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LYAPUNOV-TYPE INEQUALITIES FOR NONLINEAR SYSTEMS INVOLVING THE $(p_1, p_2, \ldots, p_n)$-LAPLACIAN

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Abstract. We prove some generalized Lyapunov-type inequalities for n-dimensional Dirichlet nonlinear systems. We extend the results obtained by Çakmak and Tiryaki [6] for a parameter $1 < p_k < 2$. As an application, we obtain lower bounds for the eigenvalues of the corresponding system.

1. Introduction

In 1907, Lyapunov [9] obtained the remarkable inequality

$$\int_a^b |f_1(s)|ds \geq \frac{4}{b-a}, \tag{1.1}$$

if Hill's equation

$$x_1'' + f_1(t)x_1 = 0 \tag{1.2}$$

has a real nontrivial solution $x_1(t)$ such that $x_1(a) = 0 = x_1(b)$, where $a, b \in \mathbb{R}$ with $a < b$ are consecutive zeros and x_1 is not identically zero on $[a, b]$, where f_1 is a real-valued continuous function defined on $\mathbb{R}$. We know that the constant 4 in the right hand side of inequality (1.1) cannot be replaced by a larger number (see [7, p. 345]).

Since this result has proved to be a useful tool in oscillation theory, disconjugacy, eigenvalue problems and numerous other applications in the study of various properties of solutions for differential equations, many proofs and generalizations or improvements of it have appeared in the literature. For authors, who contributed to the Lyapunov-type inequalities, we refer to [1-19].

Here, we give some inequalities which are useful in the comparison of our main results. We know that since the function $h(x) = x^{p_k - 1}$ is concave for $x > 0$ and $1 < p_k < 2$, Jensen's inequality $h(\frac{\omega+v}{2}) \geq \frac{1}{2}[h(\omega) + h(v)]$ with $\omega = \frac{1}{c_k - a}$ and $v = \frac{1}{b-c_k}$ for $k = 1, 2, \ldots, n$ implies

$$2^{2-p_k}[\frac{1}{c_k - a} + \frac{1}{b - c_k}]^{p_k - 1} \geq \frac{1}{(c_k - a)^{p_k - 1}} + \frac{1}{(b - c_k)^{p_k - 1}} = m_1(c_k) \tag{1.3}$$

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for $1 < p_k < 2$, $k = 1, 2, \dots, n$. If $p_k > 2$ for $k = 1, 2, \dots, n$, then the function $h(x) = x^{p_k - 1}$ is convex for $x > 0$. Thus, the inequality (1.3) is reversed, i.e.

$$\frac{1}{(c_k - a)^{p_k - 1}} + \frac{1}{(b - c_k)^{p_k - 1}} \geq 2^{2 - p_k} \big[\frac{1}{c_k - a} + \frac{1}{b - c_k}\big]^{p_k - 1} = m_2(c_k) \tag{1.4}$$

for $p_k > 2$, $k = 1, 2, \dots, n$. Moreover, if we obtain the minimum of the right hand side of inequalities (1.3) and (1.4) for $c_k \in (a, b)$, $k = 1, 2, \dots, n$, then it is easy to see that

$$\min_{a < c_k < b} m_i(c_k) = m_i(\frac{a + b}{2}) = \frac{2^{p_k}}{(b - a)^{p_k - 1}} \tag{1.5}$$

for $i = 1, 2$ and $k = 1, 2, \dots, n$.

In 2006, Napoli and Pinasco [10] obtained the following inequality

$$\Big(\int_a^b f_1(s)ds\Big)^{\alpha_1/p_1} \Big(\int_a^b f_2(s)ds\Big)^{\alpha_2/p_2} \geq \frac{2^{\alpha_1 + \alpha_2}}{(b - a)^{\alpha_1 + \alpha_2 - 1}}, \tag{1.6}$$

if the quasilinear system

$$\begin{aligned} -(\phi_{p_1}(x_1'))' &= f_1(t)|x_1|^{\alpha_1 - 2} x_1 |x_2|^{\alpha_2} \\ -(\phi_{p_2}(x_2'))' &= f_2(t)|x_1|^{\alpha_1} |x_2|^{\alpha_2 - 2} x_2 \end{aligned} \tag{1.7}$$

has a real nontrivial solution $(x_1(t), x_2(t))$ such that $x_1(a) = x_1(b) = 0 = x_2(a) = x_2(b)$ where $a, b \in \mathbb{R}$ with $a < b$ consecutive zeros, and x_k for $k = 1, 2$ are not identically zero on $[a, b]$, where $\phi_\alpha(u) = |u|^{\alpha - 2} u$, f_1 and f_2 are real-valued positive continuous functions defined on $\mathbb{R}$, $1 < p_1, p_2 < +\infty$ and the nonnegative parameters α_1, α_2 satisfy $\frac{\alpha_1}{p_1} + \frac{\alpha_2}{p_2} = 1$.

