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EXISTENCE AND MULTIPLICITY OF POSITIVE PERIODIC SOLUTIONS FOR FIRST-ORDER SINGULAR SYSTEMS WITH IMPULSE EFFECTS

BIAN-XIA YANG

Abstract. In this article, we consider the existence and multiplicity of positive periodic solutions for a first-order singular system with impulse effects. The proof of our main result is based on Krasnoselskii's fixed point theorem in a cone.

1. Introduction

Impulsive differential equations have wide applicability in physics, population dynamics, ecology, biological systems, biotechnology, industrial robotic, pharmacokinetics, optimal control, etc. The reason for this applicability arises from the fact that impulsive differential problems are an appropriate model for describing process which at certain moments change their state rapidly and which cannot be described using the classical differential equation. Therefore, the study of impulsive differential equation has gained prominence and it is a rapidly growing field, see [1, 2, 4, 5, 6] and the references therein.

In 2008, Chu and Nieto [4] studied first-order impulsive periodic boundary-value problem (BVP)

$$\begin{gathered} u'(t)+a(t)u(t)=f(t,u(t))+e(t), \quad t\in \mathbb{J}', \\ u(t_k^+)=u(t_k^-)+I_k(u(t_k)), \quad k=1,\dots,p, \quad u(0)=u(1), \end{gathered} \tag{1.1}$$

where $0=t_0<t_1<\dots<t_p<t_{p+1}=1$, $\mathbb{J}'=[0,1]\backslash\{t_1,\dots,t_p\}$, $a,e\in C(\mathbb{R},\mathbb{R})$ are 1-periodic functions, $I_k\in C(\mathbb{R},\mathbb{R})$, $k=1,\dots,p$. The nonlinearity function $f(t,u)\in C(\mathbb{J}'\times\mathbb{R})$ is 1-periodic in t, and $f(t,u)$ is left continuous at $t=t_k$, the right limit $f(t_k^+,u)$ exists. Using the Leray-Schauder nonlinear alternative and a truncation technique, under some conditions, they obtained the existence of at least one non-trivial 1-periodic solution of (1.1).

In 2011, Wang [9] studied the first-order nonautonomous singular n-dimensional system

$$u_i'(t)+a_i(t)u_i(t)=\lambda b_i(t)f_i(u_1(t),\dots,u_n(t)), \quad i=1,\dots n. \tag{1.2}$$

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By using the fixed point theorem in cones, the author established the following result, under the assumptions:

- (A1) $a_i, b_i \in C(\mathbb{R}, \mathbb{R}_+)$ are ω-periodic functions such that $\int_0^\omega a_i(t)dt > 0$ and $\int_0^\omega b_i(t)dt > 0$, for $i = 1, \dots, n$;
- (A2) $f_i \in C(\mathbb{R}_+^n \backslash \{0\}, \mathbb{R}_+ \backslash \{0\})$, $i = 1, \dots, n$, and $\lim_{|\mathbf{u}| \to 0} f_j(\mathbf{u}) = \infty$ for some $j = 1, \dots, n$.

Theorem 1.1. *Let* (A1), (A2) *hold. Then*

- (i) *there exists a* $\lambda_0 > 0$, *such that* (1.2) *has a positive* ω*-periodic solution for* $0 < \lambda < \lambda_0$;
- (ii) *if* $\lim_{|\mathbf{u}| \to \infty} \frac{f_i(\mathbf{u})}{|\mathbf{u}|} = 0$, $i = 1, \dots, n$, *then, for all* $\lambda > 0$, (1.2) *has a positive* ω*-periodic solution;*
- (iii) *if* $\lim_{|\mathbf{u}| \to \infty} \frac{f_i(\mathbf{u})}{|\mathbf{u}|} = \infty, i = 1, \dots, n$, *then, for sufficiently small* $\lambda > 0$, (1.2) *has two positive* ω*-periodic solutions.*

Here $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}_+^n = \Pi_{i=1}^n \mathbb{R}_+$, $\mathbf{u} = (u_1, u_2, \dots, u_n) \in \mathbb{R}_+^n$, $|\mathbf{u}| = \sum_{i=1}^n |u_i|$.

