Electronic Journal of Differential Equations, Vol. 2013 (2013), No. 105, pp. 1–21.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE AND PERMANENCE OF ALMOST PERIODIC SOLUTIONS FOR LESLIE-GOWER PREDATOR-PREY MODEL WITH VARIABLE DELAYS

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Abstract. By constructing a suitable Lyapunov functional and using almost periodic functional hull theory, we study the almost periodic dynamic behavior of a discrete Leslie-Gower predator-prey model with constant and variable delays. Based on the permanence result, sufficient conditions are established for the existence and uniqueness of globally attractive almost periodic solution. A example and a numerical simulation are given to illustrate the our results.

1. Introduction

Leslie [12, 13] introduced a predator-prey model where the "carrying capacity" of the predator's environment is proportional to the number of prey. Leslie stresses the fact that there are upper limits to the rates of increase of both prey and predator, which are not recognized in the Lotka-Volterra model. These upper limits can be approached under favorable conditions: for the predator, when the number of prey per predator is large; for the prey, when the number of predators (and perhaps the number of prey also) is small. In the case of continuous time, these considerations lead to the model

$$\begin{aligned} x_1' &= x_1\big(r_1 - b_1x_1 - a_1x_2\big),\\ x_2' &= x_2\big(r_2 - a_2\frac{x_2}{x_1}\big), \end{aligned} \tag{1.1}$$

which are known as Leslie-Gower predator-prey model [20]. System (1.1) is one of the simplest having maximum growth rates which each population approaches under favorable conditions.

It is well known that time delays of one type or another have been incorporated into mathematical models of population dynamics due to maturation time, capturing time or other reasons. In general, delay differential equations exhibit much more complicated dynamics than ordinary differential equations since a time delay could cause a stable equilibrium to become unstable and cause the populations to fluctuate. We refer to the monographs of Cushing [4], Gopalsamy [7], Kuang [11] for general delayed biological systems and to Beretta and Kuang [1, 2], Faria and Magalhaes [6], Gopalsamy [8, 9], May [19], Song and Wei [23], Xiao and Ruan [24], Liu

2000 *Mathematics Subject Classification.* 34K14, 34K20, 39A11.
Key words and phrases. Variable delay; permanence; almost periodicity; Leslie-Gower model.

Submitted July 23, 2012. Published April 24, 2013.

and Yuan [17] and the references cited therein for studies on delayed predator-prey systems. Some scholars have explored the dynamics of the following Leslie-Gower models with constant delays [2, 18, 25, 26, 27]:

$$\begin{gathered} x_1'(t) = x_1(t)\big(r_1 - b_1 x_1(t-\tau) - a_1 x_2(t)\big), \\ x_2'(t) = x_2(t)\big(r_2 - a_2\frac{x_2(t)}{x_1(t)}\big); \end{gathered} \tag{1.2}$$

$$\begin{gathered} x_1'(t) = x_1(t)\big(r_1 - b_1 x_1(t-\tau) - a_1 x_2(t)\big), \\ x_2'(t) = x_2(t)\big(r_2 - a_2\frac{x_2(t-\tau)}{x_1(t)}\big); \end{gathered} \tag{1.3}$$

$$\begin{gathered} x_1'(t) = x_1(t)\big(r_1 - b_1 x_1(t) - a_1 x_2(t)\big), \\ x_2'(t) = x_2(t)\big(r_2 - a_2\frac{x_2(t-\tau)}{x_1(t-\tau)}\big); \end{gathered} \tag{1.4}$$

$$\begin{gathered} x_1'(t) = x_1(t)\big(r_1 - b_1 x_1(t) - a_1 x_2(t-\tau_1)\big), \\ x_2'(t) = x_2(t)\big(r_2 - a_2\frac{x_2(t)}{x_1(t-\tau_2)}\big); \end{gathered} \tag{1.5}$$

where τ, τ_1 and τ_2 are nonnegative constants.

Firstly, in the real world, the delays in differential equations of biological phenomena are usually time-varying. Thus, it is worthwhile continuing to discuss the Leslie-Gower predator-prey model with time-varying delays. Secondly, many authors [3, 12, 14, 15, 16, 21, 22, 28, 30] have argued that the discrete time models governed by difference equations are more appropriate than the continuous ones when the populations have nonoverlapping generations. Discrete time models can also provide efficient computational models of continuous models for numerical simulations. Thirdly, Leslie-Gower predator-prey models have not been well studied yet in the sense that most results are for models with constant environment [2, 18, 25, 26, 27]. This means that the models have been assumed to be autonomous, that is, all biological or environmental parameters have been assumed to be constants in time. However, this is rarely the ease in real life, because many biological and environmental parameters do vary in time (e.g., naturally subject to seasonal fluctuations). When this is taken into account, a model must be nonautonomous, which is more difficult to analyze in general. But, in doing so, one should also take advantage of the properties of those varying parameters. For example, one may assume the parameters are periodic or almost periodic for seasonal reasons. Based on the above points, we consider the following discrete Leslie-Gower predator-prey model with pure and variable delays:

$$\begin{gathered} x_1(n+1) = x_1(n)\exp\big\{r_1(n) - b_1(n)x_1(n-\lfloor c_1(n)\rfloor) - a_1(n)x_2(n-\lfloor c_3(n)\rfloor)\big\}, \\ x_2(n+1) = x_2(n)\exp\Big\{r_2(n) - a_2(n)\frac{x_2(n-\lfloor c_2(n)\rfloor)}{x_1(n-\lfloor c_4(n)\rfloor)}\Big\}, \end{gathered} \tag{1.6}$$

where $\{r_i(n)\}$, $\{b_1(n)\}$, $\{a_i(n)\}$ and $\{c_j(n)\}$ are bounded nonnegative almost periodic sequences, $i=1,2$, $j=1,2,3,4$, $\lfloor a\rfloor$ denotes the algebraically largest integer which does not exceed a. Under the assumptions of almost periodicity of the coefficients of (1.6), our purpose of this paper is to establish sufficient conditions for the existence and uniqueness of globally attractive almost periodic solution of (1.6) by constructing a suitable Lyapunov functional and almost periodic functional hull theory. Obviously, systems (1.2)-(1.5) are special cases of (1.6).

For any bounded sequence $\{f(n)\}$ defined on $\mathbb{Z}$, let $f^u = \sup_{n\in\mathbb{Z}}\{f(n)\}$, $f^l = \inf_{n\in\mathbb{Z}}\{f(n)\}$.

Throughout this paper, we assume that

(H1) $0 < r_i^l \le r_i(n) \le r_i^u$, $0 < a_i^l \le a_i(n) \le a_i^u$ and $0 < b_1^l \le b_1(n) \le b_1^u$, $\forall n \in \mathbb{Z}$, $i = 1, 2$.

Let $\bar{c}_i := \sup_{n\in\mathbb{Z}}\lfloor c_i(n)\rfloor$, $\underline{c}_i := \inf_{n\in\mathbb{Z}}\lfloor c_i(n)\rfloor$, $i = 1, 2, 3, 4$, $c_0 = \sum_{i=1}^4 \bar{c}_i$. We consider system (1.6) together with the initial conditions

$$x_i(\theta) = \varphi_i(\theta) \ge 0, \quad \theta \in [-c_0, 0]_{\mathbb{Z}}, \quad \varphi_i(0) > 0, \quad i = 1, 2. \tag{1.7}$$

One can easily show that the solutions of (1.6) with initial conditions (1.7) are defined and remain positive for $n \in \mathbb{Z}^+ := [0, +\infty)_{\mathbb{Z}}$.

The organization of this article is as follows. In Section 2, we give some basic definitions and necessary lemmas which will be used in later sections. In Section 3, permanence of (1.6) is considered. In Section 4, global attractivity of (1.6) is investigated by constructing a suitable Lyapunov functional. In Section 5, some sufficient conditions are established for the existence and uniqueness of almost periodic solution of (1.6) by using almost periodic functional hull theory. An example and numerical simulation are given in Section 6.

2. Preliminaries

Let us state the following definitions and lemmas, which will be useful in proving our main result.

Definition 2.1 ([29]). A sequence $x : \mathbb{Z} \to \mathbb{R}$ is called an *almost periodic sequence* if the ϵ-translation set of x,

$$E\{\epsilon, x\} = \{\tau \in \mathbb{Z} : |x(n+\tau) - x(n)| < \epsilon, \ \forall n \in \mathbb{Z}\}$$

is a relatively dense set in $\mathbb{Z}$ for all $\epsilon > 0$; that is, for any given $\epsilon > 0$, there exists an integer $l(\epsilon) > 0$ such that each interval of length $l(\epsilon)$ contains an integer $\tau \in E\{\epsilon, x\}$ such that

$$|x(n+\tau) - x(n)| < \epsilon, \quad \forall n \in \mathbb{Z}.$$

The value τ is called the ϵ-translation number or ϵ-almost period.

Definition 2.2 ([29]). Let $f : \mathbb{Z} \times \mathbb{D} \to \mathbb{R}$, where $\mathbb{D}$ is an open set in $C := \{\phi : [-\tau, 0]_{\mathbb{Z}} \to \mathbb{R}\}$. $f(n, \phi)$ is said to be almost periodic in n uniformly for $\phi \in \mathbb{D}$, or uniformly almost periodic for short, if for any $\epsilon > 0$ and any compact set $\mathbb{S}$ in $\mathbb{D}$, there exists a positive integer $l(\epsilon, \mathbb{S})$ such that any interval of length $l(\epsilon, \mathbb{S})$ contains a integer τ for which

$$|f(n+\tau, \phi) - f(n, \phi)| < \epsilon, \quad \forall n \in \mathbb{Z}, \ \phi \in \mathbb{S}.$$

The value τ is called the ϵ-translation number of $f(n, \phi)$.

Definition 2.3 ([29]). The hull of f, denoted by $H(f)$, is defined by

$$H(f) = \{g(n, x) : \lim_{k\to\infty} f(n + \tau_k, x) = g(n, x) \text{ uniformly on } \mathbb{Z} \times \mathbb{S}\}$$

for some sequence $\{\tau_k\}$, where $\mathbb{S}$ is any compact set in $\mathbb{D}$.