In 2010, Çakmak and Tiryaki [6] obtained the following inequality

$$\prod_{k=1}^n \Big(\int_a^b f_k^+(s)ds\Big)^{\alpha_k/p_k} \geq \prod_{k=1}^n \big[\frac{1}{(c_k - a)^{p_k - 1}} + \frac{1}{(b - c_k)^{p_k - 1}}\big]^{\alpha_k/p_k}, \tag{1.8}$$

where $|x_k(c_k)| = \max_{a < t < b} |x_k(t)|$ and $f_k^+(t) = \max\{0, f_k(t)\}$ for $k = 1, 2, \dots, n$, if the n-dimensional problem

$$\begin{aligned} -(\phi_{p_1}(x_1'))' &= f_1(t)|x_1|^{\alpha_1 - 2} x_1 |x_2|^{\alpha_2} \dots |x_n|^{\alpha_n} \\ -(\phi_{p_2}(x_2'))' &= f_2(t)|x_1|^{\alpha_1} |x_2|^{\alpha_2 - 2} x_2 \dots |x_n|^{\alpha_n} \\ &\dots \\ -(\phi_{p_n}(x_n'))' &= f_n(t)|x_1|^{\alpha_1} |x_2|^{\alpha_2} \dots |x_n|^{\alpha_n - 2} x_n \end{aligned} \tag{1.9}$$

has a real nontrivial solution $(x_1(t), x_2(t), \dots, x_n(t))$ satisfying the Dirichlet boundary conditions

$$x_k(a) = 0 = x_k(b) \tag{1.10}$$

where $a, b \in \mathbb{R}$ with $a < b$ consecutive zeros, $x_k \not\equiv 0$ for $k = 1, 2, \dots, n$ on $[a, b]$. Here, $n \in \mathbb{N}$, $\phi_\alpha(u) = |u|^{\alpha - 2} u$, f_k are real-valued continuous functions defined on $\mathbb{R}$, $1 < p_k < +\infty$ and the nonnegative parameters α_k satisfy $\sum_{k=1}^n \frac{\alpha_k}{p_k} = 1$ for $k = 1, 2, \dots, n$. Using (1.5) in the inequality (1.8), Çakmak and Tiryaki [6] also obtained the inequality

$$\prod_{k=1}^n \Big(\int_a^b f_k^+(s)ds\Big)^{\alpha_k/p_k} \geq \frac{2^{\sum_{k=1}^n \alpha_k}}{(b - a)^{(\sum_{k=1}^n \alpha_k) - 1}}. \tag{1.11}$$

Recently, Yang et al [19] obtained the inequality

$$\int_a^b f_k(s)ds \geq \frac{2^{p_k}}{(b-a)^{p_k-1}} H_k, \tag{1.12}$$

where

$$H_k = \frac{M_k^{p_k-1}}{g_k(M_1, M_2, \dots, M_n)} \tag{1.13}$$

with $M_k = |x_k(c_k)| = \max_{a<t<b} |x_k(t)|$ for $k = 1, 2, \dots, n$, at least one inequality in (1.12) is also strict, if the following nonlinear system involving $(p_1, p_2, \dots, p_n)$-Laplacian operators

$$\begin{gathered}
(\phi_{p_1}(x_1'))' + F_1(t, x_1, x_2, \dots, x_n) = 0 \\
(\phi_{p_2}(x_2'))' + F_2(t, x_1, x_2, \dots, x_n) = 0 \\
\dots \\
(\phi_{p_n}(x_n'))' + F_n(t, x_1, x_2, \dots, x_n) = 0
\end{gathered} \tag{1.14}$$

has a real nontrivial solution $(x_1(t), x_2(t), \dots, x_n(t))$ satisfying the boundary condition (1.10), where $n \in \mathbb{N}$, $\phi_\alpha(u) = |u|^{\alpha-2}u$, $1 < p_k < +\infty$ and $F_k \in C([a,b]\times\mathbb{R}^n, \mathbb{R})$ for $k = 1, 2, \dots, n$, under the following hypothesis:

(C1) There exist the functions $f_k \in C([a,b], [0,\infty))$ and $g_k \in C(\mathbb{R}^n, [0,\infty))$ for $k = 1, 2, \dots, n$ such that

$$|F_k(t, x_1, x_2, \dots, x_n)| \leq f_k(t) g_k(x_1, x_2, \dots, x_n) \tag{1.15}$$

and

$$g_k(x_1, x_2, \dots, x_n) \text{ is monotonic nondecreasing in each variable} \tag{1.16}$$

for $k = 1, 2, \dots, n$.

