

Then, using (3.2) and (3.4) again, we see that

$$\begin{aligned}
a''(t) &\geq 4(1+\delta)\int_\Omega \sum_{i=1}^{2}(u_i)_t^2 dx + 4\delta\sum_{i=1}^{2}\|\nabla u_i\|_2^2 + 4\delta\Big(\sum_{i=1}^{2}\|u_i\|_2^2 - \frac{1+2\delta}{\delta}E(0)\Big) \\
&\quad + 4(1+2\delta)\int_0^t \sum_{i=1}^{2}\|(u_i)_t\|_2^2 dt.
\end{aligned}$$

Therefore, from Lemma 3.2, we obtain (3.12). □

Now, we are in a position to state and prove our main result.

Theorem 3.4. *Assume that* (A1)-(A4) *hold. Also assume that* (ϕ_1,φ_1), $(\phi_2,\varphi_2)\in H_0^1(\Omega)\times L^2(\Omega)$ *satisfy the assumptions of Lemma 3.2. Then the local solution* $(u_1(t), u_2(t))$ *of* (1.1)-(1.3) *blows up at finite time* T^* *in the sense of* (3.1). *Moreover, if*

$$2\delta\int_\Omega(\phi_1\varphi_1+\phi_2\varphi_2)dx>\|\phi_1\|_2^2+\|\phi_2\|_2^2,$$

then the finite time T^* *is estimated by*

$$T^*\le\frac{\|\phi_1\|_2^2+\|\phi_2\|_2^2}{2\delta\int_\Omega(\phi_1\varphi_1+\phi_2\varphi_2)dx-(\|\phi_1\|_2^2+\|\phi_2\|_2^2)}. \tag{3.15}$$

Proof. We first note that

$$2\int_0^t\int_\Omega u_i(u_i)_t\,dx\,dt=\|u_i\|_2^2-\|\phi_i\|_2^2. \tag{3.16}$$

By Hölder inequality and Young's inequality,from (3.16) we have

$$\|u_i\|_2^2\le\|\phi_i\|_2^2+\int_0^t\|u_i\|_2^2dt+\int_0^t\|(u_i)_t\|_2^2dt,\quad i=1,2. \tag{3.17}$$

Next, we will find the estimate for the life span of $a(t)$. Let

$$J(t)=\big[a(t)+(T_1-t)\sum_{i=1}^2\|\phi_i\|_2^2\big]^{-\delta},\quad\text{for } t\in[0,T_1], \tag{3.18}$$

where $T_1>0$ is a certain constant which will be specified later. Then we have

$$J'(t)=-\delta J(t)^{1+\frac1\delta}(a'(t)-\sum_{i=1}^2\|\phi_i\|_2^2), \tag{3.19}$$

$$J''(t)=-\delta J(t)^{1+\frac2\delta}V(t), \tag{3.20}$$

where

$$V(t)=a''(t)\big[a(t)+(T_1-t)\sum_{i=1}^2\|\phi_i\|_2^2\big]-(1+\delta)\Big(a'(t)-\sum_{i=1}^2\|\phi_i\|_2^2\Big)^2. \tag{3.21}$$

For simplicity of calculation, for $i=1,2$, we denote

$$P_i=\int_\Omega u_i^2dx,\quad Q_i=\int_0^t\|u_i\|_2^2dt,\quad R_i=\int_\Omega(u_i)_t^2dx,\quad S_i=\int_0^t\|(u_i)_t\|_2^2dt.$$

From (3.13) (3.16), and Hölder inequality, we obtain

$$\begin{aligned}a'(t)&=\sum_{i=1}^2\Big(\int_\Omega 2u_i(u_i)_tdx+\|\phi_i\|_2^2\Big)+2\sum_{i=1}^2\int_0^t\int_\Omega u_i(u_i)_t\,dx\,dt\\&\le 2(\sqrt{R_1P_1}+\sqrt{Q_1S_1}+\sqrt{R_2P_2}+\sqrt{Q_2S_2})+\sum_{i=1}^2\|\phi_i\|_2^2.\end{aligned} \tag{3.22}$$

By (3.12), we have

$$a''(t)\ge 4(1+\delta)(R_1+S_1+R_2+S_2). \tag{3.23}$$

Thus, from (3.22), (3.23), (3.21) and (3.18), we obtain

$$V(t)\ge[4(1+\delta)(R_1+S_1+R_2+S_2)]J(t)^{-1/\delta}$$

$$- 4(1+\delta)(\sqrt{R_1P_1}+\sqrt{Q_1S_1}+\sqrt{R_2P_2}+\sqrt{Q_2S_2})^2.$$

Further, by $(3.18) and (3.11)$, we deduce that

$$V(t) \geq 4(1+\delta)\big[(R_1+S_1+R_2+S_2)(T_1-t)\sum_{i=1}^{2}\|\phi_i\|_2^2+\Theta(t)\big],$$

where

$$\begin{aligned}\Theta(t) &= (R_1+S_1+R_2+S_2)(P_1+Q_1+P_2+Q_2)\\ &\quad - (\sqrt{R_1P_1}+\sqrt{Q_1S_1}+\sqrt{R_2P_2}+\sqrt{Q_2S_2})^2.\end{aligned}$$