Definition 2.4. Suppose that (x_1, x_2) is any solution of (1.6). (x_1, x_2) is said to be a strictly positive solution on $\mathbb{Z}$ if for $n \in \mathbb{Z}$,

$$0 < \inf_{n\in\mathbb{Z}} x_i(n) \le \sup_{n\in\mathbb{Z}} x_i(n) < \infty, \quad i = 1, 2.$$

Lemma 2.5 ([29]). *A sequence $\{x(n)\}$ is almost periodic if and only if for any sequence $\{h'_k\} \subset \mathbb{Z}$ there exists a subsequence $\{h_k\} \subset \{h'_k\}$ such that $x(n + h_k)$ converges uniformly on $n \in \mathbb{Z}$ as $k \to +\infty$. Furthermore, the limit sequence is also an almost periodic sequence.*

3. Permanence

In this section, we obtain the following permanence result of (1.6).

Lemma 3.1. *Assume that* (H1) *holds, then every solution (x_1, x_2) of* (1.6) *satisfies*

$$\limsup_{n\to\infty} x_i(n) \le M_i, \quad i = 1, 2,$$

where

$$M_1 := \min\Big((\frac{r_1}{b_1})^u \exp\{r_1^u(\bar{c}_1 + 1)\}, \frac{\exp\{r_1^u(\bar{c}_1 + 1) - 1\}}{b_1^l}\Big),$$
$$M_2 := \min\Big((\frac{r_2}{a_2})^u M_1 \exp\{r_2^u(\bar{c}_2 + 1)\}, \frac{M_1 \exp\{r_1^u(\bar{c}_1 + 1) - 1\}}{a_2^l}\Big).$$

Proof. Let (x_1, x_2) be any positive solution of (1.6) with initial conditions (1.7). From the first equation of (1.6) it follows that

$$x_1(n + 1) \le x_1(n) \exp\{r_1(n)\} \le x_1(n) \exp\{r_1^u\},$$

which yields

$$x_1(n - \lfloor c_1(n)\rfloor) \ge x_1(n) \exp\{-r_1^u \bar{c}_1\},$$

which implies

$$x_1(n + 1) \le x_1(n) \exp\big[r_1(n) - b_1(n) \exp\{-r_1^u \bar{c}_1\}x_1(n)\big]. \tag{3.1}$$

First, we present two cases to prove that

$$\limsup_{n\to\infty} x_1(n) \le M_1.$$

Case I. There exists a $l_0 \in \mathbb{Z}^+$ such that $x_1(l_0 + 1) \ge x_1(l_0)$. Then by (3.1),

$$r_1(l_0) - b_1(l_0) \exp\{-r_1^u \bar{c}_1\}x_1(l_0) \ge 0,$$

which implies

$$x_1(l_0) \le (\frac{r_1}{b_1})^u \exp\{r_1^u \bar{c}_1\} \le M_1.$$

On the one hand, from (3.1),

$$x_1(l_0 + 1) \le x_1(l_0) \exp\{r_1^u\} \le (\frac{r_1}{b_1})^u \exp\{r_1^u(\bar{c}_1 + 1)\}; \tag{3.2}$$

on the other hand, from (3.1),

$$\begin{aligned} x(l_0 + 1) &\le \frac{b_1(l_0) \exp\{-r_1^u \bar{c}_1\}x_1(l_0)}{b_1(l_0) \exp\{-r_1^u \bar{c}_1\}} \exp\big[r_1^u - b_1(l_0) \exp\{-r_1^u \bar{c}_1\}x_1(l_0)\big] \\ &\le \frac{\exp\{r_1^u(\bar{c}_1 + 1) - 1\}}{b_1^l}, \end{aligned} \tag{3.3}$$

here we used
$$\max_{x>0} x\exp\{r_1^u - x\} = \exp\{r_1^u - 1\}.$$
Together with (3.2)-(3.3), we have
$$x_1(l_0+1) \le M_1 := \min\big((\frac{r_1}{b_1})^u \exp\{r_1^u(\bar{c}_1+1)\}, \frac{\exp\{r_1^u(\bar{c}_1+1)-1\}}{b_1^l}\big). \tag{3.4}$$
We claim that
$$x_1(n) \le M_1, \quad \forall n \ge l_0.$$
In fact if there exists an integer $k_0 \ge l_0+2$ such that $x_1(k_0) > M_1$, and letting $_1$ be the least integer between l_0 and k_0 such that $x_1(_1) = \max_{l_0\le n\le k_0}\{x_1(n)\}$, then $_1 \ge l_0+2$ and $x_1(_1) > x_1(_1-1)$, which implies from the argument as that in (3.4) that
$$x_1(_1) \le M_1 < x_1(k_0).$$
This is impossible. This proves the claim.

Case II. $x_1(n) \ge x_1(n+1)$, $\forall n \in \mathbb{Z}^+$. In particular, $\lim_{n\to\infty} x_1(n)$ exists, denoted by $\bar{x}_1$. Taking limit in the first equation of (1.6) gives
$$\lim_{n\to\infty}\big[r_1(n) - b_1(n)x_1(n-\lfloor c_1(n)\rfloor) - a_1(n)x_2(n-\lfloor c_2(n)\rfloor)\big] = 0.$$
Hence $\bar{x}_1 \le (\frac{r_1}{b_1})^u \le M_1$. This proves the claim.

From the two claims above, $\limsup_{n\to\infty} x_1(n) \le M_1$. For arbitrary $\epsilon > 0$, there exists $n_0 \in \mathbb{Z}^+$ such that
$$x_1(n) \le M_1+\epsilon \quad \text{for } n \ge n_0.$$
For $n > n_0 + 2c_0$, from the second equation in (1.6), we have
$$x_2(n+1) \le x_2(n)\exp\big\{r_2(n) - \frac{a_2(n)x_2(n-\lfloor c_2(n)\rfloor)}{M_1+\epsilon}\big\} \le x_2(n)\exp\{r_2^u\},$$
which yields
$$x_2(n-\lfloor c_2(n)\rfloor) \ge x_2(n)\exp\{-r_2^u\bar{c}_2\},$$
which implies
$$x_2(n+1) \le x_2(n)\exp\big[r_2(n) - \frac{a_2(n)\exp\{-r_2^u\bar{c}_2\}x_2(n)}{M_1+\epsilon}\big].$$
Similar to the above argument as x_1, we can easily obtain that
$$\limsup_{n\to\infty} x_2(n) \le M_2 := \min\Big((\frac{r_2}{a_2})^u M_1\exp\{r_2^u(\bar{c}_2+1)\}, \frac{M_1\exp\{r_1^u(\bar{c}_1+1)-1\}}{a_2^l}\Big).$$
This completes the proof. □

Lemma 3.2. *Assume that* (H1) *and the following condition hold:*

(H2) $r_1^l > a_1^u M_2$.

Then every solution (x_1, x_2) *of* (1.6) *satisfies*
$$\liminf_{n\to\infty} x_i(n) \ge m_i, \quad i = 1,2,$$
where
$$\alpha := \exp\big(\big[b_1^u M_1 + a_1^u M_2 - r_1^l\big]\bar{c}_1\big), \quad \beta := \exp\{\big[\frac{a_2^u M_2}{m_1} - r_2^l\big]\bar{c}_2\},$$
$$m_1 := \frac{r_1^l - a_1^u M_2}{b_1^u\alpha}\exp\{r_1^l - a_1^u M_2 - b_1^u\alpha M_1\}, \quad m_2 := \frac{r_2^l m_1}{\beta a_2^u}\exp\{r_2^l - \frac{a_2^u\beta M_2}{m_1}\}.$$

Proof. From the definition of M_1, we obtain

$$r_1^l - b_1^u M_1 - a_1^u M_2 \le r_1^l - b_1^u M_1 \le r_1^l - b_1^u (\frac{r_1}{b_1})^u \le r_1^l - r_1^u \le 0,$$

which implies $\alpha \ge 1$.

By Lemma 3.1 and (H2), for an arbitrary $\epsilon > 0$, there exists $n_1 \in \mathbb{Z}^+$ such that

$$x_i(n) \le M_i + \epsilon, \quad r_1^l > a_1^u(M_2 + \epsilon), \quad \forall n \ge n_1, \ i = 1, 2.$$

For $n > n_1 + c_0$, from the first equation of (1.6), we have

$$x_1(n+1) \ge x_1(n) \exp\{r_1^l - b_1^u(M_1 + \epsilon) - a_1^u(M_2 + \epsilon)\}.$$

So

$$x_1(n - \lfloor c_1(n) \rfloor) \le x_1(n) \exp\left\{[b_1^u(M_1 + \epsilon) + a_1^u(M_2 + \epsilon) - r_1^l]\bar{c}_1\right\} := x_1(n)\alpha(\epsilon),$$

where

$$\alpha(\epsilon) := \exp\left\{[b_1^u(M_1 + \epsilon) + a_1^u(M_2 + \epsilon) - r_1^l]\bar{c}_1\right\} \ge 1.$$

From the first equation of (1.6), we have

$$x_1(n+1) \ge x_1(n) \exp\{r_1^l - a_1^u(M_2 + \epsilon) - b_1^u \alpha(\epsilon) x_1(n)\}, \quad \forall n \ge n_0 + c_0. \tag{3.5}$$

Next, we present two cases to prove that

$$\liminf_{n\to\infty} x_1(n) \ge m_1.$$

Case I. There exists a $l_0 \ge n_0 + c_0$ such that $x_1(l_0 + 1) \le x_1(l_0)$. Then from (3.5),

$$r_1^l - a_1^u(M_2 + \epsilon) - b_1^u \alpha(\epsilon) x_1(l_0) \le 0,$$

which implies

$$x_1(l_0) \ge \frac{r_1^l - a_1^u(M_2 + \epsilon)}{b_1^u \alpha(\epsilon)}.$$

In view of (3.5), we can easily obtain that

$$\begin{aligned} x_1(l_0 + 1) &\ge \frac{r_1^l - a_1^u(M_2 + \epsilon)}{b_1^u \alpha(\epsilon)} \exp\{r_1^l - a_1^u(M_2 + \epsilon) - b_1^u \alpha(\epsilon)(M_1 + \epsilon)\} \\ &:= m_1(\epsilon) \le \frac{r_1^l - a_1^u(M_2 + \epsilon)}{b_1^u \alpha(\epsilon)}. \end{aligned}$$

We claim that

$$x_1(n) \ge m_1(\epsilon) \quad \text{for } n \ge l_0.$$

By way of contradiction, assume that there exists a $p_0 \ge l_0$ such that $x_1(p_0) < m_1(\epsilon)$. Then $p_0 \ge l_0 + 2$. Let $p_1 \ge l_0 + 2$ be the smallest integer such that $x_1(p_0) < m_1(\epsilon)$. Then $x_1(p_1 - 1) > x_1(p_1)$. The above argument produces that $x_1(p_1) \ge m_1(\epsilon)$, a contradiction. This proves the claim.