Yang et al [19] claim that the inequality (1.11) with $f_k(t) > 0$ for $k = 1, 2, \dots, n$ of Çakmak and Tiryaki [6] can be obtained by using the inequality (1.12) under the following conditions

$$F_k(t, x_1, x_2, \dots, x_n) = f_k(t) g_k(x_1, x_2, \dots, x_n), \quad k = 1, 2, \dots, n, \tag{1.17}$$

where $g_k(x_1, x_2, \dots, x_n) = |z_k(x_1, x_2, \dots, x_n)|$ with

$$\begin{gathered}
z_1(x_1, x_2, \dots, x_n) = |x_1|^{\alpha_1-2}x_1|x_2|^{\alpha_2}\dots|x_n|^{\alpha_n} \\
z_2(x_1, x_2, \dots, x_n) = |x_1|^{\alpha_1}|x_2|^{\alpha_2-2}x_2\dots|x_n|^{\alpha_n} \\
\dots \\
z_n(x_1, x_2, \dots, x_n) = |x_1|^{\alpha_1}|x_2|^{\alpha_2}\dots|x_n|^{\alpha_n-2}x_n,
\end{gathered} \tag{1.18}$$

where $\alpha_k \geq 0$ for $k = 1, 2, \dots, n$ such that $\sum_{k=1}^n \frac{\alpha_k}{p_k} = 1$. It is easy to see from (1.16) that the nondecreasing condition on each variable of g_k with (1.18) for $k = 1, 2, \dots, n$ is not satisfied. Therefore, [19, Remarks 1–3, Corollary 3] fail. So, [19, Corollary 3] does not apply to this example.

Now, we present the following hypothesis instead of (C1):

(C1*) There exist the functions $f_k \in C([a,b], [0,\infty))$ and $g_k \in C(\mathbb{R}^n, [0,\infty))$ for $k = 1, 2, \dots, n$ such that

$$|F_k(t, x_1, x_2, \dots, x_n)| \leq f_k(t) g_k(|x_1|, |x_2|, \dots, |x_n|) \tag{1.19}$$

and $g_k(u_1, u_2, \dots, u_n)$ is monotonic nondecreasing in each variable u_i, such that either $g_k(0, 0, \dots, 0) = 0$ or $g_k(u_1, u_2, \dots, u_n) > 0$ for at least one $u_i \neq 0$ for $i = 1, 2, \dots, n$, for $k = 1, 2, \dots, n$.

It is clear that if the hypothesis (C1) is replaced by (C1*) for system (1.14), then (1.11) with $f_k(t) > 0$ for $k = 1, 2, \dots, n$ of Çakmak and Tiryaki [6] obtain by using inequality (1.12) under the condition $\alpha_k \geq 1$ for $k = 1, 2, \dots, n$.

In this article, our purpose is to obtain Lyapunov-type inequalities for system (1.14) similar to the ones given in Yang et al [19] by imposing somewhat different conditions on the function F_k for $k = 1, 2, \dots, n$, and improve and generalize the results of Çakmak and Tiryaki [6] when $1 < p_k < 2$ for $k = 1, 2, \dots, n$. In addition, the positivity conditions on the function f_k for $k = 1, 2, \dots, n$ in hypothesis (C1) are dropped. We also obtain a better lower bound for the eigenvalues of corresponding system as an application.

We derive some Lyapunov-type inequalities for system (1.14), where all components of the solution $(x_1(t), x_2(t), \dots, x_n(t))$ have consecutive zeros at the points $a, b \in \mathbb{R}$ with $a < b$ in $I = [t_0, \infty) \subset \mathbb{R}$. For system (1.14), we also derive some Lyapunov-type inequalities which relate not only points a and b in I at which all components of the solution $(x_1(t), x_2(t), \dots, x_n(t))$ have consecutive zeros but also a point in (a, b) where all components of the solution $(x_1(t), x_2(t), \dots, x_n(t))$ are maximized.