Inspired by [4, 9], in this paper, we are concerned with the existence and multiplicity of the positive 1-periodic solution of the following first-order singular n-dimensional system with impulse effect

$$
\begin{gathered}
u_i'(t) + a_i(t)u_i(t) = \lambda b_i(t) f_i(t, u_1(t), \dots, u_n(t)) + \lambda e_i(t), \quad t \in \mathbb{J}', \\
u_i(t_k^+) = u_i(t_k^-) + \lambda I_i^k(u_1(t_k), \dots, u_n(t_k)), \quad k = 1, \dots, p, \\
u_i(0) = u_i(1), \quad i = 1, \dots, n,
\end{gathered} \tag{1.3}
$$

where $\lambda > 0$ is a parameter, $\mathbb{J}'$ is defined as above. By a positive 1-periodic solution, we mean a positive 1-periodic function in $C^1(\mathbb{R}, \mathbb{R}^n)$ solving corresponding systems (1.3) and each component is positive for all t.

We will use the following assumptions:

- (H1) $a_i, e_i \in C(\mathbb{R}, \mathbb{R})$, $b_i \in C(\mathbb{R}, \mathbb{R}_+ \backslash \{0\})$ are 1-periodic functions and $\int_0^1 a_i(t)dt > 0$ for $i = 1, \dots, n$;
- (H2) $f_i \in C(\mathbb{J}' \times (\mathbb{R}_+^n \backslash \{0\}), \mathbb{R}_+ \backslash \{0\})$ is 1-periodic in t. Moreover, $f_i(t, u)$ is left continuous at $t = t_k$ and the right limit $f_i(t_k^+, u)$ exists, $i = 1, \dots, n$;
- (H3) $I_i^k \in C(\mathbb{R}_+^n, \mathbb{R}_+)$, $k = 1, \dots, p$, $i = 1, \dots, n$.

Using Krasnoseskii's fixed point theorem in cone, we obtain the following result.

Theorem 1.2. *Let* (H1)–(H3) *hold. Assume that* $\lim_{|\mathbf{u}| \to 0} f_i(t, \mathbf{u}) = \infty$, $i = 1, \dots, n$ *uniformly with respect to* $t \in [0, 1]$. *Then*

- (i) *there exists a* $\lambda_1 > 0$, *such that* (1.3) *has a positive* 1*-periodic solution for* $0 < \lambda < \lambda_1$;
- (ii) *if* $\lim_{|\mathbf{u}| \to \infty} \frac{f_i(t, \mathbf{u})}{|\mathbf{u}|} = 0$ *and* $\lim_{|\mathbf{u}| \to \infty} f_i(t, \mathbf{u}) = \infty$ *uniformly with respect to* $t \in [0, 1]$, $\lim_{mathbfu \to \infty} \frac{I_i^k(\mathbf{u})}{|\mathbf{u}|} = 0$ *for* $i = 1, \dots, n$, $k = 1, \dots, p$, *then, there exists* $\lambda_2 > 0$, *such that* (1.3) *has a positive* 1*-periodic solution for* $\lambda > \lambda_2$;
- (iii) *if* $\lim_{|\mathbf{u}| \to \infty} \frac{f_i(t, \mathbf{u})}{|\mathbf{u}|} = \infty, i = 1, \dots, n$ *uniformly with respect to* $t \in [0, 1]$, *then, for sufficiently small* $\lambda > 0$, (1.3) *has two positive* 1*-periodic solutions.*

We remark that e_i may take negative values in this paper; nevertheless, we still obtain the existence and multiplicity of positive 1-periodic solution of (1.3).

Remark 1.3. If $I_k = 0$ for $k = 1, \dots, p$, $f_i(t, \mathbf{u}) = f_i(\mathbf{u})$, $e_i = 0$ for $i = 1, \dots, n$, then system (1.3) reduces to (1.2). In this case, we need only $\lim_{|\mathbf{u}|\to 0} f_j(\mathbf{u}) = \infty$ for some $j = 1, \dots, n$; so (i), (iii) of Theorem 1.2 reduce to the (i), (iii) of Theorem 1.1, respectively. Hence, Theorem 1.2 extends Theorem 1.1.

If $n = 1$, $\lambda = 1$, $b_i(t) = 1$, then system (1.3) reduces to (1.1). So, Theorem 1.2 partially improves the result of [4].

The rest of this paper is organized as follows. In Section 2, some notation and preliminaries are given. In Section 3, we give the proof of main result. At last, an example is presented to illustrate the main result.