Case II. We assume that $x_1(n) < x_1(n+1)$, for all $n \ge n_0 + c_0$. Then $\lim_{n\to\infty} x_1(n)$ exists, denoted by $\underline{x}_1$. Taking limit in the first equation of (1.6) gives

$$\lim_{n\to\infty} \left[r_1(n) - b_1(n) x_1(n - \lfloor c_1(n) \rfloor) - a_1(n) x_2(n - \lfloor c_2(n) \rfloor)\right] = 0.$$

Hence $\underline{x}_1 \ge \frac{r_1^l - a_1^u M_2}{b_1^u} \ge m_1(\epsilon)$ and $\lim_{\epsilon\to 0} m_1(\epsilon) = m_1$. This proves the claim.

From the two claims above, $\liminf_{n\to\infty} x_1(n) \ge m_1$. There exists two positive constants ϵ_0 and n_2 such that

$$x_i(n) \le M_i + \epsilon_0, \quad x_1(n) > m_1 - \epsilon_0 > 0, \quad \forall n > n_2 + c_0, \ i = 1, 2.$$

From the second equation of (1.6), we have

$$x_2(n+1) \geq x_2(n)\exp\big\{r_2^l - \frac{a_2^u(M_2+\epsilon_0)}{m_1-\epsilon_0}\big\}, \quad \forall n > n_2 + c_0.$$

So

$$x_2(n - \lfloor c_2(n) \rfloor) \leq x_2(n)\exp\big\{[\frac{a_2^u(M_2+\epsilon_0)}{m_1-\epsilon_0} - r_2^l]\bar{c}_2\big\} := x_2(n)\beta(\epsilon_0)$$

for all $n > n_2 + c_0$, where

$$\beta(\epsilon_0) := \exp\{[\frac{a_2^u(M_2+\epsilon_0)}{m_1-\epsilon_0} - r_2^l]\bar{c}_2\} \geq 1.$$

From the second equation of (1.6), we have

$$x_2(n+1) \geq x_2(n)\exp\big\{r_2^l - \frac{a_2^u\beta(\epsilon_0)x_2(n)}{m_1-\epsilon_0}\big\}, \quad \forall n \geq n_0 + c_0. \tag{3.6}$$

Similar to the above analysis for x_1, we can obtain

$$\liminf_{n\to\infty} x_2(n) \geq m_2 := \frac{r_2^l m_1}{\beta a_2^u}\exp\big\{r_2^l - \frac{a_2^u\beta M_2}{m_1}\big\}.$$

The proof is complete. □

By Lemmas 3.1 and 3.2, we can easily show the following result.

Theorem 3.3. *Assume that* (H1)–(H2) *hold, then every solution* (x_1, x_2) *of* (1.6) *satisfies*

$$m_i \leq \liminf_{n\to\infty} N_i(n) \leq \limsup_{n\to\infty} N_i(n) \leq M_i, \quad i = 1, 2.$$

That is, (1.6) *is permanent.*

4. Global attractivity

In this section, we investigate the global attractivity of (1.6). Define a function $\chi : [0,\infty)_{\mathbb{Z}} \to \{0,1\}$ as follows:

$$\chi(s) := \begin{cases} 0, & \text{if } s = 0, \\ 1, & \text{if } s \in [1,\infty)_{\mathbb{Z}}. \end{cases}$$

Let

$$\mu_1 := \exp\{r_1^u - b_1^l m_1 - a_1^l m_2\}, \quad \mu_2 := \exp\{r_2^u - \frac{a_2^l m_2}{M_1}\},$$
$$\nu_1 := \max\{\mu_1, 1\}, \quad \nu_2 := \max\{\mu_2, 1\},$$
$$\delta_1 := \max\{r_1^u, b_1^u M_1 + a_1^u M_2\}, \quad \delta_2 := \max\{r_2^u, \frac{a_2^u M_2}{m_1}\}.$$

Theorem 4.1. *Assume that* (H1)—(H2) *hold. Suppose further that*

(H3) *there exist two positive constants* λ_1 *and* λ_2 *such that* $\min\{\Theta_1, \Theta_2\} > 0$, *where*

$$\Theta_1 := \lambda_1 \min[b_1^l, \frac{2}{M_1} - b_1^u] - \lambda_1 M_1 \mu_1 (b_1^u)^2 \chi(\bar{c}_1)\bar{c}_1(\bar{c}_1 - \underline{c}_1 + 1) - \lambda_1\nu_1\delta_1 b_1^u \chi(\bar{c}_1)\bar{c}_1$$
$$- \frac{\lambda_2\chi(\bar{c}_2)\bar{c}_2(\bar{c}_4 - \underline{c}_4 + 1)M_2^2\mu_2(a_2^u)^2}{m_1^3} - \frac{\lambda_2 a_2^u(\bar{c}_4 - \underline{c}_4 + 1)M_2}{m_1^2}$$

and

$$\begin{aligned}\Theta_2 :=& \lambda_2 \min[\frac{a_2^l}{M_1}, \frac{2}{M_2} - \frac{a_2^u}{m_1}] - \frac{\lambda_2 a_2^u \chi(\bar{c}_2)\bar{c}_2 \nu_2 \delta_2}{m_1} \\ &- \frac{\lambda_2 \chi(\bar{c}_2)\bar{c}_2(\bar{c}_2 - \underline{c}_2 + 1)M_2\mu_2(a_2^u)^2}{m_1^2} - \lambda_1 M_1 \mu_1 a_1^u b_1^u \chi(\bar{c}_1)\bar{c}_1(\bar{c}_3 - \underline{c}_3 + 1) \\ &- \lambda_1 a_1^u(\bar{c}_3 - \underline{c}_3 + 1).\end{aligned}$$

Then (1.6) *is globally attractive.*

Proof. From condition (H3), there exist small positive constants $\epsilon < \min\{m_1, m_2\}$ and λ such that

$$\begin{aligned}\Theta_1(\epsilon) :=& \lambda_1 \min\big[b_1^l, \frac{2}{M_1+\epsilon} - b_1^u\big] - \lambda_1(M_1+\epsilon)\mu_1(\epsilon)(b_1^u)^2\chi(\bar{c}_1)\bar{c}_1(\bar{c}_1 - \underline{c}_1 + 1) \\ &- \lambda_1\nu_1(\epsilon)\delta_1(\epsilon)b_1^u\chi(\bar{c}_1)\bar{c}_1 - \lambda_2\frac{\chi(\bar{c}_2)\bar{c}_2(\bar{c}_4 - \underline{c}_4 + 1)(M_2+\epsilon)^2\mu_2(\epsilon)(a_2^u)^2}{(m_1-\epsilon)^3} \\ &- \lambda_2\frac{a_2^u(\bar{c}_4 - \underline{c}_4 + 1)(M_2+\epsilon)}{(m_1-\epsilon)^2} > \lambda,\end{aligned}$$

$$\begin{aligned}\Theta_2(\epsilon) :=& \lambda_2 \min\big[\frac{a_2^l}{M_1+\epsilon}, \frac{2}{M_2+\epsilon} - \frac{a_2^u}{m_1-\epsilon}\big] - \lambda_2\frac{a_2^u\chi(\bar{c}_2)\bar{c}_2\nu_2(\epsilon)\delta_2(\epsilon)}{m_1-\epsilon} \\ &- \lambda_2\frac{\chi(\bar{c}_2)\bar{c}_2(\bar{c}_2 - \underline{c}_2 + 1)(M_2+\epsilon)\mu_2(\epsilon)(a_2^u)^2}{(m_1-\epsilon)^2} \\ &- \lambda_1(M_1+\epsilon)\mu_1(\epsilon)a_1^u b_1^u\chi(\bar{c}_1)\bar{c}_1(\bar{c}_3 - \underline{c}_3 + 1) - \lambda_1 a_1^u(\bar{c}_3 - \underline{c}_3 + 1) > \lambda,\end{aligned}$$

where

$$\mu_1(\epsilon) := \exp\{r_1^u - b_1^l(m_1-\epsilon) - a_1^l(m_2-\epsilon)\}, \quad \mu_2(\epsilon) := \exp\{r_2^u - \frac{a_2^l(m_2-\epsilon)}{M_1+\epsilon}\},$$
$$\nu_1(\epsilon) := \max\{\mu_1(\epsilon), 1\}, \quad \nu_2(\epsilon) := \max\{\mu_2(\epsilon), 1\},$$
$$\delta_1(\epsilon) := \max\{r_1^u, b_1^u(M_1+\epsilon) + a_1^u(M_2+\epsilon)\}, \quad \delta_2(\epsilon) := \max\big\{r_2^u, \frac{a_2^u(M_2+\epsilon)}{m_1-\epsilon}\big\}.$$

Suppose that (x_1, x_2) and (y_1, y_2) are two positive solutions of (1.6). By Theorem 3.3, there exists a constant $N_0 > 0$ such that

$$m_i - \epsilon \le x_i(n),\, y_i(n) \le M_i + \epsilon, \quad n \ge N_0,\ i = 1, 2.$$

Let

$$V_{11}(n) = |\ln x_1(n) - \ln y_1(n)|.$$

In view of (1.6), we have

$$\begin{aligned}V_{11}(n+1) &= |\ln x_1(n+1) - \ln y_1(n+1)| \\ &= \big|[\ln x_1(n) - \ln y_1(n)] - b_1(n)[x_1(n - \lfloor c_1(n)\rfloor) - y_1(n - \lfloor c_1(n)\rfloor)] \\ &\quad - a_1(n)[x_2(n - \lfloor c_3(n)\rfloor) - y_2(n - \lfloor c_3(n)\rfloor)]\big| \\ &= \big|[\ln x_1(n) - \ln y_1(n)] - b_1(n)[x_1(n) - y_1(n)] \\ &\quad + b_1(n)\chi(\bar{c}_1)\big([x_1(n) - x_1(n - \lfloor c_1(n)\rfloor)] + [y_1(n) - y_1(n - \lfloor c_1(n)\rfloor)]\big) \\ &\quad - a_1(n)[x_2(n - \lfloor c_3(n)\rfloor) - y_2(n - \lfloor c_3(n)\rfloor)]\big|.\end{aligned} \tag{4.1}$$