Since our attention is restricted to the Lyapunov-type inequalities for system of differential equations, we shall assume the existence of the nontrivial solution $(x_1(t), x_2(t), \dots, x_n(t))$ of system (1.14).

2. Main results

We give the following hypothesis for system (1.14).

(C2) There exist the functions $f_k \in C([a, b], \mathbb{R})$ and $g_k \in C(\mathbb{R}^n, [0, \infty))$ such that

$$F_k(t, x_1, x_2, \dots, x_n)x_k \leq f_k(t)g_k(|x_1|, |x_2|, \dots, |x_n|) \tag{2.1}$$

and

$$\begin{gathered} g_k(u_1, u_2, \dots, u_n) \text{ is monotonic nondecreasing in each} \\ \text{variable } u_i \text{ such that either } g_k(0, 0, \dots, 0) = 0 \text{ or} \\ g_k(u_1, u_2, \dots, u_n) > 0 \text{ for at least one } u_i \neq 0,\ i = 1, 2, \dots, n, \end{gathered} \tag{2.2}$$

for $k = 1, 2, \dots, n$.

One of the main results of this article is the following theorem, whose proof is different from the that of [19, Theorem 1] and modified that of [13, Theorem 2.1].

Theorem 2.1. *Assume that hypothesis* (C2) *is satisfied. If* (1.14) *has a real nontrivial solution* $(x_1(t), x_2(t), \dots, x_n(t))$ *satisfying the boundary condition* (1.10)*, then the inequalities*

$$\int_a^b f_k^+(s)ds \geq 2^{2-p_k}\Big[\frac{1}{c_k - a} + \frac{1}{b - c_k}\Big]^{p_k-1} M_k H_k \tag{2.3}$$

hold, where $f_k^+(t) = \max\{0, f_k(t)\}$*, and* H_k*,* M_k *for* $k = 1, 2, \dots, n$ *are as in* (1.13)*. Moreover, at least one inequality in* (2.3) *is strict.*

Proof. Let the boundary condition (1.10) hold; i.e., $x_k(a) = 0 = x_k(b)$ for $k = 1, 2, \dots, n$ where $n \in \mathbb{N}$, $a, b \in \mathbb{R}$ with $a < b$ consecutive zeros and x_k for $k =$

$1, 2, \dots, n$ are not identically zero on $[a, b]$. Thus, by Rolle's theorem, we can choose $c_k \in (a, b)$ such that

$$M_k = \max_{a<t<b} |x_k(t)| = |x_k(c_k)| \quad \text{and} \quad x_k'(c_k) = 0$$

for $k = 1, 2, \dots, n$. By using $x_k(a) = 0$ and Hölder's inequality, we obtain

$$|x_k(c_k)| \le \int_a^{c_k} |x_k'(s)| ds \le (c_k - a)^{(p_k-1)/p_k} \Big(\int_a^{c_k} |x_k'(s)|^{p_k} ds \Big)^{1/p_k} \tag{2.4}$$

and hence

$$|x_k(c_k)|^{p_k} \le (c_k - a)^{p_k-1} \int_a^{c_k} |x_k'(s)|^{p_k} ds \tag{2.5}$$

for $k = 1, 2, \dots, n$ and $c_k \in (a, b)$. Similarly, by using $x_k(b) = 0$ and Hölder's inequality, we obtain

$$|x_k(c_k)|^{p_k} \le (b - c_k)^{p_k-1} \int_{c_k}^b |x_k'(s)|^{p_k} ds \tag{2.6}$$

for $k = 1, 2, \dots, n$ and $c_k \in (a, b)$. Multiplying the inequalities (2.5) and (2.6) by $(b - c_k)^{p_k-1}$ and $(c_k - a)^{p_k-1}$ for $k = 1, 2, \dots, n$, respectively, we obtain

$$(b - c_k)^{p_k-1} |x_k(c_k)|^{p_k} \le [(c_k - a)(b - c_k)]^{p_k-1} \int_a^{c_k} |x_k'(s)|^{p_k} ds \tag{2.7}$$

and

$$(c_k - a)^{p_k-1} |x_k(c_k)|^{p_k} \le [(c_k - a)(b - c_k)]^{p_k-1} \int_{c_k}^b |x_k'(s)|^{p_k} ds \tag{2.8}$$

for $k = 1, 2, \dots, n$ and $c_k \in (a, b)$. Thus, adding the inequalities (2.7) and (2.8), we have