2. Preliminaries

Denote

$$\begin{aligned} PC[0,1] = \Big\{ u : u &\text{ is continuous on } \mathbb{J}', \text{ left continuous at } t = t_k, \\ &\text{and the right limit } u(t_k^+) \text{ exists for } k = 1, \dots, p. \end{aligned}$$

Let $E = \Pi_{i=1}^n PC[0,1]$ which is a Banach space under the norm

$$\|\mathbf{u}\| = \sum_{i=1}^n \sup_{t\in[0,1]} |u_i(t)|.$$

Denote the cone

$$\begin{aligned} K = \big\{ \mathbf{u} = (u_1, \dots, u_n) \in E : u_i(t) \geq 0,\ t \in [0,1], i = 1, \dots, n, \text{ and} \\ \min_{t\in[0,1]} |\mathbf{u}(t)| \geq \sigma \|\mathbf{u}\| \big\}, \end{aligned}$$

where

$$\sigma = \min_{i=1,\dots,n} \{\sigma_i\}, \quad \sigma_i = \frac{m_i}{M_i}, i = 1, \dots, n. \tag{2.1}$$

The constants m_i, M_i will be defined by (2.3) below.

Let $T_\lambda : K\backslash\{0\} \to E$ be a map with components $(T_\lambda^1, \dots, T_\lambda^n)$:

$$T_\lambda^i \mathbf{u}(t) = \lambda \int_0^1 G_i(t,s)\Big[b_i(s) f_i(s, \mathbf{u}(s)) + e_i(s)\Big] ds + \lambda \sum_{k=1}^p G_i(t, t_k) I_i^k(\mathbf{u}(t_k)), \tag{2.2}$$

where

$$G_i(t,s) = \begin{cases} \frac{e^{-A_i(t)+A_i(s)}}{1-e^{-A_i(1)}}, & 0 \leq s \leq t \leq 1, \\ \frac{e^{-A_i(1)-A_i(t)+A_i(s)}}{1-e^{-A_i(1)}}, & 0 \leq t < s \leq 1, \end{cases}$$

with $A_i(t) = \int_0^t a_i(s) ds$, (see [4] for details). It is easy to see that (H1) implies that $G_i(t,s) > 0$.

Clearly, $\mathbf{u} \in E\backslash\{0\}$ is a solution of (1.3) if and only if it is a fixed point of T_λ. Also note that each component $u_i(t)$ of any nonnegative periodic solution $\mathbf{u}(t)$ is strictly positive for all t because of the positiveness of $G_i(t,s)$ and assumptions (H1)–(H3).

For convenience, throughout this paper, we denote

$$M_i = \sup_{t,s\in[0,1]} G_i(t,s), \quad m_i = \inf_{t,s\in[0,1]} G_i(t,s) \tag{2.3}$$

and

$$|\mathbf{u}| = \sum_{i=1}^n |u_i|, \quad \text{where } \mathbf{u} = (u_1, u_2, \dots, u_n) \in \mathbb{R}^n.$$

For $r > 0$, define $\Omega_r = \{\mathbf{u} \in K : \|\mathbf{u}\| < r\}$. Then $\partial\Omega_r = \{\mathbf{u} \in K : \|\mathbf{u}\| = r\}$. We now look at several properties of the operator T_λ.

Lemma 2.1. *Assume that* (H1)–(H3) *hold.*

(i) *If* $\lim_{|\mathbf{u}|\to 0} f_i(t,\mathbf{u}) = \infty$ *uniformly with respect to* $t \in [0,1]$ *for* $i = 1,\dots,n$, *then there is a* $\delta > 0$, *such that for* $r \in (0,\delta)$, $T_\lambda : \bar{\Omega}_r\backslash\{0\} \to K$ *is completely continuous.*

(ii) *If* $\lim_{|\mathbf{u}|\to\infty} f_i(t,\mathbf{u}) = \infty$ *uniformly with respect to* $t \in [0,1]$ *for* $i = 1,\dots,n$, *then there is a* $\Delta > 0$, *such that for* $R > \Delta$, $T_\lambda : K\backslash\Omega_R \to K$ *is completely continuous.*