Define

$$
\begin{aligned}
P_1(n) &:= r_1(n) - b_1(n)x_1(n - \lfloor c_1(n)\rfloor) - a_1(n)x_2(n - \lfloor c_3(n)\rfloor), \quad \forall n \in \mathbb{Z},\\
Q_1(n) &:= r_1(n) - b_1(n)y_1(n - \lfloor c_1(n)\rfloor) - a_1(n)y_2(n - \lfloor c_3(n)\rfloor), \quad \forall n \in \mathbb{Z}.
\end{aligned}
$$

In view of (1.6), we obtain

$$
\begin{aligned}
&\big|[x_1(n) - x_1(n - \lfloor c_1(n)\rfloor)] + [y_1(n) - y_1(n - \lfloor c_1(n)\rfloor)]\big|\\
&= \big|\sum_{s=n-\lfloor c_1(n)\rfloor}^{n-1} \big[x_1(s+1) - y_1(s+1)\big] - \sum_{s=n-\lfloor c_1(n)\rfloor}^{n-1} \big[x_1(s) - y_1(s)\big]\big|\\
&= \big|\sum_{s=n-\lfloor c_1(n)\rfloor}^{n-1} \big[x_1(s)e^{P_1(s)} - y_1(s)e^{Q_1(s)}\big] - \sum_{s=n-\lfloor c_1(n)\rfloor}^{n-1} \big[x_1(s) - y_1(s)\big]\big|\\
&= \big|\sum_{s=n-\lfloor c_1(n)\rfloor}^{n-1} x_1(s)\big[e^{P_1(s)} - e^{Q_1(s)}\big] + \sum_{s=n-\lfloor c_1(n)\rfloor}^{n-1} \big[x_1(s) - y_1(s)\big]\big[e^{Q_1(s)} - 1\big]\big|\\
&\le \sum_{s=n-\bar{c}_1}^{n-1} x_1(s)\xi_1(s)\big[b_1(s)\big|x_1(s - \lfloor c_1(s)\rfloor) - y_1(s - \lfloor c_1(s)\rfloor)\big|\\
&\quad + a_1(s)\big|x_2(s - \lfloor c_3(s)\rfloor) - y_2(s - \lfloor c_3(s)\rfloor)\big|\big]\\
&\quad + \sum_{s=n-\bar{c}_1}^{n-1} \xi_2(s)\big|r_1(s) - b_1(s)y_1(s - \lfloor c_1(s)\rfloor)\\
&\quad - a_1(s)y_2(s - \lfloor c_3(s)\rfloor)\big|\big|x_1(s) - y_1(s)\big|\\
&\le \sum_{s=n-\bar{c}_1}^{n-1} \sum_{k=\underline{c}_1}^{\bar{c}_1} (M_1+\epsilon)\mu_1(\epsilon)b_1^u\big|x_1(s-k) - y_1(s-k)\big|\\
&\quad + \sum_{s=n-\bar{c}_1}^{n-1} \sum_{k=\underline{c}_3}^{\bar{c}_3} (M_1+\epsilon)\mu_1(\epsilon)a_1^u\big|x_2(s-k) - y_2(s-k)\big|\\
&\quad + \sum_{s=n-\bar{c}_1}^{n-1} \nu_1(\epsilon)\delta_1(\epsilon)\big|x_1(s) - y_1(s)\big|\\
&\le \sum_{s=n-k-\bar{c}_1}^{n-k-1} \sum_{k=\underline{c}_1}^{\bar{c}_1} (M_1+\epsilon)\mu_1(\epsilon)b_1^u\big|x_1(s) - y_1(s)\big|\\
&\quad + \sum_{s=n-k-\bar{c}_1}^{n-k-1} \sum_{k=\underline{c}_3}^{\bar{c}_3} (M_1+\epsilon)\mu_1(\epsilon)a_1^u\big|x_2(s) - y_2(s)\big|\\
&\quad + \sum_{s=n-\bar{c}_1}^{n-1} \nu_1(\epsilon)\delta_1(\epsilon)\big|x_1(s) - y_1(s)\big|,
\end{aligned}
\tag{4.2}
$$

where $\xi_1(s)$ lies between $e^{P_1(s)}$ and $e^{Q_1(s)}$, $\xi_2(s)$ lies between $e^{Q_1(s)}$ and 1, $s = n - \lfloor c_1(n)\rfloor, \dots, n-1$, for$n > N_0 + c_0$.

By (4.1) and (4.2), we have

$$
\begin{aligned}
\Delta V_{11}(n) &= V_{11}(n+1) - V_{11}(n) \\
&\le -|\ln x_1(n) - \ln y_1(n)| + \big|[\ln x_1(n) - \ln y_1(n)] - b_1(n)[x_1(n) - y_1(n)]\big| \\
&\quad + b_1(n)\chi(\bar c_1)\big|[x_1(n) - x_1(n - \lfloor c_1(n)\rfloor)] + [y_1(n) - y_1(n - \lfloor c_1(n)\rfloor)]\big| \\
&\quad + a_1(n)\big|x_2(n - \lfloor c_3(n)\rfloor) - y_2(n - \lfloor c_3(n)\rfloor)\big| \\
&\le -\Big[\frac{1}{\sigma_1(n)} - \Big|\frac{1}{\sigma_1(n)} - b_1(n)\Big|\Big]\big|x_1(n) - y_1(n)\big| \\
&\quad + \sum_{s=n-k-\bar c_1}^{n-k-1} \sum_{k=\underline{c}_1}^{\bar c_1} (M_1+\epsilon)\mu_1(\epsilon)(b_1^u)^2\chi(\bar c_1)\big|x_1(s) - y_1(s)\big| \\
&\quad + \sum_{s=n-k-\bar c_1}^{n-k-1} \sum_{k=\underline{c}_3}^{\bar c_3} (M_1+\epsilon)\mu_1(\epsilon)a_1^u b_1^u\chi(\bar c_1)\big|x_2(s) - y_2(s)\big| \\
&\quad + \sum_{s=n-\bar c_1}^{n-1} \nu_1(\epsilon)\delta_1(\epsilon)b_1^u\chi(\bar c_1)\big|x_1(s) - y_1(s)\big| + \sum_{s=n-\bar c_3}^{n-\underline{c}_3} a_1^u\big|x_2(s) - y_2(s)\big| \\
&\le -\min\Big[b_1^l, \frac{2}{M_1+\epsilon} - b_1^u\Big]\big|x_1(n) - y_1(n)\big| \\
&\quad + \sum_{s=n-k-\bar c_1}^{n-k-1} \sum_{k=\underline{c}_1}^{\bar c_1} (M_1+\epsilon)\mu_1(\epsilon)(b_1^u)^2\chi(\bar c_1)\big|x_1(s) - y_1(s)\big| \\
&\quad + \sum_{s=n-k-\bar c_1}^{n-k-1} \sum_{k=\underline{c}_3}^{\bar c_3} (M_1+\epsilon)\mu_1(\epsilon)a_1^u b_1^u\chi(\bar c_1)\big|x_2(s) - y_2(s)\big| \\
&\quad + \sum_{s=n-\bar c_1}^{n-1} \nu_1(\epsilon)\delta_1(\epsilon)b_1^u\chi(\bar c_1)\big|x_1(s) - y_1(s)\big| + \sum_{s=n-\bar c_3}^{n-\underline{c}_3} a_1^u\big|x_2(s) - y_2(s)\big|,
\end{aligned}
\tag{4.3}
$$

here we used

$$
|x_1(n) - y_1(n)| = \sigma_1(n)|\ln x_1(n) - \ln y_1(n)|,
$$

where $\sigma_1(n)$ lies between $x_1(n)$ and $y_1(n)$, $\forall n > N_0 + c_0$. Let

$$
\begin{gathered}
V_{12}(n) = \sum_{t=0}^{\bar c_1-1} \sum_{s=n-k-\bar c_1+t}^{n-1} \sum_{k=\underline{c}_1}^{\bar c_1} (M_1+\epsilon)\mu_1(\epsilon)(b_1^u)^2\chi(\bar c_1)\big|x_1(s) - y_1(s)\big|, \\
V_{13}(n) = \sum_{t=0}^{\bar c_1-1} \sum_{s=n-k-\bar c_1+t}^{n-1} \sum_{k=\underline{c}_3}^{\bar c_3} (M_1+\epsilon)\mu_1(\epsilon)a_1^u b_1^u\chi(\bar c_1)\big|x_2(s) - y_2(s)\big|, \\
V_{14}(n) = \sum_{t=0}^{\bar c_1-1} \sum_{s=n-\bar c_1+t}^{n-1} \nu_1(\epsilon)\delta_1(\epsilon)b_1^u\chi(\bar c_1)\big|x_1(s) - y_1(s)\big|, \\
V_{15}(n) = \sum_{t=0}^{\bar c_3-\underline{c}_3} \sum_{s=n-\bar c_3+t}^{n-1} a_1^u\big|x_2(s) - y_2(s)\big|.
\end{gathered}
$$