$$|x_k(c_k)|^{p_k} [(b - c_k)^{p_k-1} + (c_k - a)^{p_k-1}] \le [(c_k - a)(b - c_k)]^{p_k-1} \int_a^b |x_k'(s)|^{p_k} ds \tag{2.9}$$

for $k = 1, 2, \dots, n$ and $c_k \in (a, b)$. It is easy to see that the functions $z_k(x) = (b - x)^{p_k-1} + (x - a)^{p_k-1}$ take the minimum values at $\frac{a+b}{2}$; i.e.,

$$z_k(x) \ge \min_{a<x<b} z_k(x) = z_k(\frac{a + b}{2}) = 2(\frac{b - a}{2})^{p_k-1}$$

for $k = 1, 2, \dots, n$. Thus, we obtain

$$|x_k(c_k)|^{p_k} [2(\frac{b - a}{2})^{p_k-1}] \le [(c_k - a)(b - c_k)]^{p_k-1} \int_a^b |x_k'(s)|^{p_k} ds \tag{2.10}$$

and hence

$$2M_k^{p_k} = 2|x_k(c_k)|^{p_k} \le [\frac{2}{b - a}(c_k - a)(b - c_k)]^{p_k-1} \int_a^b |x_k'(s)|^{p_k} ds \tag{2.11}$$

for $k = 1, 2, \dots, n$ and $c_k \in (a, b)$. Multiplying the k-th equation of system (1.14) by $x_k(t)$, integrating from a to b by using integration by parts and taking into account that $x_k(a) = 0 = x_k(b)$ and the inequalities (2.1) for $k = 1, 2, \dots, n$, then the monotonicity of g_k yields

$$\begin{aligned} \int_a^b |x_k'(s)|^{p_k} ds &= \int_a^b F_k(s, x_1(s), x_2(s), \dots, x_n(s)) x_k(s) ds \\ &\le \int_a^b f_k(s) g_k(|x_1(s)|, |x_2(s)|, \dots, |x_n(s)|) ds \\ &\le \int_a^b f_k^+(s) g_k(|x_1(s)|, |x_2(s)|, \dots, |x_n(s)|) ds \\ &= g_k(M_1, M_2, \dots, M_n) \int_a^b f_k^+(s) ds. \end{aligned} \tag{2.12}$$

Then, using (2.12) in (2.11), we have

$$\int_a^b f_k^+(s) ds \ge \frac{2M_k^{p_k}}{g_k(M_1, M_2, \dots, M_n)} \Big[\frac{b-a}{2(c_k - a)(b - c_k)}\Big]^{p_k - 1} \tag{2.13}$$

for $k = 1, 2, \dots, n$. Since $(x_1(t), x_2(t), \dots, x_n(t))$ is a nontrivial solution of system (1.14), it is easy to see that at least one inequality in (2.13) is strict, which completes the proof. □

Another main result of this paper is the following theorem whose proof is almost the same to that of [19, Theorem 1]; hence it is omitted.

Theorem 2.2. *Let all the assumptions of Theorem 2.1 hold. Then the inequality*

$$\int_a^b f_k^+(s) ds \ge \Big[\frac{1}{(c_k - a)^{p_k - 1}} + \frac{1}{(b - c_k)^{p_k - 1}}\Big] M_k H_k \tag{2.14}$$

holds, where $f_k^+(t)$, H_k and M_k for $k = 1, 2, \dots, n$ are as in Theorem 2.1. Moreover, at least one inequality in (2.14) is strict.

Remark 2.3. The right-hand side of inequalities (2.3) in Theorem 2.1 or (2.14) in Theorem 2.2 shows that c_k, for $k = 1, 2, \dots, n$, cannot be too close to a or b, since the exponents satisfy $1 < p_k < +\infty$ for $k = 1, 2, \dots, n$. We have $\int_a^b f_k^+(s) ds < \infty$ for $k = 1, 2, \dots, n$, but

$$\lim_{c_k \to a^+,\, c_k \to b^-} \Big[\frac{1}{c_k - a} + \frac{1}{b - c_k}\Big]^{p_k - 1} = \infty, \text{ or}$$
$$\lim_{c_k \to a^+,\, c_k \to b^-} \Big[\frac{1}{(c_k - a)^{p_k - 1}} + \frac{1}{(b - c_k)^{p_k - 1}}\Big] = \infty$$

for $k = 1, 2, \dots, n$.

Now, according to the value of p_k, we compare Theorem 2.1 with Theorem 2.2 as follows.

