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SELFADJOINT EXTENSIONS OF A SINGULAR MULTIPOINT DIFFERENTIAL OPERATOR OF FIRST ORDER

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Abstract. In this work, we describe all selfadjoint extensions of the minimal operator generated by linear singular multipoint symmetric differential expression $l = (l_1, l_2, l_3)$, $l_k = i\frac{d}{dt} + A_k$ with selfadjoint operator coefficients A_k, $k = 1, 2, 3$ in a Hilbert space. This is done as a direct sum of Hilbert spaces of vector-functions

$$L_2(H, (-\infty, a_1)) \oplus L_2(H, (a_2, b_2)) \oplus L_2(H, (a_3, +\infty))$$

where $-\infty < a_1 < a_2 < b_2 < a_3 < +\infty$. Also, we study the structure of the spectrum of these extensions.

1. Introduction

Many problems arising in modeling processes in multi-particle quantum mechanics, in quantum field theory, in multipoint boundary value problems for differential equations, and in the physics of rigid bodies use selfadjoint extensions of symmetric differential operators as a direct sum of Hilbert spaces [1, 11, 12].

The general theory of selfadjoint extensions of symmetric operators in Hilbert spaces and their spectral theory have deeply been investigated by many mathematicians; see for example [3, 6, 8, 9]. Applications of this theory to two-point differential operators in Hilbert space of functions have been even continued up to date.

It is well-known that for the existence of selfadjoint extension of any linear closed densely defined symmetric operator B in a Hilbert space H, necessary and sufficient condition is a equality of deficiency indices $m(B) = n(B)$, where $m(B) = \dim\ker(B^* + i)$, $n(B) = \dim\ker(B^* - i)$.

However multipoint situations may occur in different tables in the following sense. Let B_1 and B_2 be minimal operators generated by the linear differential expression $i\frac{d}{dt}$ in the Hilbert space of functions $L_2(-\infty, a)$ and $L_2(b, +\infty)$, $a < b$, respectively. In this case, it is known that

$$(m(B_1), n(B_1)) = (0, 1), \quad (m(B_2), n(B_2)) = (1, 0).$$

Consequently, B_1 and B_2 are maximal symmetric operators, but they are not selfadjoint. However, direct sum $B = B_1 \oplus B_2$ of operators in the direct sum

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$\mathfrak{H} = L_2(-\infty, a) \oplus L_2(b, +\infty)$ spaces have equal defect numbers $(1, 1)$. Then by the general theory [8] it has a selfadjoint extension. On the other hand, it can be easily shown that

$$u_2(b) = e^{i\varphi} u_1(a), \quad \varphi \in [0, 2\pi), \quad u = (u_1, u_2), \quad u_1 \in D(B_1^*), \quad u_2 \in D(B_2^*).$$

In the singular cases, there has been no investigation so far. However, in physical and technical processes, many of the problems resulting from the examination of the solution is of great importance in singular cases.

The selfadjoint extension theory for ode's is known for any number of intervals, finite or infinite, and any order expressions, see [4]. This theory is based on the GKN (Glazmann-Krein-Naimark) Theory [7].

In this work in section 2, by the method of Calkin-Gorbachuk (see [2, 6, 9]), we describe all selfadjoint extensions of the minimal operator generated by singular multipoint symmetric differential operator of first order in the direct sum of Hilbert space

$$L_2(H, (-\infty, a_1)) \oplus L_2(H, (a_2, b_2)) \oplus L_2(H, (a_3, +\infty)),$$

where $-\infty < a_1 < a_2 < b_2 < a_3 < +\infty$ in terms of boundary values.In section 3, the spectrum of such extensions is studied.