Also we obtain

$$\begin{aligned}\Delta V_{12}(n) &= V_{12}(n+1) - V_{12}(n)\\ &= (M_1+\epsilon)\mu_1(\epsilon)(b_1^u)^2\chi(\bar{c}_1)\bar{c}_1(\bar{c}_1 - \underline{c}_1 + 1)\big|x_1(n) - y_1(n)\big|\\ &\quad - \sum_{s=n-k-\bar{c}_1}^{n-k-1}\sum_{k=\underline{c}_1}^{\bar{c}_1}(M_1+\epsilon)\mu_1(\epsilon)(b_1^u)^2\chi(\bar{c}_1)\big|x_1(s) - y_1(s)\big|,\end{aligned} \tag{4.4}$$

$$\begin{aligned}\Delta V_{13}(n) &= V_{13}(n+1) - V_{13}(n)\\ &= (M_1+\epsilon)\mu_1(\epsilon)a_1^u b_1^u\chi(\bar{c}_1)\bar{c}_1(\bar{c}_3 - \underline{c}_3 + 1)\big|x_2(n) - y_2(n)\big|\\ &\quad - \sum_{s=n-k-\bar{c}_1}^{n-k-1}\sum_{k=\underline{c}_3}^{\bar{c}_3}(M_1+\epsilon)\mu_1(\epsilon)a_1^u b_1^u\chi(\bar{c}_1)\big|x_2(s) - y_2(s)\big|,\end{aligned} \tag{4.5}$$

$$\begin{aligned}\Delta V_{14}(n) &= V_{14}(n+1) - V_{14}(n)\\ &= \nu_1(\epsilon)\delta_1(\epsilon)b_1^u\chi(\bar{c}_1)\bar{c}_1\big|x_1(n) - y_1(n)\big|\\ &\quad - \sum_{s=n-\bar{c}_1}^{n-1}\nu_1(\epsilon)\delta_1(\epsilon)b_1^u\chi(\bar{c}_1)\big|x_1(s) - y_1(s)\big|,\end{aligned} \tag{4.6}$$

$$\begin{aligned}\Delta V_{15}(n) &= V_{15}(n+1) - V_{15}(n)\\ &= a_1^u(\bar{c}_3 - \underline{c}_3 + 1)\big|x_2(n) - y_2(n)\big| - \sum_{s=n-\bar{c}_3}^{n-\underline{c}_3} a_1^u\big|x_2(s) - y_2(s)\big|.\end{aligned} \tag{4.7}$$

Define

$$V_1(n) = V_{11}(n) + V_{12}(n) + V_{13}(n) + V_{14}(n) + V_{15}(n).$$

From (4.3)-(4.7) it follows that

$$\begin{aligned}\Delta V_1(n) \le -\{&\min\big[b_1^l, \frac{2}{M_1+\epsilon} - b_1^u\big] - (M_1+\epsilon)\mu_1(\epsilon)(b_1^u)^2\chi(\bar{c}_1)\bar{c}_1(\bar{c}_1 - \underline{c}_1 + 1)\\ &- \nu_1(\epsilon)\delta_1(\epsilon)b_1^u\chi(\bar{c}_1)\bar{c}_1\}\big|x_1(n) - y_1(n)\big|\\ &+ \{(M_1+\epsilon)\mu_1(\epsilon)a_1^u b_1^u\chi(\bar{c}_1)\bar{c}_1(\bar{c}_3 - \underline{c}_3 + 1)\\ &+ a_1^u(\bar{c}_3 - \underline{c}_3 + 1)\}\big|x_2(n) - x_2(n)\big|, \quad \forall n > N_0 + c_0.\end{aligned} \tag{4.8}$$

Let

$$V_{21}(n) = |\ln x_2(n) - \ln y_2(n)|.$$

From (1.6), we have

$$\begin{aligned}V_{21}(n+1) &= |\ln x_2(n+1) - \ln y_2(n+1)|\\ &= \Big|[\ln x_2(n) - \ln y_2(n)] - a_2(n)\big[\frac{x_2(n-\lfloor c_2(n)\rfloor)}{x_1(n-\lfloor c_4(n)\rfloor)} - \frac{y_2(n-\lfloor c_2(n)\rfloor)}{y_1(n-\lfloor c_4(n)\rfloor)}\big]\Big|.\end{aligned} \tag{4.9}$$

Further, it follows that

$$\begin{aligned}&\big[\frac{x_2(n-\lfloor c_2(n)\rfloor)}{x_1(n-\lfloor c_4(n)\rfloor)} - \frac{y_2(n-\lfloor c_2(n)\rfloor)}{y_1(n-\lfloor c_4(n)\rfloor)}\big]\\ &= \frac{x_2(n-\lfloor c_2(n)\rfloor)y_1(n-\lfloor c_4(n)\rfloor) - y_2(n-\lfloor c_2(n)\rfloor)x_1(n-\lfloor c_4(n)\rfloor)}{x_1(n-\lfloor c_4(n)\rfloor)y_1(n-\lfloor c_4(n)\rfloor)}\\ &= \frac{\big[x_2(n-\lfloor c_2(n)\rfloor) - y_2(n-\lfloor c_2(n)\rfloor)\big]}{x_1(n-\lfloor c_4(n)\rfloor)}\end{aligned}$$

$$- \frac{y_2(n-\lfloor c_2(n)\rfloor)\big[x_1(n-\lfloor c_4(n)\rfloor)-y_1(n-\lfloor c_4(n)\rfloor)\big]}{x_1(n-\lfloor c_4(n)\rfloor)y_1(n-\lfloor c_4(n)\rfloor)},$$

which from (4.9) implies

$$\begin{aligned}
V_{21}(n+1) &\le \Big|[\ln x_2(n)-\ln y_2(n)] - \frac{a_2(n)\big[x_2(n-\lfloor c_2(n)\rfloor)-y_2(n-\lfloor c_2(n)\rfloor)\big]}{x_1(n-\lfloor c_4(n)\rfloor)}\Big| \\
&\quad + \frac{a_2(n)y_2(n-\lfloor c_2(n)\rfloor)\big|x_1(n-\lfloor c_4(n)\rfloor)-y_1(n-\lfloor c_4(n)\rfloor)\big|}{x_1(n-\lfloor c_4(n)\rfloor)y_1(n-\lfloor c_4(n)\rfloor)} \\
&\le \Big|[\ln x_2(n)-\ln y_2(n)] - \frac{a_2(n)\big[x_2(n)-y_2(n)\big]}{x_1(n-\lfloor c_4(n)\rfloor)}\Big| \\
&\quad + \frac{a_2(n)\chi(\bar{c}_2)\big|[x_2(n)-x_2(n-\lfloor c_2(n)\rfloor)]+[y_2(n)-y_2(n-\lfloor c_2(n)\rfloor)]\big|}{x_1(n-\lfloor c_4(n)\rfloor)} \\
&\quad + \frac{a_2(n)y_2(n-\lfloor c_2(n)\rfloor)\big|x_1(n-\lfloor c_4(n)\rfloor)-y_1(n-\lfloor c_4(n)\rfloor)\big|}{x_1(n-\lfloor c_4(n)\rfloor)y_1(n-\lfloor c_4(n)\rfloor)}.
\end{aligned} \tag{4.10}$$

Define

$$\begin{gathered}
P_2(n) := r_2(n) - a_2(n)\frac{x_2(n-\lfloor c_2(n)\rfloor)}{x_1(n-\lfloor c_4(n)\rfloor)}, \quad \forall n\in\mathbb{Z}, \\
Q_2(n) := r_2(n) - a_2(n)\frac{y_2(n-\lfloor c_2(n)\rfloor)}{y_1(n-\lfloor c_4(n)\rfloor)}, \quad \forall n\in\mathbb{Z}.
\end{gathered}$$

By (1.6), we obtain

$$\begin{aligned}
&\big|[x_2(n)-x_2(n-\lfloor c_2(n)\rfloor)]+[y_2(n)-y_2(n-\lfloor c_2(n)\rfloor)]\big| \\
&= \Big|\sum_{s=n-\lfloor c_2(n)\rfloor}^{n-1} x_2(s)\big[e^{P_2(s)}-e^{Q_2(s)}\big] + \sum_{s=n-\lfloor c_2(n)\rfloor}^{n-1}\big[x_2(s)-y_2(s)\big]\big[e^{Q_2(s)}-1\big]\Big| \\
&\le \sum_{s=n-\bar{c}_2}^{n-1} x_2(s)\xi_1'(s)a_2(s)\Big|\frac{x_2(s-\lfloor c_2(s)\rfloor)}{x_1(s-\lfloor c_4(s)\rfloor)} - \frac{y_2(n-\lfloor c_2(s)\rfloor)}{y_1(s-\lfloor c_4(s)\rfloor)}\Big| \\
&\quad + \sum_{s=n-\bar{c}_2}^{n-1}\xi_2'(s)\Big|r_2(s)-a_2(s)\frac{y_2(s-\lfloor c_2(s)\rfloor)}{y_1(s-\lfloor c_4(s)\rfloor)}\Big|\big|x_2(s)-y_2(s)\big| \\
&\le \sum_{s=n-\bar{c}_2}^{n-1} x_2(s)\xi_1'(s)\frac{a_2(s)\big|x_2(s-\lfloor c_2(s)\rfloor)-y_2(s-\lfloor c_2(s)\rfloor)\big|}{x_1(s-\lfloor c_4(s)\rfloor)} \\
&\quad + \sum_{s=n-\bar{c}_2}^{n-1} x_2(s)\xi_1'(s)\frac{a_2(s)y_2(s-\lfloor c_2(s)\rfloor)\big|x_1(s-\lfloor c_4(s)\rfloor)-y_1(s-\lfloor c_4(s)\rfloor)\big|}{x_1(s-\lfloor c_4(s)\rfloor)y_1(s-\lfloor c_4(s)\rfloor)} \\
&\quad + \sum_{s=n-\bar{c}_2}^{n-1}\xi_2'(s)\Big|r_2(s)-a_2(s)\frac{y_2(s-\lfloor c_2(s)\rfloor)}{y_1(s-\lfloor c_4(s)\rfloor)}\Big|\big|x_2(s)-y_2(s)\big| \\
&\le \sum_{s=n-k-\bar{c}_2}^{n-k-1}\sum_{k=\underline{c}_2}^{\bar{c}_2}\frac{(M_2+\epsilon)\mu_2(\epsilon)a_2^u}{m_1-\epsilon}\big|x_2(s)-y_2(s)\big| \\
&\quad + \sum_{s=n-k-\bar{c}_2}^{n-k-1}\sum_{k=\underline{c}_4}^{\bar{c}_4}\frac{(M_2+\epsilon)^2\mu_2(\epsilon)a_2^u}{(m_1-\epsilon)^2}\big|x_1(s)-y_1(s)\big|
\end{aligned}$$

$$
+\sum_{s=n-\bar{c}_2}^{n-1}\nu_2(\epsilon)\delta_2(\epsilon)\big|x_2(s)-y_2(s)\big|, \tag{4.11}
$$

where $\xi_1'(s)$ lies between $e^{P_2(s)}$ and $e^{Q_2(s)}$, $\xi_2'(s)$ lies between $e^{Q_2(s)}$ and 1, $s = n-\lfloor c_2(n)\rfloor,\dots,n-1$, for $n > N_0 + c_0$.