Remark 2.4. It is easy to see from inequality (1.3) that if we take $1 < p_k < 2$, for $k = 1, 2, \dots, n$, then inequality (2.3) is better than (2.14) in the sense that (2.14) follows from (2.3), but not conversely. Similarly, from inequality (1.4), if $p_k > 2$, for $k = 1, 2, \dots, n$, then inequality (2.14) is better than (2.3) in the sense that (2.3) follows from (2.14), but not conversely. Moreover, if $p_k = 2$ or $c_k = \frac{a+b}{2}$ for $k = 1, 2, \dots, n$, then Theorem 2.1 is exactly the same as Theorem 2.2.

By using (1.5) in Theorem 2.1 or 2.2, we obtain the following result.

Theorem 2.5. *Let all the assumptions of Theorem 2.1 hold. Then the inequality*

$$\int_a^b f_k^+(s)ds \geq \frac{2^{p_k}}{(b-a)^{p_k-1}} M_k H_k \tag{2.15}$$

holds, where $f_k^+(t)$, H_k and M_k for $k = 1, 2, \dots, n$ are as in Theorem 2.1. Moreover, at least one inequality in (2.15) *is strict.*

Now, we present the following hypothesis which gives the importance of our theorems for system (1.9).

(C3) There exist the functions $f_k \in C([a, b], \mathbb{R})$ and $g_k \in C(\mathbb{R}^n, [0, \infty))$ such that

$$F_k(t, x_1, x_2, \dots, x_n)x_k = f_k(t)g_k(|x_1|, |x_2|, \dots, |x_n|) \tag{2.16}$$

and

$g_k(u_1, u_2, \dots, u_n)$ is monotonic nondecreasing in each variable u_i such that either $g_k(0, 0, \dots, 0) = 0$ or $g_k(u_1, u_2, \dots, u_n) > 0$ for at least one $u_i \neq 0$ for $i = 1, 2, \dots, n$, (2.17)

where $g_k(|x_1|, |x_2|, \dots, |x_n|) = x_k z_k(x_1, x_2, \dots, x_n)$ with (1.18) for $k = 1, 2, \dots, n$ such that $\alpha_k \geq 0$ and $\sum_{k=1}^n \frac{\alpha_k}{p_k} = 1$.

It is easy to see that system (1.14) with hypothesis (C3) reduces to system (1.9). Since

$$\prod_{k=1}^n (M_k H_k)^{\alpha_k/p_k} = 1, \tag{2.18}$$

we have the following results from Theorems 2.1 and 2.2, respectively.

Theorem 2.6. *Assume that hypothesis* (C3) *is satisfied. If* (1.14) *has a real nontrivial solution* $(x_1(t), x_2(t), \dots, x_n(t))$ *satisfying the boundary condition* (1.10), *then*

$$\prod_{k=1}^n \Big(\int_a^b f_k^+(s)ds\Big)^{\alpha_k/p_k} \geq \prod_{k=1}^n \big[2^{2-p_k}(\frac{1}{c_k - a} + \frac{1}{b - c_k})^{p_k-1}\big]^{\alpha_k/p_k}, \tag{2.19}$$

where $|x_k(c_k)| = \max_{a<t<b} |x_k(t)|$ and $f_k^+(t) = \max\{0, f_k(t)\}$ for $k = 1, 2, \dots, n$. Moreover, at least one inequality in (2.19) *is strict.*

Theorem 2.7. *Let all the assumptions of Theorem 2.6 hold. Then the inequality*

$$\prod_{k=1}^n \Big(\int_a^b f_k^+(s)ds\Big)^{\alpha_k/p_k} \geq \prod_{k=1}^n \big[\frac{1}{(c_k - a)^{p_k-1}} + \frac{1}{(b - c_k)^{p_k-1}}\big]^{\alpha_k/p_k} \tag{2.20}$$

holds, where c_k and $f_k^+(t)$ for $k = 1, 2, \dots, n$ are as in Theorem 2.6. Moreover, at least one inequality in (2.20) *is strict.*

By using (1.5) in Theorem 2.6 or 2.7 and (2.18) in Theorem 2.5, we have the following result.