2. Description of selfadjoint extensions

Let H be a separable Hilbert space and $a_1, a_2, b_2, a_3 \in \mathbb{R}$, $a_1 < a_2 < b_2 < a_3$. In the Hilbert space $L_2(H, (-\infty, a_1)) \oplus L_2(H, (a_2, b_2)) \oplus L_2(H, (a_3, +\infty))$ of vector-functions let us consider the linear multipoint differential expression

$$l(u) = (l_1(u_1), l_2(u_2), l_3(u_3)) = (iu_1' + A_1 u_1, iu_2' + A_2 u_2, iu_3' + A_3 u_3),$$

where $u = (u_1, u_2, u_3)$, $A_k : D(A_k) \subset H \to H$, $k = 1, 2, 3$ are linear selfadjoint operators in H. In the linear manifold $D(A_k) \subset H$ introduce the inner product

$$(f, g)_{k,+} := (A_k f A_k, g)_H + (f, g)_H, \quad f, g \in D(A_k),\ k = 1, 2, 3.$$

For $k = 1, 2, 3$, $D(A_k)$ is a Hilbert space under the positive norm $\|\cdot\|_{k,+}$ with respect to the Hilbert space H. It is denoted by $H_{k,+}$. Denote $H_{k,-}$ a Hilbert space with the negative norm. It is clear that an operator A_k is continuous from $H_{k,+}$ to H and that its adjoint operator $\tilde{A}_k : H \to H_{k,-}$ is a extension of the operator A_k. On the other hand, $\tilde{A}_k : D(\tilde{A}_k) = H \subset H_{k,-1} \to H_{k,-1}$ is a linear selfadjoint operator.

In the direct sum, $L_2(H, (-\infty, a_1)) \oplus L_2(H, (a_2, b_2)) \oplus L_2(H, (a_3, +\infty))$ define by

$$\tilde{l}(u) = (\tilde{l}_1(u_1), \tilde{l}_2(u_2), \tilde{l}_3(u_3)), \tag{2.1}$$

where $u = (u_1, u_2, u_3)$ and $\tilde{l}_1(u_1) = iu_1' + \tilde{A}_1 u_1$, $\tilde{l}_2(u_2) = iu_2' + \tilde{A}_2 u_2$, $\tilde{l}_3(u_3) = iu_3' + \tilde{A}_3 u_3$.

The minimal L_{10} (L_{20} and L_{30}) and maximal L_1 (L_2 and L_3) operators generated by differential expression $\tilde{l}_1$ ($\tilde{l}_2$ and $\tilde{l}_3$) in $L_2(H, (-\infty, a_1))$ ($L_2(H, (a_2, b_2))$ and $L_2(H, (b, +\infty))$) have been investigation in [5].

The operators $L_0 = L_{10} \oplus L_{20} \oplus L_{30}$ and $L = L_1 \oplus L_2 \oplus L_3$ in the space $L_2 = L_2(H, (-\infty, a_1)) \oplus L_2(H, (a_2, b_2)) \oplus L_2(H, (a_3, +\infty))$ are called minimal and maximal (multipoint) operators generated by the differential expression (2.1), respectively. Note that the operator L_0 is symmetric and $L_0^* = L$ in L_2. On the other hand, it is clear that, $m(L_{10}) = 0$, $n(L_{10}) = \dim H$, $m(L_{20}) = \dim H$, $n(L_{20}) = \dim H$, $m(L_{30}) = \dim H$, $n(L_{30}) = 0$.

Consequently, $m(L_0) = n(L_0) = 2\dim H > 0$. Hence, the minimal operator L_0 has a selfadjoint extension [8]. For example, the differential expression $\tilde{l}(u)$ with the boundary condition $u(a_1) = u(a_3)$, $u(a_2) = u(b_2)$ generates a selfadjoint operator in L_2.

All selfadjoint extensions of the minimal operator L_0 in L_2 in terms of the boundary values are described.

Note that space of boundary values has an important role in the theory of selfadjoint extensions of linear symmetric differential operators [6, 9].

Let $B : D(B) \subset \mathcal{H} \to \mathcal{H}$ be a closed densely defined symmetric operator in the Hilbert space $\mathcal{H}$, having equal finite or infinite deficiency indices. A triplet $(\mathfrak{H}, \gamma_1, \gamma_2)$, where $\mathfrak{H}$ is a Hilbert space, γ_1 and γ_2 are linear mappings of $D(B^*)$ into $\mathfrak{H}$, is called a space of boundary values for the operator B if for any $f, g \in D(B^*)$

$$(B^*f, g)_{\mathcal{H}} - (f, B^*g)_{\mathcal{H}} = (\gamma_1(f), \gamma_2(g))_{\mathfrak{H}} - (\gamma_2(f), \gamma_1(g))_{\mathfrak{H}},$$

while for any $F, G \in \mathfrak{H}$, there exists an element $f \in D(B^*)$, such that $\gamma_1(f) = F$ and $\gamma_2(f) = G$.