By (4.10) and (4.11), we have

$$
\begin{aligned}
\Delta V_{21}(n) &= V_{21}(n+1)-V_{21}(n)\\
&\le -|\ln x_2(n)-\ln y_2(n)| + \big|[\ln x_2(n)-\ln y_2(n)] - \frac{a_2(n)\big[x_2(n)-y_2(n)\big]}{x_1(n-\lfloor c_4(n)\rfloor)}\big|\\
&\quad + \frac{a_2(n)\chi(\bar{c}_2)\big|[x_2(n)-x_2(n-\lfloor c_2(n)\rfloor)]+[y_2(n)-y_2(n-\lfloor c_2(n)\rfloor)]\big|}{x_1(n-\lfloor c_4(n)\rfloor)}\\
&\quad + \frac{a_2(n)y_2(n-\lfloor c_2(n)\rfloor)\big|x_1(n-\lfloor c_4(n)\rfloor)-y_1(n-\lfloor c_4(n)\rfloor)\big|}{x_1(n-\lfloor c_4(n)\rfloor)y_1(n-\lfloor c_4(n)\rfloor)}\\
&\le -\big[\frac{1}{\sigma_2(n)} - \big|\frac{1}{\sigma_2(n)} - \frac{a_2(n)}{x_1(n-\lfloor c_4(n)\rfloor)}\big|\big]\big|x_2(n)-y_2(n)\big|\\
&\quad + \sum_{s=n-k-\bar{c}_2}^{n-k-1}\sum_{k=\underline{c}_2}^{\bar{c}_2}\frac{\chi(\bar{c}_2)(M_2+\epsilon)\mu_2(\epsilon)(a_2^u)^2}{(m_1-\epsilon)^2}\big|x_2(s)-y_2(s)\big|\\
&\quad + \sum_{s=n-k-\bar{c}_2}^{n-k-1}\sum_{k=\underline{c}_4}^{\bar{c}_4}\frac{\chi(\bar{c}_2)(M_2+\epsilon)^2\mu_2(\epsilon)(a_2^u)^2}{(m_1-\epsilon)^3}\big|x_1(s)-y_1(s)\big|\\
&\quad + \sum_{s=n-\bar{c}_2}^{n-1}\frac{a_2^u\chi(\bar{c}_2)\nu_2(\epsilon)\delta_2(\epsilon)}{m_1-\epsilon}\big|x_2(s)-y_2(s)\big|\\
&\quad + \frac{a_2^u(M_2+\epsilon)}{(m_1-\epsilon)^2}\sum_{s=n-\bar{c}_4}^{n-\underline{c}_4}\big|x_1(s)-y_1(s)\big|\\
&\le -\min\big[\frac{a_2^l}{M_1+\epsilon}, \frac{2}{M_2+\epsilon} - \frac{a_2^u}{m_1-\epsilon}\big]\big|x_2(n)-y_2(n)\big|\\
&\quad + \sum_{s=n-k-\bar{c}_2}^{n-k-1}\sum_{k=\underline{c}_2}^{\bar{c}_2}\frac{\chi(\bar{c}_2)(M_2+\epsilon)\mu_2(\epsilon)(a_2^u)^2}{(m_1-\epsilon)^2}\big|x_2(s)-y_2(s)\big|\\
&\quad + \sum_{s=n-k-\bar{c}_2}^{n-k-1}\sum_{k=\underline{c}_4}^{\bar{c}_4}\frac{\chi(\bar{c}_2)(M_2+\epsilon)^2\mu_2(\epsilon)(a_2^u)^2}{(m_1-\epsilon)^3}\big|x_1(s)-y_1(s)\big|\\
&\quad + \sum_{s=n-\bar{c}_2}^{n-1}\frac{a_2^u\chi(\bar{c}_2)\nu_2(\epsilon)\delta_2(\epsilon)}{m_1-\epsilon}\big|x_2(s)-y_2(s)\big|\\
&\quad + \frac{a_2^u(M_2+\epsilon)}{(m_1-\epsilon)^2}\sum_{s=n-\bar{c}_4}^{n-\underline{c}_4}\big|x_1(s)-y_1(s)\big|.
\end{aligned} \tag{4.12}
$$

Here we used that

$$
|x_2(n)-y_2(n)| = \sigma_2(n)|\ln x_2(n)-\ln y_2(n)|,
$$

where $\sigma_2(n)$ lies between $x_2(n)$ and $y_2(n)$, for $n > N_0 + c_0$. Let

$$V_2(n) = V_{21}(n) + V_{22}(n) + V_{23}(n) + V_{24}(n) + V_{25}(n),$$

where

$$\begin{gathered}
V_{22}(n) = \sum_{t=0}^{\bar{c}_2-1} \sum_{s=n-k-\bar{c}_2+t}^{n-1} \sum_{k=\underline{c}_2}^{\bar{c}_2} \frac{\chi(\bar{c}_2)(M_2+\epsilon)\mu_2(\epsilon)(a_2^u)^2}{(m_1-\epsilon)^2}\big|x_2(s)-y_2(s)\big|, \\
V_{23}(n) = \sum_{t=0}^{\bar{c}_2-1} \sum_{s=n-k-\bar{c}_2+t}^{n-1} \sum_{k=\underline{c}_4}^{\bar{c}_4} \frac{\chi(\bar{c}_2)(M_2+\epsilon)^2\mu_2(\epsilon)(a_2^u)^2}{(m_1-\epsilon)^3}\big|x_1(s)-y_1(s)\big|, \\
V_{24}(n) = \sum_{t=0}^{\bar{c}_2-1} \sum_{s=n-\bar{c}_2+t}^{n-1} \frac{a_2^u\chi(\bar{c}_2)\nu_2(\epsilon)\delta_2(\epsilon)}{m_1-\epsilon}\big|x_2(s)-y_2(s)\big|, \\
V_{25}(n) = \frac{a_2^u(M_2+\epsilon)}{(m_1-\epsilon)^2} \sum_{t=0}^{\bar{c}_4-\underline{c}_4} \sum_{s=n-\bar{c}_4+t}^{n-1} \big|x_1(s)-y_1(s)\big|.
\end{gathered}$$

By a similar argument as that in (4.8), we obtain

$$\begin{aligned}
\Delta V_2(n) &\le -\{\min\big[\frac{a_2^l}{M_1+\epsilon}, \frac{2}{M_2+\epsilon} - \frac{a_2^u}{m_1-\epsilon}\big] - \frac{a_2^u\chi(\bar{c}_2)\bar{c}_2\nu_2(\epsilon)\delta_2(\epsilon)}{m_1-\epsilon} \\
&\quad - \frac{\chi(\bar{c}_2)\bar{c}_2(\bar{c}_2-\underline{c}_2+1)(M_2+\epsilon)\mu_2(\epsilon)(a_2^u)^2}{(m_1-\epsilon)^2}\}\big|x_2(n)-y_2(n)\big| \\
&\quad + \{\frac{\chi(\bar{c}_2)\bar{c}_2(\bar{c}_4-\underline{c}_4+1)(M_2+\epsilon)^2\mu_2(\epsilon)(a_2^u)^2}{(m_1-\epsilon)^3} \\
&\quad + \frac{a_2^u(\bar{c}_4-\underline{c}_4+1)(M_2+\epsilon)}{(m_1-\epsilon)^2}\}\big|x_1(n)-y_1(n)\big|, \quad \forall n > N_0 + c_0.
\end{aligned} \tag{4.13}$$

We construct a Lyapunov functional as follows:

$$V(n) = \lambda_1 V_1(n) + \lambda_2 V_2(n),$$

which from (4.8) and (4.13) implies

$$\begin{aligned}
\Delta V(n) &\le -\{\lambda_1 \min\big[b_1^l, \frac{2}{M_1+\epsilon} - b_1^u\big] - \lambda_1(M_1+\epsilon)\mu_1(\epsilon)(b_1^u)^2\chi(\bar{c}_1)\bar{c}_1(\bar{c}_1-\underline{c}_1+1) \\
&\quad - \lambda_1\nu_1(\epsilon)\delta_1(\epsilon)b_1^u\chi(\bar{c}_1)\bar{c}_1 - \lambda_2\frac{\chi(\bar{c}_2)\bar{c}_2(\bar{c}_4-\underline{c}_4+1)(M_2+\epsilon)^2\mu_2(\epsilon)(a_2^u)^2}{(m_1-\epsilon)^3} \\
&\quad - \lambda_2\frac{a_2^u(\bar{c}_4-\underline{c}_4+1)(M_2+\epsilon)}{(m_1-\epsilon)^2}\}\big|x_1(n)-y_1(n)\big| \\
&\quad - \{\lambda_2 \min\big[\frac{a_2^l}{M_1+\epsilon}, \frac{2}{M_2+\epsilon} - \frac{a_2^u}{m_1-\epsilon}\big] - \lambda_2\frac{a_2^u\chi(\bar{c}_2)\bar{c}_2\nu_2(\epsilon)\delta_2(\epsilon)}{m_1-\epsilon} \\
&\quad - \lambda_2\frac{\chi(\bar{c}_2)\bar{c}_2(\bar{c}_2-\underline{c}_2+1)(M_2+\epsilon)\mu_2(\epsilon)(a_2^u)^2}{(m_1-\epsilon)^2} \\
&\quad - \lambda_1(M_1+\epsilon)\mu_1(\epsilon)a_1^u b_1^u\chi(\bar{c}_1)\bar{c}_1(\bar{c}_3-\underline{c}_3+1) - \lambda_1 a_1^u(\bar{c}_3-\underline{c}_3+1)\} \\
&\quad \times \big|x_2(n)-y_2(n)\big| \\
&\le -\lambda\big[\big|x_1(n)-y_1(n)\big| + \big|x_2(n)-y_2(n)\big|\big], \quad \forall n > N_0 + c_0.
\end{aligned} \tag{4.14}$$

Taking $n_2 \in (N_0 + c_0, \infty)_{\mathbb{Z}}$, and adding on both sides of (4.14) over $[n_2, n]_{\mathbb{Z}}$, we have

$$V(n+1) + \lambda \sum_{s=n_2}^{n} |x_1(s) - y_1(s)| + \lambda \sum_{s=n_2}^{n} |x_2(s) - y_2(s)| \le V(n_2) < \infty.$$

Therefore,

$$\lim_{n\to+\infty} \sup \sum_{s=n_2}^{n} |x_1(s) - y_1(s)| + \lim_{n\to+\infty} \sup \sum_{s=n_2}^{n} |x_2(s) - y_2(s)| \le \frac{V(n_2)}{\lambda} < \infty.$$

From the above inequality one could easily deduce that

$$\lim_{n\to+\infty} |x_1(s) - y_1(s)| = 0 \quad \text{and} \quad \lim_{n\to+\infty} |x_2(s) - y_2(s)| = 0.$$

This completes the proof. $\square$

When $\bar{c}_1 = \bar{c}_2 = 0$, system (1.6) reduces to

$$\begin{gathered} x_1(n+1) = x_1(n) \exp\{r_1(n) - b_1(n)x_1(n) - a_1(n)x_2(n - \lfloor c_3(n) \rfloor)\}, \\ x_2(n+1) = x_2(n) \exp\{r_2(n) - a_2(n) \frac{x_2(n)}{x_1(n - \lfloor c_4(n) \rfloor)}\}. \end{gathered} \tag{4.15}$$

From Theorem 4.1, we can easily obtain the following theorem.