(iii) *If* $T_\lambda : K\backslash\{0\} \to K$, *then for* $\mathbf{u} \in K$ *with* $\|\mathbf{u}\| = r$, *we have*

$$\|T_\lambda\mathbf{u}\| \ge \frac{\lambda\hat{m}_r}{2}\sum_{i=1}^n m_i\int_0^1 b_i(s)ds, \tag{2.4}$$

$$\|T_\lambda\mathbf{u}\| \le \lambda\sum_{i=1}^n M_i\Big(\hat{M}_r\int_0^1 b_i(s)ds + \int_0^1 |e_i(s)|ds + p\tilde{M}_r\Big), \tag{2.5}$$

where $\hat{m}_r = \min\{f_i(t,\mathbf{u}) : t \in [0,1], \mathbf{u} \in \mathbb{R}^n_+$ *with* $\sigma r \le |\mathbf{u}| \le r, i = 1,\dots,n\}$,

$$\hat{M}_r = \max\{f_i(t,\mathbf{u}) : t \in [0,1], \mathbf{u} \in \mathbb{R}^n_+ \text{ with } \sigma r \le |\mathbf{u}| \le r, i = 1,\dots,n\},$$

$$\tilde{M}_r = \max\{I_i^k(u) : \mathbf{u} \in \mathbb{R}^n_+ \quad \text{with } \sigma r \le |\mathbf{u}| \le r,\ k = 1,\dots,p, i = 1,\dots,n\}.$$

Proof. (i) We split $b_i(t)f_i(t,\mathbf{u})+e_i(t)$ into two terms $\frac{1}{2}b_i(t)f_i(t,\mathbf{u})$ and $\frac{1}{2}b_i(t)f_i(t,\mathbf{u}) + e_i(t)$. Then the first term is always positive and used to carry out the estimates of the operator. We will make the second term $\frac{1}{2}b_i(t)f_i(t,\mathbf{u}) + e_i(t)$ positive by choosing appropriate domains of f_i.

Noting that $b_i(t)$ is continuous and positive on $[0,1]$, and $\lim_{|\mathbf{u}|\to 0} f_i(t,\mathbf{u}) = \infty$, for $i = 1,\dots,n$, there exists $\delta > 0$, such that

$$f_i(t,\mathbf{u}) \ge 2\frac{\max_{t\in[0,1]}\{|e_i(t)|\} + 1}{\min_{t\in[0,1]} b_i(t)}, \quad t \in [0,1],\ \mathbf{u} \in \mathbb{R}^n,\ 0 < |\mathbf{u}| \le \delta.$$

Now for $r \in (0,\delta)$ and $\mathbf{u} \in \bar{\Omega}_r\backslash\{0\}, t \in [0,1]$, we have

$$\begin{aligned} b_i(t)f_i(t,\mathbf{u}(t)) + e_i(t) &\ge \frac{1}{2}b_i(t)f_i(t,\mathbf{u}(t)) + e_i(t) \\ &\ge b_i(t)\frac{\max_{t\in[0,1]}\{|e_i(t)|\} + 1}{\min_{t\in[0,1]}\{b_i(t)\}} + e_i(t) > 0, \end{aligned}$$

and

$$\begin{aligned} \min_{t\in[0,1]}(T^i_\lambda\mathbf{u})(t) &\ge \lambda\int_0^1 m_i\Big[b_i(s)f_i(s,\mathbf{u}(s)) + e_i(s)\Big]ds + \lambda m_i\sum_{k=1}^p I_i^k(\mathbf{u}(t_k)) \\ &= \lambda\sigma_i\int_0^1 M_i\Big[b_i(s)f_i(s,\mathbf{u}(s)) + e_i(s)\Big]ds + \lambda\sigma_i\sum_{k=1}^p M_iI_i^k(\mathbf{u}(t_k)) \\ &\ge \sigma_i\sup_{t\in[0,1]}|T^i_\lambda\mathbf{u}|. \end{aligned}$$

Thus, $T_\lambda(\bar{\Omega}_r\backslash\{0\}) \subset K$. According to Arzela-Ascoli theorem and the hypothesis (H1)-(H3), we know that $T_\lambda : \bar{\Omega}_r\backslash\{0\} \to K$ is completely continuous.