By Schwartz inequality, $\Theta(t)$ is nonnegative. Hence, we have

$$V(t) \geq 0, \quad \text{for } t \geq 0. \tag{3.24}$$

Therefore by (3.20) and (3.24), we obtain $J''(t) \leq 0$ for $t \geq 0$, and then

$$J(t) \leq J(0) + J'(0)t, \quad \text{for } t \geq 0. \tag{3.25}$$

Also, we note that

$$J(0) > 0 \quad \text{and} \quad J'(0) < 0$$

due to (3.18), (3.19) and (3.6). Hence, if we choose $T_1 \geq -J(0)/J'(0)$, from (3.25), there exists a finite time $T^* \leq T_1$ such that

$$\lim_{t\to T^{*-}} J(t) = 0.$$

Then, it follows from the definition on $J(t)$ by (3.18) that

$$\lim_{t\to T^{*-}} \sum_{i=1}^{2}\Big(\|u_i\|_2^2 + \int_0^t \|u_i(s)\|_2^2 ds\Big) = \infty,$$

which implies that

$$\lim_{t\to T^{*-}} \sum_{i=1}^{2}\|u_i\|_2^2 = \infty.$$

Moreover, if

$$2\delta\int_\Omega(\phi_1\varphi_1+\phi_2\varphi_2)dx > \|\phi_1\|_2^2+\|\phi_2\|_2^2,$$

the upper bound T^* can be estimated as

$$T^* \leq \frac{\|\phi_1\|_2^2+\|\phi_2\|_2^2}{2\delta\int_\Omega(\phi_1\varphi_1+\phi_2\varphi_2)dx - (\|\phi_1\|_2^2+\|\phi_2\|_2^2)}.$$

This completes the proof. □

Example 3.5. Consider the system (1.1)-(1.3) with

$$f_1(u_1,u_2) = u_1^2u_2, \quad f_2(u_1,u_2) = u_1u_2^2;$$

that is, we consider the problem

$$\begin{gathered}(u_1)_{tt} - \Delta u_1 + u_1 + (u_1)_t = u_1^2u_2 \quad \text{in } \Omega\times[0,T),\\ (u_2)_{tt} - \Delta u_2 + u_2 + (u_2)_t = u_2^2u_1 \quad \text{in } \Omega\times[0,T),\\ u_1(x,0)=\phi_1, \quad u_2(x,0)=\phi_2, \quad x\in\Omega,\\ (u_1)_t(x,0)=\varphi_1, \quad (u_2)_t(x,0)=\varphi_2, \quad x\in\Omega,\\ u_1(x,t)=0, \quad u_2(x,t)=0, \quad x\in\partial\Omega\times(0,T).\end{gathered} \tag{3.26}$$

By (3.2) and (3.3), we have

$$E(t) = \frac{1}{2}\sum_{i=1}^{2}\big[\|(u_i)_t\|_2^2 + \|\nabla u_i\|_2^2 + \|u_i\|_2^2\big] - \frac{1}{2}\|u_1^2u_2^2\|_2^2,$$

$$I(t) = \sum_{i=1}^{2}\big[\|\nabla u_i\|_2^2 + \|u_i\|_2^2\big] - 2\int_\Omega u_1^2u_2^2dx,$$

and assumption (A4) is satisfied with $\delta = 1/2$. To apply Theorem 3.4, we need to check that the initial data set that satisfies conditions $E(0) > 0$, $I(0) < 0$ and

$$\|\phi_1\|_2^2 + \|\phi_2\|_2^2 > 4E(0), \tag{3.27}$$

by (3.5) is not empty. Setting

$$\begin{gathered}\alpha = \|\phi_1\|_2^2 + \|\phi_2\|_2^2, \quad \beta = \|\nabla\phi_1\|_2^2 + \|\nabla\phi_2\|_2^2, \\ \gamma = \|\phi_1\phi_2\|_2^2, \quad \lambda = \|\varphi_1\|_2^2 + \|\varphi_2\|_2^2.\end{gathered} \tag{3.28}$$

Then the above conditions $E(0) > 0$, $I(0) < 0$ and (3.27) read as follows

$$E(0) = \frac{1}{2}(\alpha+\beta+\lambda) - \frac{1}{2}\gamma > 0, \tag{3.29}$$

$$I(0) = \alpha+\beta-2\gamma < 0, \tag{3.30}$$

$$\alpha > 2(\alpha+\beta+\lambda) - 2\gamma. \tag{3.31}$$

Having (3.30) in mind, we choose ϕ_1 and ϕ_2 such that

$$\alpha+\beta = 2\gamma - \varepsilon\gamma, \tag{3.32}$$

with $0 < \varepsilon < 2$. Thus (3.30) is satisfied. At this moment, we consider two cases: (i) $0 < \varepsilon \le 1$ and (ii) $1 < \varepsilon < 2$.