Note that any symmetric operator with equal deficiency indices has at least one space of boundary values [6].

Firstly, note that the following proposition which validity of this claim can be easily proved.

Lemma 2.1. *The triplet (H, γ_1, γ_2), where*

$$\gamma_1 : D((L_{10} \oplus 0 \oplus L_{30})^*) \to H, \quad \gamma_1(u) = \frac{1}{i\sqrt{2}}(u_1(a_1) + u_3(a_3)),$$
$$\gamma_2 : D((L_{10} \oplus 0 \oplus L_{30})^*) \to H, \quad \gamma_2(u) = \frac{1}{\sqrt{2}}(u_1(a_1) - u_3(a_3)),$$
$$u = (u_1, u_2, u_3) \in D((L_{10} \oplus 0 \oplus L_{30})^*)$$

is a space of boundary values of the minimal operator $L_{10} \oplus 0 \oplus L_{30}$ in the direct sum $L_2(H, (-\infty, a_1)) \oplus 0 \oplus L_2(H, (a_3, +\infty))$.

Proof. For arbitrary $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ from $D((L_{10} \oplus 0 \oplus L_{30})^*)$ the validity of the equality

$$\begin{aligned}&(Lu, v)_{L_2(H,(-\infty,a_1))\oplus 0\oplus L_2(H,(a_3,+\infty))} - (u, Lv)_{L_2(H,(-\infty,a_1))\oplus 0\oplus L_2(H,(a_3,+\infty))}\\ &= (\gamma_1(u), \gamma_2(v))_H - (\gamma_2(u), \gamma_1(v))_H\end{aligned}$$

can be easily verified. Now for any given elements $f, g \in H$, we will find the function $u = (u_1, u_2, u_3) \in D((L_{10} \oplus 0 \oplus L_{30})^*)$ such that

$$\gamma_1(u) = \frac{1}{i\sqrt{2}}(u_1(a_1) + u_3(a_3)) = f \quad \text{and} \quad \gamma_2(u) = \frac{1}{\sqrt{2}}(u_1(a_1) - u_3(a_3)) = g;$$

that is,

$$u_1(a_1) = (if + g)/\sqrt{2} \quad \text{and} \quad u_3(a_3) = (if - g)/\sqrt{2}.$$

If we choose the functions $u_1(t), u_3(t)$ in the form

$$u_1(t) = \int_{-\infty}^{t} e^{s-a_1} ds (if + g)/\sqrt{2} \quad \text{with } t < a_1;$$
$$u_2(t) = 0, \quad \text{with } a_2 < t < b_2;$$

$$u_3(t) = \int_t^\infty e^{a_3 - t} ds (if - g)/\sqrt{2} \quad \text{with } t > a_3$$

then it is clear that $(u_1, u_2, u_3) \in D((L_{10} \oplus 0 \oplus L_{30})^*)$ and $\gamma_1(u) = f$, $\gamma_2(u) = g$. □

Furthermore, using the result which is obtained in [5] the next assertion is proved.

Lemma 2.2. *The triplet* (H, Γ_1, Γ_2),

$$\Gamma_1 : D((0 \oplus L_{20} \oplus 0)^*) \to H, \quad \Gamma_1(u) = \frac{1}{i\sqrt{2}}(u_2(a_2) + u_2(b_2)),$$
$$\Gamma_2 : D((0 \oplus L_{20} \oplus 0)^*) \to H, \quad \Gamma_2(u) = \frac{1}{\sqrt{2}}(u_2(a_2) - u_2(b_2)),$$
$$u = (u_1, u_2, u_3) \in D((0 \oplus L_{20} \oplus 0)^*)$$

is a space of boundary values of the minimal operator $0 \oplus L_0 \oplus 0$ *in the direct sum* $0 \oplus L_2(H, (a_2, b_2)) \oplus 0$.

The following result can be easily established.