(ii) If $\lim_{|\mathbf{u}|\to\infty} f_i(t,\mathbf{u}) = \infty$, there is an $\hat{R} > 0$, such that

$$f_i(t,\mathbf{u}) \geq 2\frac{\max_{t\in[0,1]}\{|e_i(t)|\}+1}{\min_{t\in[0,1]}\{b_i(t)\}}, \quad t\in[0,1],\ \mathbf{u}\in\mathbb{R}^n,\ |\mathbf{u}|\geq\hat{R}.$$

Let $\Delta = \frac{\hat{R}}{\sigma}$. Then for $R > \Delta$, $\mathbf{u} \in K\backslash\Omega_R$, we have that $\min_{t\in[0,1]}|\mathbf{u}(t)| \geq \sigma\|\mathbf{u}\| \geq \hat{R}$, and therefore

$$b_i(t)f_i(t,\mathbf{u}) + e_i(t) \geq \frac{1}{2}b_i(t)f_i(t,\mathbf{u}) + e_i(t) > 0, \quad t\in[0,1].$$

Similar to (i), we have that $T_\lambda : K\backslash\Omega_R \to K$ is completely continuous.

(iii) If $\mathbf{u} \in K$ with $\|\mathbf{u}\| = r$, then for $t \in [0,1]$, $\sigma r \leq |\mathbf{u}(t)| \leq r$, so $\hat{m}_r \leq f_i(t,\mathbf{u}(t)) \leq \hat{M}_r$, $t\in[0,1]$, and $I_i^k(u) \leq \tilde{M}_r$, $k = 1,\dots,p$, $i = 1,\dots,n$. By the definition of $T_\lambda\mathbf{u}$, we have

$$\begin{aligned}
\|T_\lambda\mathbf{u}\| &= \sum_{i=1}^n \sup_{t\in[0,1]} T_\lambda^i\mathbf{u}(t) \\
&\geq \frac{1}{2}\lambda\sum_{i=1}^n m_i\int_0^1 b_i(s)f_i(s,\mathbf{u}(s))ds \\
&\geq \frac{\lambda\hat{m}_r}{2}\sum_{i=1}^n m_i\int_0^1 b_i(s)ds,
\end{aligned}$$

and

$$\begin{aligned}
\|T_\lambda\mathbf{u}\| &= \sum_{i=1}^n \sup_{t\in[0,1]} T_\lambda^i\mathbf{u}(t) \\
&\leq \lambda\sum_{i=1}^n M_i\Big(\int_0^1 b_i(s)f_i(s,\mathbf{u}(s))ds + \int_0^1 |e_i(s)|ds + \sum_{k=1}^p I_i^k(\mathbf{u}(t_k))\Big) \\
&\leq \lambda\sum_{i=1}^n M_i\Big(\hat{M}_r\int_0^1 b_i(s)ds + \int_0^1 |e_i(s)|ds + p\tilde{M}_r\Big).
\end{aligned}$$

□

The following well-known fixed point theorem is crucial in our arguments.

Lemma 2.2 ([7, 8])**.** *Let E be a Banach space and K a cone in E. Assume that Ω_1, Ω_2 are bounded open subsets of E with $0 \in \Omega_1, \bar{\Omega}_1 \subset \Omega_2$, and let*

$$T : K\cap(\bar{\Omega}_2\backslash\Omega_1) \to K$$

be completely continuous operator such that either

(i) $\|Tu\| \geq \|u\|, u \in K\cap\partial\Omega_1$ *and* $\|Tu\| \leq \|u\|, u\in K\cap\partial\Omega_2$*; or*
(ii) $\|Tu\| \leq \|u\|, u \in K\cap\partial\Omega_1$ *and* $\|Tu\| \geq \|u\|, u\in K\cap\partial\Omega_2$*.*

Then T has a fixed point in $K\cap(\bar{\Omega}_2\backslash\Omega_1)$.

3. Proof of main results

Proof of Theorem 1.2. (i) By Lemma 2.1 (i), there is a $\delta > 0$, such that if $0 < r < \delta$, then $T_\lambda : \bar{\Omega}_r\backslash\{0\} \to K$ is completely continuous. Now, for a fixed number

$r_1 \in (0, \delta)$, if we choose

$$\lambda_1 = \frac{r_1}{\sum_{i=1}^n M_i\Big(\hat{M}_{r_1}\int_0^1 b_i(s)ds + \int_0^1 |e_i(s)|ds + p\tilde{M}_{r_1}\Big)},$$

for $\lambda \in (0, \lambda_1)$, (2.5) implies

$$\|T_\lambda \mathbf{u}\| < \|\mathbf{u}\|, \quad \mathbf{u} \in \partial\Omega_{r_1}. \tag{3.1}$$