Case (i) $0 < \varepsilon \le 1$. In this case, we further require ϕ_1 and ϕ_2 to satisfy $\alpha > -2(\varepsilon-1)\gamma$, and then, select λ such that

$$0 < \lambda < \frac{\alpha}{2} + (\varepsilon-1)\gamma. \tag{3.33}$$

Substituting (3.32) into (3.29) and $0 < \varepsilon \le 1$, we see that

$$2E(0) = \alpha+\beta+\lambda-\gamma = \lambda - (\varepsilon-1)\gamma > 0,$$

this implies that (3.29) is achieved. Since $\lambda < \frac{\alpha}{2} + (\varepsilon-1)\gamma$ by (3.33), we deduce that

$$\alpha > 2\lambda - 2(\varepsilon-1)\gamma = 2(\alpha+\beta+\lambda) - 2\gamma,$$

where the last equality is derived due to (3.32). Thus (3.31) is obtained.

Case (ii) $1 < \varepsilon < 2$. In this case, we select λ such that

$$(\varepsilon-1)\gamma < \lambda < \frac{\alpha}{2} + (\varepsilon-1)\gamma. \tag{3.34}$$

Similarly as in part (i), we see that the conditions (3.29)-(3.31) are satisfied. Therefore, from above arguments, the set of all initial data which satisfy the conditions $E(0) > 0$, $I(0) < 0$ and (3.27) is not empty.

Furthermore, although $\|\phi_1\|_2^2 + \|\phi_2\|_2^2 > 4E(0)$ gives an upper bound of the initial energy $E(0)$. $E(0)$ can be chosen to be arbitrary positive provided that $\alpha = \|\phi_1\|_2^2 + \|\phi_2\|_2^2$ is large enough and β, γ can be also larger accordingly to make sure (3.29)-(3.31) is still satisfied.

Next, we give an example to illustrate the above discussion is workable. Consider the problem (3.26) with $\Omega = (0,4)$,

$$\phi_1(x) = \begin{cases} x, & 0 < x < 1, \\ x^2, & 1 \le x < 3, \\ -9x+36, & 3 \le x < 4; \end{cases} \quad \phi_2(x) = \begin{cases} 3x, & 0 < x < 1, \\ 3, & 1 \le x < 3, \\ -3x+12, & 3 \le x < 4. \end{cases}$$

Then, from (3.28) and (3.32), we have the following data

$$\alpha = \|\phi_1\|_2^2 + \|\phi_2\|_2^2 = 99.73, \quad \beta = \|\nabla\phi_1\|_2^2 + \|\nabla\phi_2\|_2^2 = 134.67$$
$$\gamma = \|\phi_1\phi_2\|_2^2 = 583.2, \quad \varepsilon = 1.598.$$

Now, based on (3.34), choose λ such that

$$348.8 = (\varepsilon - 1)\gamma < \lambda = \|\varphi_1\|_2^2 + \|\varphi_2\|_2^2 < \frac{\alpha}{2} + (\varepsilon - 1)\gamma = 398.65.$$

Then

$$E(0) = \frac{1}{2}(\alpha+\beta+\lambda) - \frac{1}{2}\gamma = \frac{1}{2}(\lambda - 348.8) > 0,$$
$$I(0) = \alpha + \beta - 2\gamma = -932 < 0,$$
$$2(\alpha+\beta+\lambda) - 2\gamma = 2(\lambda - 348.8) < \alpha.$$

Thus Theorem 3.4 is applicable.

Example 3.6. Consider the system (1.1)-(1.3) in $\mathbb{R}^3$ with

$$f_1(u_1,u_2) = 4\lambda(u_1+\alpha u_2)^3 + 2\beta u_1 u_2^2, \quad f_2(u_1,u_2) = 4\alpha\lambda(u_1+\alpha u_2)^3 + 2\beta u_1^2 u_2.$$

Assume that $\lambda > 0$, $\beta > 0$ and α is any real number. Now we have

$$F(u_1,u_2) = \lambda(u_1+\alpha u_2)^4 + 2\beta u_1^2 u_2^2.$$

We see that (A4) is satisfied if $0 < \delta \le 1/2$. Thus Theorem 3.4 is applicable.

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