Corollary 2.8. *Let all the assumptions of Theorem 2.6 hold. Then the inequality*

$$\prod_{k=1}^n \Big(\int_a^b f_k^+(s)ds\Big)^{\alpha_k/p_k} \geq \frac{2^{\sum_{k=1}^n \alpha_k}}{(b-a)^{(\sum_{k=1}^n \alpha_k)-1}} \tag{2.21}$$

holds, where $f_k^+(t)$ for $k = 1,\ 2, \dots, n$ is as in Theorem 2.6. Moreover, at least one inequality in (2.21) *is strict.*

Remark 2.9. It is easy to see from (1.3) that if we take $1 < p_k < 2$ for $k = 1, 2, \dots, n$, then (2.19) is better than (1.8) in the sense that (1.8) follows from (2.19), but not conversely. Similarly, from (1.4), if $p_k > 2$ for $k = 1, 2, \dots, n$, then (1.8) is better than (2.19) in the sense that (2.19) follows from (1.8), but not conversely.

Remark 2.10. It is easy to see that inequality (2.20) is exactly the same as (1.8), and (2.21) is exactly the same as (1.11).

Remark 2.11. When $\alpha_k = p_k$ for $k = 1, 2, \dots, n$, and for $i \neq k$, $\alpha_i = 0$ for $i = 1, 2, \dots, n$ in system (1.9), we obtain the result for the case of a single equation from Theorems 2.6, 2.7 or Corollary 2.8.

Remark 2.12. Since $|f(x)| \geq f^+(x)$, the integrals of $\int_a^b f_k^+(s)ds$ for $k = 1, 2, \dots, n$ in the above results can also be replaced by $\int_a^b |f_k(s)|ds$ for $k = 1, 2, \dots, n$, respectively.

3. Applications

In this section, we present some applications of the Lyapunov-type inequalities obtained in Section 2.

Firstly, we give the same example of Yang et al [19] which gives the importance of our results. Note that our Corollary 2.8 is applicable to the following example, but [19, Corollary 3] is not applicable to it, since the nondecreasing condition on each variable of g_k for $k = 1, 2, \dots, n$ is not satisfied.

Example 3.1. Consider the quasilinear system

$$\begin{gathered}(\phi_{p_1}(x_1'))' + f_1(t)(3 + \sin 2x_1)|x_1|^{\alpha_1 - 2}x_1|x_2|^{\alpha_2 - 1}x_2 = 0\\ (\phi_{p_2}(x_2'))' + f_2(t)(1 + \sin^2 2x_2)|x_1|^{\alpha_1 - 1}x_1|x_2|^{\alpha_2 - 2}x_2 = 0,\end{gathered} \tag{3.1}$$

where $\phi_\alpha(u) = |u|^{\alpha - 2}u$, $p_1, p_2 > 1$, $\alpha_1, \alpha_2 \geq 0$ with $\frac{\alpha_1}{p_1} + \frac{\alpha_2}{p_2} = 1$, f_1 and f_2 are nonnegative continuous functions on $[a, b]$. Assume that system (3.1) has a real nontrivial solution $(x_1(t), x_2(t))$ satisfying the Dirichlet boundary condition $x_1(a) = x_1(b) = 0 = x_2(a) = x_2(b)$. Since

$$\begin{gathered}F_1(t, x_1, x_2)x_1 \leq 4f_1(t)|x_1|^{\alpha_1}|x_2|^{\alpha_2} \quad \text{and}\\ F_2(t, x_1, x_2)x_2 \leq 2f_2(t)|x_1|^{\alpha_1}|x_2|^{\alpha_2},\end{gathered} \tag{3.2}$$

where $g_k(u_1, u_2) = u_1^{\alpha_1}u_2^{\alpha_2}$ for $k = 1, 2$ which are satisfied the nondecreasing condition on each variable u_i for $i = 1, 2$, we have the following inequalities

$$4\int_a^b f_1(s)ds > \frac{2^{p_1}}{(b-a)^{p_1 - 1}}M_1H_1, \quad 2\int_a^b f_2(s)ds > \frac{2^{p_2}}{(b-a)^{p_2 - 1}}M_2H_2 \tag{3.3}$$

with $M_1H_1 = M_1^{p_1 - \alpha_1}M_2^{-\alpha_2}$ and $M_2H_2 = M_1^{-\alpha_1}M_2^{p_2 - \alpha_2}$ from Theorem 2.6. Hence, we have

$$\Big(\int_a^b f_1(s)ds\Big)^{\frac{\alpha_1}{p_1}}\Big(\int_a^b f_2(s)ds\Big)^{\frac{\alpha_2}{p_2}} > \frac{2^{\alpha_1 + \alpha_2 - \frac{\alpha_1}{p_1} - 1}}{(b-a)^{\alpha_1 + \alpha_2 - 1}} \tag{3.4}$$

from Corollary 2.8.