Lemma 2.3. *Every selfadjoint extension of* L_0 *in*

$$L_2 = L_2(H, (-\infty, a_1)) \oplus L_2(H, (a_2, b_2)) \oplus L_2(H, (a_3, +\infty))$$

is a direct sum of selfadjoint extensions of the minimal operator $L_{10} \oplus 0 \oplus L_{30}$ *in* $L_2(H, (-\infty, a_1)) \oplus 0 \oplus L_2(H, (a_3, +\infty))$ *and minimal operator* $0 \oplus L_0 \oplus 0$ *in* $0 \oplus L_2(H, (a_2, b_2)) \oplus 0$.

Finally, using the method in [6] the following result can be deduced.

Theorem 2.4. *If* $\tilde{L}$ *is a selfadjoint extension of the minimal operator* L_0 *in* L_2, *then it generates by differential expression* (2.1) *and boundary conditions*

$$u_3(a_3) = W_1 u_1(a_1),$$
$$u_2(b_2) = W_2 u_2(a_2),$$

where $W_1, W_2 : H \to H$ *are a unitary operators. Moreover, the unitary operators* W_1, W_2 *in* H *are determined uniquely by the extension* $\tilde{L}$; *i.e.* $\tilde{L} = L_{W_1 W_2}$ *and vice versa.*

3. The spectrum of the selfadjoint extensions

In this section the structure of the spectrum of the selfadjoint extension $L_{W_1 W_2}$ in L_2 will be investigated. In this case by the Lemma 2.3 it is clear that

$$L_{W_1 W_2} = L_{W_1} \oplus L_{W_2},$$

where L_{W_1} and L_{W_2} are selfadjoint extensions of the minimal operators $L_0(1,0,1) = L_{10} \oplus 0 \oplus L_{30}$ and $L_0(0,1,0) = 0 \oplus L_0 \oplus 0$ in the Hilbert spaces $L_2(1,0,1) = L_2(H, (-\infty, a_1)) \oplus 0 \oplus L_2(H, (a_3, +\infty))$ and $L_2(0,1,0) = 0 \oplus L_2(H, (a_2, b_2)) \oplus 0$, respectively.

First, we have to prove the following result.

Theorem 3.1. *The point spectrum of any selfadjoint extension* L_{W_1} *in the Hilbert space* $L_2(1,0,1)$ *is empty; i.e.,*

$$\sigma_p(L_{W_1}) = \emptyset.$$

Proof. Let us consider the following problem for the spectrum of the selfadjoint extension L_{W_1} of the minimal operator $L_0(1,0,1)$ in the Hilbert space $L_2(1,0,1)$,

$$L_{W_1}u = \lambda u, \quad u = (u_1, 0, u_3) \in L_2(1,0,1);$$

that is,

$$\begin{gathered}
\tilde{l}_1(u_1) = iu_1' + \tilde{A}_1 u_1 = \lambda u_1, \quad u_1 \in L_2(H,(-\infty,a_1)),\\
\tilde{l}_3(u_3) = iu_3' + \tilde{A}_3 u_3 = \lambda u_3, \quad u_3 \in L_2(H,(a_3,+\infty)), \quad \lambda \in \mathbb{R},\\
u_3(a_3) = W_1 u_1(a_1).
\end{gathered}$$

The general solution of this problem is

$$\begin{gathered}
u_1(\lambda;t) = e^{i(\tilde{A}_1-\lambda)(t-a_1)} f_1^*, \quad t < a_1,\\
u_3(\lambda;t) = e^{i(\tilde{A}_3-\lambda)(t-a_3)} f_3^*, \quad t > a_3,\\
f_3^* = W_1 f_1^*, \quad f_1^*, f_3^* \in H.
\end{gathered}$$

It is clear that for the $f_1^* \neq 0$, $f_3^* \neq 0$ the functions $u_1(\lambda;.) \notin L_2(H,(-\infty,a_1))$, $u_2(\lambda;.) \notin L_2(H,(a_3,+\infty))$. So for every unitary operator W_1 we have $\sigma_p(L_{W_1}) = \emptyset$. □

Since residual spectrum of any selfadjoint operator in any Hilbert space is empty, it is sufficient to investigate the continuous spectrum of the selfadjoint extensions L_{W_1} of the minimal operator $L_0(1,0,1)$ in the Hilbert space $L_2(1,0,1)$.