On the other hand, for $\lambda \in (0, \lambda_1)$, in view of the assumption $\lim_{|\mathbf{u}|\to 0} f_i(t, \mathbf{u}) = \infty$, there is a positive number $r_2 < r_1$, such that

$$f_i(t, \mathbf{u}) \geq \eta|\mathbf{u}|, \quad t \in [0, 1],\ \mathbf{u} \in \mathbb{R}^n \text{ with } 0 < |\mathbf{u}| \leq r_2,$$

where $\eta > 0$ is chosen so that

$$\frac{\lambda\eta\sigma}{2}\min_{i=1,\dots,n}\{m_i\int_0^1 b_i(s)ds\} > 1.$$

Thus, for $\mathbf{u} \in \partial\Omega_{r_2}$, we have

$$f_i(t, \mathbf{u}(t)) \geq \eta|\mathbf{u}(t)|, \quad t \in [0, 1].$$

and

$$\begin{aligned}
\|T_\lambda \mathbf{u}\| &\geq \sup_{t\in[0,1]} T_\lambda^i \mathbf{u}(t) \\
&= \sup_{t\in[0,1]} \lambda\Big(\int_0^1 G_i(t,s)\big[b_i(s)f_i(s,\mathbf{u}(s)) + e_i(s)\big]ds + \sum_{k=1}^p G_i(t,t_k)I_i^k(\mathbf{u}(t_k))\Big) \\
&\geq \frac{1}{2}\lambda\sup_{t\in[0,1]}\int_0^1 G_i(t,s)b_i(s)f_i(s,\mathbf{u}(s))ds \\
&\geq \frac{1}{2}\lambda m_i\int_0^1 b_i(s)f_i(s,\mathbf{u}(s))ds \\
&\geq \frac{1}{2}\eta\lambda m_i\int_0^1 b_i(s)|\mathbf{u}(s)|ds \\
&\geq \frac{1}{2}\eta\lambda m_i\sigma\int_0^1 b_i(s)ds\|\mathbf{u}\| > \|\mathbf{u}\|.
\end{aligned} \tag{3.2}$$

So from Lemma 2.2, (3.1), (3.2), we obtain that T_λ has a fixed point $\mathbf{u} \in \bar{\Omega}_{r_1}\backslash\Omega_{r_2}$. The fixed point $\mathbf{u}$ is the desired positive 1-periodic solution of (1.3).

(ii) According to Lemma 2.1 (ii), there is a $\Delta > 0$, such that for $R > \Delta$, $T_\lambda : K\backslash\Omega_R \to K$ is completely continuous. Now for a fixed number $R_1 > \Delta$, if we choose

$$\lambda_2 = \frac{2R_1}{\hat{m}_{R_1}\sum_{i=1}^n m_i\int_0^1 b_i(s)ds},$$

for $\lambda > \lambda_2$, (2.4) means that

$$\|T_\lambda \mathbf{u}\| \geq \frac{\lambda\hat{m}_{R_1}}{2}\sum_{i=1}^n m_i\int_0^1 b_i(s)ds > R_1 = \|\mathbf{u}\|, \quad \mathbf{u} \in \partial\Omega_{R_1}. \tag{3.3}$$

On the other hand. Since $\lim_{|\mathbf{u}|\to\infty}\frac{f_i(t,\mathbf{u})}{|\mathbf{u}|}=0$, $\lim_{|\mathbf{u}|\to\infty}\frac{I_i^k(\mathbf{u})}{|\mathbf{u}|}=0$, for a fixed $\lambda>\lambda_2$, we can choose

$$R_2>\max\big\{2R_1,\quad 2\lambda\sum_{i=1}^n M_i\int_0^1|e_i(s)|ds\big\},$$

so that

$$f_i(t,\mathbf{u})\le\epsilon|\mathbf{u}| \text{ and } I_i^k(\mathbf{u})\le\epsilon|\mathbf{u}| \text{ for } t\in[0,1],\ \mathbf{u}\in\mathbb{R}^n \text{ with } |\mathbf{u}|\ge\sigma R_2,$$

where the constant $\epsilon>0$ satisfies

$$\lambda\epsilon\sum_{i=1}^n M_i\Big(\int_0^1 b_i(s)ds+p\Big)<\frac{1}{2}.$$

From the definition of T_λ, for $\mathbf{u}\in\partial\Omega_{R_2}$, we have

$$\begin{aligned}
&\|T_\lambda\mathbf{u}\|\\
&=\sum_{i=1}^n\sup_{t\in[0,1]}T_\lambda^i\mathbf{u}(t)\\
&\le\lambda\sum_{i=1}^n M_i\Big(\int_0^1 b_i(s)f_i(s,\mathbf{u}(s))ds+\int_0^1|e_i(s)|ds+\sum_{k=1}^p I_i^k(\mathbf{u}(t_k))\Big)\\
&\le\lambda\sum_{i=1}^n M_i\Big(r_2\epsilon\int_0^1 b_i(s)ds+\int_0^1|e_i(s)|ds+pr_2\epsilon\Big)<R_2=\|\mathbf{u}\|.
\end{aligned}\tag{3.4}$$