Theorem 4.2. *Assume that (H1)–(H2) hold. Suppose further that*

(H4) *there exist two positive constants λ_1 and λ_2 such that* $\min\{\Lambda_1, \Lambda_2\} > 0$, *where*

$$\begin{gathered} \Lambda_1 := \lambda_1 \min\big[b_1^l, \frac{2}{M_1} - b_1^u\big] - \frac{\lambda_2 a_2^u(\bar{c}_4 - \underline{c}_4 + 1)M_2}{m_1^2}, \\ \Lambda_2 := \lambda_2 \min\big[\frac{a_2^l}{M_1}, \frac{2}{M_2} - \frac{a_2^u}{m_1}\big] - \lambda_1 a_1^u(\bar{c}_3 - \underline{c}_3 + 1). \end{gathered}$$

Then (4.15) *is globally attractive.*

5. Almost periodic solution

In this section, we study the existence and uniqueness of a globally attractive almost periodic solution of (1.6) by using almost periodic functional hull theory.

Let $\{\tau_k\}$ be any integer valued sequence such that $\tau_k \to \infty$ as $k \to \infty$. By Lemma 2.5, taking a subsequence if necessary, we have

$$r_i(n+\tau_k) \to r_i^*(n), \quad a_i(n+\tau_k) \to a_i^*(n), \quad b_1(n+\tau_k) \to b_1^*(n), \quad c_j(n+\tau_k) \to c_j^*(n)$$

as $k \to \infty$ for $n \in \mathbb{Z}$, $i = 1, 2$, $j = 1, 2, 3, 4$. Then we get a hull equations for (1.6) as follows:

$$\begin{gathered} x_1(n+1) = x_1(n) \exp\{r_1^*(n) - b_1^*(n)x_1(n - \lfloor c_1^*(n) \rfloor) - a_1^*(n)x_2(n - \lfloor c_3^*(n) \rfloor)\}, \\ x_2(n+1) = x_2(n) \exp\{r_2^*(n) - a_2^*(n) \frac{x_2(n - \lfloor c_2^*(n) \rfloor)}{x_1(n - \lfloor c_4^*(n) \rfloor)}\}. \end{gathered} \tag{5.1}$$

By the almost periodic theory, we can conclude that if (1.6) satisfies (H1)-(H3), then the hull equations (5.1) of (1.6) also satisfies (H1)-(H3).

By [29, Theorem 3.4], it is easy to obtain the following lemma.

Lemma 5.1. *If each hull equation of* (1.6) *has a unique strictly positive solution, then* (1.6) *has a unique strictly positive almost periodic solution.*

By using Lemma 5.1, we obtain the following result.

Lemma 5.2. *If* (1.6) *satisfies (H1)-(H3), then* (1.6) *admits a unique strictly positive almost periodic solution.*

Proof. By Lemma 5.1, to prove the existence of a unique strictly positive almost periodic solution of (1.6), we only need to prove that each hull equation of (1.6) has a unique strictly positive solution.

Firstly, we prove the existence of a strictly positive solution of any hull equations (5.1). By the almost periodicity of $\{r_i(n)\}$, $\{b_1(n)\}$, $\{a_i(n)\}$ and $\{c_j(n)\}$, there exists an integer valued sequence $\{\tau_k\}$ with $\tau_k \to \infty$ as $k \to \infty$ such that

$$r_i^*(n+\tau_k) \to r_i^*(n), \quad a_i^*(n+\tau_k) \to a_i^*(n), \quad b_1^*(n+\tau_k) \to b_1^*(n), \quad c_j^*(n+\tau_k) \to c_j^*(n)$$

as $k \to \infty$ for $n \in \mathbb{Z}$, $i = 1, 2$, $j = 1, 2, 3, 4$. It is not difficult to see that solutions of equations (5.1) with initial conditions (1.7) are well defined for all $n \geq 0$ and satisfy $x_i(n) > 0$, $i = 1, 2$. Suppose that (x_1, x_2) is any positive solution of hull equations (5.1). Let ϵ be an arbitrary small positive number. It follows from Theorem 3.3 that there exists a positive integer N_1 such that

$$m_i - \epsilon \leq x_i(n) \leq M_i + \epsilon, \quad \forall n > N_1, \quad i = 1, 2.$$

Write $x_i^p(n) = x_i(n + \eta_p)$ for $n \geq N_1 - \eta_p$, $p = 1, 2, \dots$, $i = 1, 2$. For any positive integer q, it is easy to see that there exist sequences $\{x_1^p(n) : p \geq q\}$ and $\{x_2^p(n) : p \geq q\}$ such that the sequences $\{x_1^p(n)\}$ and $\{x_2^p(n)\}$ have subsequences, denoted by $\{x_1^p(n)\}$ and $\{x_2^p(n)\}$ again, converging on any finite interval of $\mathbb{Z}$ as $p \to \infty$, respectively. Thus we have sequences $\{y_1(n)\}$ and $\{y_2(n)\}$ such that

$$x_i^p(n) \to y_i(n), \quad \forall n \in \mathbb{Z} \text{ as } p \to \infty, \ i = 1, 2.$$

Combining this convergence with

$$\begin{aligned} x_1^p(n+1) &= x_1^p(n) \exp\{r_1^*(n+\eta_p) - b_1^*(n+\eta_p)x_1(n+\eta_p - \lfloor c_1^*(n+\eta_p)\rfloor) \\ &\quad - a_1^*(n+\eta_p)x_2(n+\eta_p - \lfloor c_3^*(n+\eta_p)\rfloor)\}, \\ x_2^p(n+1) &= x_2^p(n) \exp\{r_2^*(n+\eta_p) - a_2^*(n+\eta_p)\frac{x_2(n+\eta_p - \lfloor c_2^*(n+\eta_p)\rfloor)}{x_1(n+\eta_p - \lfloor c_4^*(n+\eta_p)\rfloor)}\} \end{aligned}$$

gives

$$\begin{gathered} y_1(n+1) = y_1(n) \exp\{r_1^*(n) - b_1^*(n)y_1(n - \lfloor c_1^*(n)\rfloor) - a_1^*(n)y_2(n - \lfloor c_3^*(n)\rfloor)\}, \\ y_2(n+1) = y_2(n) \exp\{r_2^*(n) - a_2^*(n)\frac{y_2(n - \lfloor c_2^*(n)\rfloor)}{y_1(n - \lfloor c_4^*(n)\rfloor)}\}. \end{gathered}$$

We can easily see that (y_1, y_2) is a solution of hull equations (5.1) and $m_i - \epsilon \leq y_i(n) \leq M_i + \epsilon$ for $n \in \mathbb{Z}$, $i = 1, 2$. Since ϵ is an arbitrary small positive number, it follows that $m_i \leq y_i(n) \leq M_i$ for $n \in \mathbb{Z}$, $i = 1, 2$, which implies that each hull equations of (1.6) has at least one strictly positive solution.

Now we prove the uniqueness of the strictly positive solution of each hull equations (5.1). Suppose that the hull equations (5.1) has two arbitrary strictly positive solutions (x_1^*, x_2^*) and (y_1^*, y_2^*), which satisfy

$$m_i \leq x_i^*, \, y_i^* \leq M_i, \quad i = 1, 2.$$

Similar to Theorem 4.1, we define a Lyapunov functional

$$V^*(n) = \lambda_1 V_1^*(n) + \lambda_2 V_2^*(n),$$

where

$$V_1^*(n) = V_{11}^*(n) + V_{12}^*(n) + V_{13}^*(n) + V_{14}^*(n) + V_{15}^*(n),$$
$$V_2^*(n) = V_{21}^*(n) + V_{22}^*(n) + V_{23}^*(n) + V_{24}^*(n) + V_{25}^*(n).$$

Here

$$\begin{gathered}
V_{11}^*(n) = |\ln x_1^*(n) - \ln y_1^*(n)|,\\
V_{12}^*(n) = \sum_{t=0}^{\bar{c}_1-1} \sum_{s=n-k-\bar{c}_1+t}^{n-1} \sum_{k=\underline{c}_1}^{\bar{c}_1} M_1\mu_1(b_1^u)^2\chi(\bar{c}_1)\big|x_1^*(s) - y_1^*(s)\big|,\\
V_{13}^*(n) = \sum_{t=0}^{\bar{c}_1-1} \sum_{s=n-k-\bar{c}_1+t}^{n-1} \sum_{k=\underline{c}_3}^{\bar{c}_3} M_1\mu_1 a_1^u b_1^u\chi(\bar{c}_1)\big|x_2^*(s) - y_2^*(s)\big|,\\
V_{14}^*(n) = \sum_{t=0}^{\bar{c}_1-1} \sum_{s=n-\bar{c}_1+t}^{n-1} \nu_1\delta_1 b_1^u\chi(\bar{c}_1)\big|x_1^*(s) - y_1^*(s)\big|,\\
V_{15}^*(n) = \sum_{t=0}^{\bar{c}_3-\underline{c}_3} \sum_{s=n-\bar{c}_3+t}^{n-1} a_1^u\big|x_2^*(s) - y_2^*(s)\big|,\\
V_{21}^*(n) = |\ln x_2^*(n) - \ln y_2^*(n)|,\\
V_{22}^*(n) = \sum_{t=0}^{\bar{c}_2-1} \sum_{s=n-k-\bar{c}_2+t}^{n-1} \sum_{k=\underline{c}_2}^{\bar{c}_2} \frac{\chi(\bar{c}_2)M_2\mu_2(a_2^u)^2}{m_1^2}\big|x_2^*(s) - y_2^*(s)\big|,\\
V_{23}^*(n) = \sum_{t=0}^{\bar{c}_2-1} \sum_{s=n-k-\bar{c}_2+t}^{n-1} \sum_{k=\underline{c}_4}^{\bar{c}_4} \frac{\chi(\bar{c}_2)M_2^2\mu_2(a_2^u)^2}{m_1^3}\big|x_1^*(s) - y_1^*(s)\big|,\\
V_{24}^*(n) = \sum_{t=0}^{\bar{c}_2-1} \sum_{s=n-\bar{c}_2+t}^{n-1} \frac{a_2^u\chi(\bar{c}_2)\nu_2\delta_2}{m_1}\big|x_2^*(s) - y_2^*(s)\big|,\\
V_{25}^*(n) = \frac{a_2^u M_2}{m_1^2}\sum_{t=0}^{\bar{c}_4-\underline{c}_4} \sum_{s=n-\bar{c}_4+t}^{n-1} \big|x_1^*(s) - y_1^*(s)\big|.
\end{gathered}$$