Secondly, we give another application of the Lyapunov-type inequalities obtained for system (1.9). Note that the lower bounds are found by using inequality (2.20) in Theorem 2.7 coincide with that of [6, Theorem 9]. Now, we present new lower bounds by using inequality (2.19) in Theorem 2.6 which give a better lower bound for the eigenvalues of following system than that of [6, Theorem 9] when $1 < p_k < 2$ for $k = 1, 2, \dots, n$.

Let λ_k for $k = 1, 2, \dots, n$ be generalized eigenvalues of system (1.9), and $r(t)$ be a positive function for all $t \in \mathbb{R}$. Therefore, system (1.9) with $f_k(t) = \lambda_k \alpha_k r(t) > 0$ for $k = 1, 2, \dots, n$ and all $t \in \mathbb{R}$ reduces to the system

$$\begin{gathered} -(|x_1'|^{p_1-2}x_1')' = \lambda_1\alpha_1 r(t)|x_1|^{\alpha_1-2}x_1|x_2|^{\alpha_2}\dots|x_n|^{\alpha_n} \\ -(|x_2'|^{p_2-2}x_2')' = \lambda_2\alpha_2 r(t)|x_1|^{\alpha_1}|x_2|^{\alpha_2-2}x_2\dots|x_n|^{\alpha_n} \\ \dots \\ -(|x_n'|^{p_n-2}x_n')' = \lambda_n\alpha_n r(t)|x_1|^{\alpha_1}|x_2|^{\alpha_2}\dots|x_n|^{\alpha_n-2}x_n\,. \end{gathered} \tag{3.5}$$

By using similar techniques to the technique in [6], we obtain the following result which gives lower bounds for the n-th eigenvalue λ_n. The proof of following theorem is based on above generalization of the Lyapunov-type inequality, as in that of [6, Theorem 9] and hence is omitted.

Theorem 3.2. *There exist a function $k_1(\lambda_1, \lambda_2, \dots, \lambda_{n-1})$ such that*

$$\lambda_n \ge k_1(\lambda_1, \lambda_2, \dots, \lambda_{n-1}) \tag{3.6}$$

for every generalized eigenvalue $(\lambda_1, \lambda_2, \dots, \lambda_n)$ of system (3.5), where $|x_k(c_k)| = \max_{a<t<b}|x_k(t)|$ for $k = 1, 2, \dots, n$ and

$$\begin{aligned} &k_1(\lambda_1, \lambda_2, \dots, \lambda_{n-1}) \\ &= \frac{1}{\alpha_n}\Big\{\prod_{k=1}^{n}\big[2^{2-p_k}(\tfrac{1}{c_k-a}+\tfrac{1}{b-c_k})^{p_k-1}\big]^{\alpha_k/p_k}\big[\prod_{k=1}^{n-1}(\lambda_k\alpha_k)^{\alpha_k/p_k}\int_a^b r(s)ds\big]^{-1}\Big\}^{p_n/\alpha_n}. \end{aligned} \tag{3.7}$$

Remark 3.3. Let $1 < p_k < 2$ for $k = 1, 2, \dots, n$. If we compare Theorem 3.2 with [6, Theorem 9], we obtain $k_1(\lambda_1, \lambda_2, \dots, \lambda_{n-1}) \ge h_1(\lambda_1, \lambda_2, \dots, \lambda_{n-1})$ since the inequality (1.3) holds. Thus, Theorem 3.2 gives a better lower bound than [6, Theorem 9].

Remark 3.4. Since k_1 is a continuous function, $k_1(\lambda_1, \lambda_2, \dots, \lambda_{n-1}) \to +\infty$ as any eigenvalue of $\lambda_k \to 0^+$ for $k = 1, 2, \dots, n-1$. Therefore, there exists a ball centered in the origin such that the generalized spectrum is contained in its exterior. Also, by rearranging terms in (3.6) we obtain

$$\prod_{k=1}^{n}\lambda_k^{\alpha_k/p_k} \ge \prod_{k=1}^{n}[2^{2-p_k}(\frac{1}{c_k-a}+\frac{1}{b-c_k})^{p_k-1}]^{\alpha_k/p_k}\Big[\prod_{k=1}^{n}\alpha_k^{\alpha_k/p_k}\int_a^b r(s)ds\Big]^{-1}. \tag{3.8}$$

It is clear that when the interval collapses, right-hand side of (3.8) approaches infinity.

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