By Lemma 2.2, (3.3), (3.4), we have that T_λ has a fixed point $\mathbf{u}\in\bar\Omega_{R_2}\backslash\Omega_{R_1}$. The fixed point $\mathbf{u}$ is the desired positive 1-periodic solution of (1.3).

(iii) Since $\lim_{|\mathbf{u}|\to0}f_i(t,\mathbf{u})=\infty$, (i) implies (1.3) has a positive periodic solutions $\mathbf{u}_1\in\bar\Omega_{r_1}\backslash\Omega_{r_2}$ for $\lambda\in(0,\lambda_1)$.

On the other hand, since $\lim_{|\mathbf{u}|\to\infty}\frac{f_i(t,\mathbf{u})}{|\mathbf{u}|}=\infty$, by Lemma 2.1 (ii), there is $\Delta>0$, such that if $R>\Delta$, $T_\lambda:K\backslash\Omega_R\to K$ is completely continuous. For a fixed number $R_3>\max\{\Delta,r_1\}$, if we choose

$$\lambda_0=\frac{R_3}{\sum_{i=1}^n M_i\Big(\hat M_{R_3}\int_0^1 b_i(s)ds+\int_0^1|e_i(s)|ds+p\tilde M_{R_3}\Big)},$$

for $\lambda<\lambda_0$, (2.5) implies

$$\|T_\lambda\mathbf{u}\|<\|\mathbf{u}\|,\quad \mathbf{u}\in\partial\Omega_{R_3}.\tag{3.5}$$

Since $\lim_{|\mathbf{u}|\to\infty}\frac{f_i(t,\mathbf{u})}{|\mathbf{u}|}=\infty$ uniformly with respect to $t\in[0,1]$, there is a positive number $\tilde r$ such that

$$f_i(t,\mathbf{u})\ge\eta|\mathbf{u}|,\quad t\in[0,1],\ \mathbf{u}\in\mathbb{R}^n \text{ with } |\mathbf{u}|\ge\tilde r,$$

where $\eta>0$ is chosen so that

$$\frac{\lambda\eta\sigma}{2}\min_{i=1,\dots,n}\{m_i\int_0^1 b_i(s)ds\}>1.$$

Let $R_4=\max\{2R_3,\frac{1}{\sigma}\tilde r\}>\Delta$. If $\mathbf{u}\in\partial\Omega_{R_4}$, then $\min_{t\in[0,1]}|\mathbf{u}(t)|\ge\sigma\|\mathbf{u}\|=\sigma R_4\ge\tilde r$, which suggests that

$$f_i(t,\mathbf{u}(t))\ge\eta|\mathbf{u}(t)|,\quad t\in[0,1].$$

Similar to (3.2), we get

$$\|T_\lambda \mathbf{u}\| \geq \lambda \Gamma \eta \|\mathbf{u}\| > \|\mathbf{u}\|, \quad \mathbf{u} \in \partial\Omega_{R_4}.$$

It follows from Lemma 2.2 that T_λ has a fixed point $\mathbf{u}_2 \in \bar{\Omega}_{R_4} \backslash \Omega_{R_3}$, which is a positive 1-periodic solution of (1.3) for $\lambda < \lambda_0$.