Similar to the argument for (4.14), one has

$$\Delta V^* \le -\lambda|x_1^*(n) - y_1^*(n)| - \lambda|x_2^*(n) - y_2^*(n)|, \quad \forall n \in \mathbb{Z}.$$

Summing both sides of the above inequality from n to 0, we have

$$\lambda\sum_{s=n}^{0}|x_1^*(s) - y_1^*(s)| + \lambda\sum_{s=n}^{0}|x_2^*(s) - y_2^*(s)| \le V^*(n) - V^*(1), \quad \forall n < 0.$$

Note that V^* is bounded. Hence we have

$$\sum_{s=-\infty}^{0}|x_1^*(s) - y_1^*(s)| < \infty, \quad \sum_{s=-\infty}^{0}|x_2^*(s) - y_2^*(s)| < \infty,$$

which imply that

$$\lim_{n\to-\infty}|x_1^*(n) - y_1^*(n)| = 0, \quad \lim_{n\to-\infty}|x_2^*(n) - y_2^*(n)| = 0.$$

Let

$$\begin{aligned} P_0 := \lambda_1\Big\{&\frac{1}{m_1}+4c_0^2(c_0+1)M_1\mu_1(b_1^u)^2\chi(\bar c_1)+c_0^2\nu_1\delta_1b_1^u\chi(\bar c_1)+c_0(c_0+1)a_1^u\Big\}\\ &+\lambda_2\Big\{\frac{1}{m_2}+4c_0^2(c_0+1)\frac{\chi(\bar c_2)M_2\mu_2(a_2^u)^2}{m_1^2}+c_0^2\frac{a_2^u\chi(\bar c_2)\nu_2\delta_2}{m_1}\\ &+c_0(c_0+1)\frac{a_2^uM_2}{m_1^2}\Big\}. \end{aligned}$$

For arbitrary $\epsilon > 0$, there exists a positive integer N_2 such that

$$|x_1^*(n)-y_1^*(n)|<\frac{\epsilon}{P_0},\quad |x_2^*(n)-y_2^*(n)|<\frac{\epsilon}{P_0},\quad \forall n<-N_2.$$

Hence, one has

$$\begin{gathered} V_{11}^*(n)\le\frac{\epsilon}{m_1P_0},\quad V_{21}^*(n)\le\frac{\epsilon}{m_2P_0},\quad \forall n<-N_2,\\ V_{12}^*(n)+V_{13}^*(n)\le 4c_0^2(c_0+1)M_1\mu_1(b_1^u)^2\chi(\bar c_1)\frac{\epsilon}{P_0},\quad \forall n<-N_2,\\ V_{14}^*(n)\le c_0^2\nu_1\delta_1b_1^u\chi(\bar c_1)\frac{\epsilon}{P_0},\quad \forall n<-N_2,\\ V_{15}^*(n)\le c_0(c_0+1)a_1^u\frac{\epsilon}{P_0},\quad \forall n<-N_2,\\ V_{22}^*(n)+V_{23}^*(n)\le 4c_0^2(c_0+1)\frac{\chi(\bar c_2)M_2\mu_2(a_2^u)^2}{m_1^2}\frac{\epsilon}{P_0},\quad \forall n<-N_2,\\ V_{24}^*(n)\le c_0^2\frac{a_2^u\chi(\bar c_2)\nu_2\delta_2}{m_1}\frac{\epsilon}{P_0},\quad \forall n<-N_2,\\ V_{25}^*(n)\le c_0(c_0+1)\frac{a_2^uM_2}{m_1^2}\frac{\epsilon}{P_0},\quad \forall n<-N_2, \end{gathered}$$

which imply

$$V^*(n)=\lambda_1V_1^*(n)+\lambda_2V_2^*(n)<\epsilon,\quad \forall n<-N_2.$$

Therefore,

$$\lim_{n\to-\infty}V^*(n)=0.$$

Note that $V^*(n)$ is a nonincreasing function on $\mathbb{Z}$, and then $V^*(n)\equiv 0$. That is,

$$x_1^*(n)=y_1^*(n),\quad x_2^*(n)=y_2^*(n),\quad \forall n\in\mathbb{Z}.$$

Therefore, each hull equation of (1.6) has a unique strictly positive solution.

In view of the above discussion, any hull equation of (1.6) has a unique strictly positive solution. By Lemma 5.1, system (1.6) has a unique strictly positive almost periodic solution. The proof is complete. □

By Theorem 4.1 and Lemma 5.2, we obtain the following theorem.

Theorem 5.3. *Suppose that* (H1)–(H3) *hold, then* (1.6) *admits a unique strictly positive almost periodic solution, which is globally attractive.*

By Theorem 4.2 and Lemma 5.2, we have the following theorem.

Theorem 5.4. *Suppose that* (H1), (H2), (H4) *hold, then* (4.15) *admits a unique strictly positive almost periodic solution, which is globally attractive.*

Corollary 5.5. *Assume that* (H1)–(H3) *hold. Suppose that the non-negative coefficients* $\{r_i(n)\}$, $\{b_1(n)\}$, $\{a_i(n)\}$ *and* $\{c_j(n)\}$ *are periodic of period* ω, $i = 1, 2$, $j = 1, 2, 3, 4$, *then system* (1.6) *admits a unique strictly positive periodic solution of period* ω, *which is globally attractive.*

Corollary 5.6. *Assume that* (H1), (H2), (H4) *hold. Suppose further that the non-negative coefficients* $\{r_i(n)\}$, $\{b_1(n)\}$, $\{a_i(n)\}$ *and* $\{c_j(n)\}$ *are periodic of period* ω, $i = 1, 2$, $j = 1, 2, 3, 4$, *then* (4.15) *admits a unique strictly positive periodic solution of period* ω, *which is globally attractive.*

6. An example

Example 6.1. Consider the following discrete Leslie-Gower predator-prey model with variable delay:

$$
\begin{aligned}
x_1(n+1) = x_1(n) \exp\big\{&0.05[1+\cos^2(\sqrt{2}n)] - 0.2x_1(n - \lfloor 1 - 0.5(-1)^n \rfloor) \\
&- 0.001x_2(n)\big\}, \\
x_2(n+1) = x_2(n) &\exp\big\{0.1 - 0.2(1+|\sin n|)\frac{x_2(n)}{x_1(n)}\big\}.
\end{aligned} \tag{6.1}
$$

Then this system is permanent and has a unique globally attractive almost periodic solution.

Corresponding to system (1.6), $r_1^u = 0.1$, $r_1^l = 0.05$, $r_2^u = r_2^l = 0.1$, $b_1^u = b_1^l = 0.2$, $a_1^u = a_1^l = 0.001$, $a_2^u = 0.4$, $a_2^l = 0.2$, $\bar{c}_1 = 1$, $\underline{c}_1 = 0$, $\bar{c}_2 = \underline{c}_2 = \bar{c}_3 = \underline{c}_3 = \bar{c}_4 = \underline{c}_4 = 0$. By easy calculations, we have $M_1 < 0.6$, $M_2 < 0.33$, $m_1 > 0.21$, $m_2 > 0.03$, $\mu_1 < 1.1$, $\nu_1 < 1.1$, and $\delta_1 < 0.12$. Taking $\lambda_1 = 1$, $\lambda_2 = \frac{1}{30}$, we have

$$
\begin{gathered}
\Theta_1 > 0.2 - 0.0528 - 0.0264 - 0.1 = 0.0208 > 0, \\
\Theta_2 > 0.011 - 0.000132 - 0.001 \approx 0.01 > 0,
\end{gathered}
$$

which implies that condition (H3) of Theorem 5.3 is satisfied. It is easy to verify that (H1)–(H2) hold and the result follows from Theorem 3.3 and Theorem 5.3 (see Figure 1).

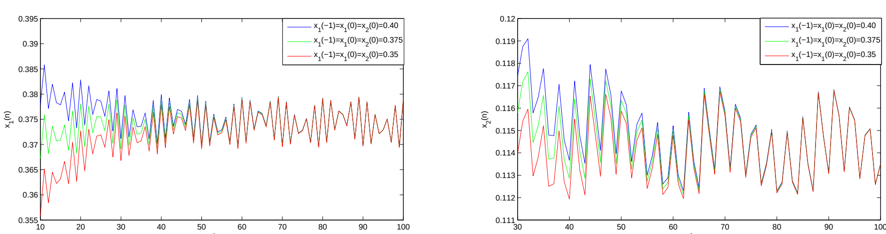

Figure 1. Solutions of (6.1) with initial values $x_1(-1) = x_1(0) = x_2(0) = 0.35$, 0.375 and 0.40, respectively

Conclusion. The aim of this article is to give sufficient conditions for the existence, uniqueness and global attractivity of positive almost periodic solution in a discrete Leslie-Gower predator-prey model with pure and variable delays. Based on the permanence result, the global attractivity of the above model is obtained by constructing a suitable Lyapunov functional. In addition, one makes use of almost periodic functional hull theory to show that the above model has a unique positive almost periodic solution.

Acknowledgements. The author would like to thank the anonymous referees for their carefully reading of this manuscript, and for their valuable comments and suggestions.

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