Noting that

$$r_2 < \|\mathbf{u}_1\| < r_1 < R_3 < \|\mathbf{u}_2\| < R_4,$$

we can conclude that $\mathbf{u}_1$ and $\mathbf{u}_2$ are the desired distinct positive periodic solutions of (1.3) for $\lambda < \min\{\lambda_0, \lambda_1\}$. $\square$

Example. We consider the first-order singular 2-dimensional system with impulse effect

$$\begin{gathered}
u_1'(t) + (\sin(2\pi t) + \frac{1}{2}) u_1(t) = \lambda(\sin(2\pi t) + 2) f_1(t, u_1, u_2) + \lambda \sin(2\pi t), \\
t \in (0,1) \backslash \{\frac{1}{2}\}, \\
u_2'(t) + (\cos(2\pi t) + \frac{1}{3}) u_2(t) = \lambda(\cos(2\pi t) + 3) f_2(t, u_1, u_2) + \lambda \cos(2\pi t), \\
t \in (0,1) \backslash \{\frac{1}{2}\}, \\
u_1(\frac{1}{2}^+) = u_1(\frac{1}{2}^-) + \lambda(u_1(\frac{1}{2}) + u_2(\frac{1}{2}))^{3/4}, \\
u_2(\frac{1}{2}^+) = u_2(\frac{1}{2}^-) + \lambda(u_1(\frac{1}{2}) + u_2(\frac{1}{2}))^{1/2}, \\
u_1(0) = u_1(1), \quad u_2(0) = u_2(1).
\end{gathered} \tag{3.6}$$

Let

$$\begin{gathered}
f_1(t, u_1, u_2) = 2 + \sin(2\pi t) + \frac{1}{u_1^2 + u_2^3} + (u_1 + u_2)^{1/2}, \\
f_2(t, u_1, u_2) = 3 + \sin(2\pi t) + \frac{1}{u_1 + u_2^2} + (u_1 + u_2)^{1/3}.
\end{gathered}$$

Comparing with (1.3), we have $n = 2$, $p = 1$, $t_1 = 1/2$. Clearly assumptions (H1)–(H3) are satisfied, we can easily check that $\lim_{|\mathbf{u}| \to 0} f_i(t, \mathbf{u}) = \infty$, $\lim_{|\mathbf{u}| \to \infty} f_i(t, \mathbf{u}) = \infty$ and $\lim_{|\mathbf{u}| \to \infty} \frac{f_i(t,\mathbf{u})}{\mathbf{u}} = 0$, $i = 1, 2$ uniformly with respect to $t \in [0, 1]$. So by (i) (ii) of Theorem 1.2, we have: there exists a $\lambda_1 > 0$, such that (3.6) has a positive 1-periodic solution for $0 < \lambda < \lambda_1$ and there exists λ_2, such that (3.6) has a positive 1-periodic solution for $\lambda > \lambda_2$.

Similarly, if we let

$$\begin{gathered}
f_1(t, u_1, u_2) = 2 + \sin(2\pi t) + \frac{1}{u_1^2 + u_2^3} + (u_1 + u_2)^2, \\
f_2(t, u_1, u_2) = 3 + \sin(2\pi t) + \frac{1}{u_1 + u_2^2} + (u_1 + u_2)^3.
\end{gathered}$$

According to (iii) of Theorem 1.2, for sufficiently small $\lambda > 0$, (3.6) has two positive 1-periodic solutions.

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References

[1] D. D. Bainov, P. S. Simeonov; *Impulsive differential equations: periodic solutions and applications*, Longman, Harlow, 1993.
[2] D. D. Bainov, P. S. Simeonov; *Systems with Impulse effect*, Ellis Horwood, Chichester, 1989.
[3] D. D. Bainov, S. G. Hristova, S. Hu, V. Lakshmikantham; *Periodic boundary-value problem for systems of first-order impulsive differential equations*, Differential Integral Equations, 2 (1989), 37-43.
[4] J. Chu, J. Nieto; *Impulsive periodic solutions of first-order singular differential equations*, Bull. London. Math. Soc. 40 (2008), 143-150.
[5] L. H. Erbe, X. Liu; *Existence of periodic solutions of impulsive differential systems*, J. Appl. Math. Stochastic Anal. 4 (1991), 137-146.
[6] M. Frigon, D. O'Regan; *Existence results for first-order impulsive differential equations*, J. Math. Anal. Appl. 193 (1995), 96-113.
[7] D. Guo, V. Lakshmikantham; *Nonlinear Problems in Abstract Cones*, Academic Press, Orlando, FL, 1988.
[8] M. Krasnoselskii; *Positive Solutions of Operator Equations*, Noordhoff, Groningen, 1964.
[9] H. Wang; *Positive periodic solutions of singular systems of first order ordinary differential equations*, Appl. Math. Comput. 8(2011), 1605-1610.

Bian-Xia Yang
School of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu 730000, China
E-mail address: yanglina7765309@163.com