Theorem 3.2. *The continuous spectrum of any selfadjoint extension L_{W_1} of the minimal operator $L_0(1,0,1)$ in the Hilbert space $L_2(1,0,1)$ is $\sigma_c(L_{W_1}) = \mathbb{R}$.*

Proof. Firstly, we search for the resolvent operator of the extension L_{W_1} generated by the differential expression $(\tilde{l}_1, 0, \tilde{l}_3)$ and the boundary condition

$$u_3(a_3) = W_1 u_1(a_1)$$

in the Hilbert space $L_2(1,0,1)$; i.e.

$$\begin{gathered}
\tilde{l}_1(u_1) = iu_1' + \tilde{A}_1 u_1 = \lambda u_1 + f_1, \quad u_1, f_1 \in L_2(H,(-\infty,a_1)),\\
\tilde{l}_3(u_3) = iu_3' + \tilde{A}_3 u_3 = \lambda u_3 + f_3, \quad u_3, f_3 \in L_2(H,(a_3,+\infty)),\\
\lambda \in \mathbb{C}, \quad \lambda_i = \operatorname{Im}\lambda > 0\\
u_3(a_3) = W_1 u_1(a_1)
\end{gathered} \tag{3.1}$$

Now, we will show that the function

$$u(\lambda;t) = (u_1(\lambda;t), 0, u_3(\lambda;t)),$$

where

$$\begin{gathered}
u_1(\lambda;t) = e^{i(\tilde{A}_1-\lambda)(t-a_1)} f_1^* + i\int_t^{a_1} e^{i(\tilde{A}_1-\lambda)(t-s)} f_1(s)ds, \quad t < a_1,\\
u_3(\lambda;t) = i\int_t^{\infty} \mathrm{e}^{i(\tilde{A}_3-\lambda)(t-s)} f_3(s)ds, \quad t > a_3,\\
f_1^* = W^*\Big(i\int_{a_3}^{\infty} \mathrm{e}^{i(\tilde{A}_3-\lambda)(t-s)(b-s)} f_3(s)ds\Big)
\end{gathered}$$

is a solution of the boundary value problem (3.1) in the Hilbert space $L_2(1,0,1)$. It is sufficient to show that

$$u_1(\lambda;t) \in L_2(H,(-\infty,a_1)),$$

$$u_3(\lambda;t)\in L_2(H,(a_3,+\infty))$$

for $\lambda_i>0$. Indeed, in this case

$$\begin{aligned}\|f_1^*\|_H^2&=\Big\|\int_{a_3}^{\infty}\mathrm{e}^{i(\tilde A_3-\lambda)(a_3-s)}f(s)ds\Big\|_H^2\le\Big(\int_{a_3}^{\infty}e^{\lambda_i(a_3-s)}\|f(s)\|_Hds\Big)^2\\&\le\Big(\int_{a_3}^{\infty}e^{2\lambda_i(a_3-s)}ds\Big)\Big(\int_{a_3}^{\infty}\|f(s)\|_H^2ds\Big)=\frac{1}{2\lambda_i}\|f\|^2_{L_2(H,(a_3,+\infty))}<\infty,\end{aligned}$$

$$\begin{aligned}\|e^{i(\tilde A_1-\lambda)(t-a_1)}f_1^*\|^2_{L_2(H,(-\infty,a_1))}&=\|e^{-i\lambda(t-a_1)}f_1^*\|^2_{L_2(H,(-\infty,a_1))}\\&=\int_{-\infty}^{a_1}\|e^{-i\lambda(t-a_1)}f_1^*\|_H^2dt\\&=\int_{-\infty}^{a_1}e^{2\lambda_i(t-a_1)}dt\|f_1^*\|_H^2\\&=\frac{1}{2\lambda_i}\|f_1^*\|_H^2<\infty\end{aligned}$$

and

$$\begin{aligned}&\Big\|i\int_t^{a_1}e^{i(\tilde A_1-\lambda)(t-s)}f_1(s)ds\Big\|^2_{L_2(H,(-\infty,a_1))}\\&\le\int_{-\infty}^{a_1}\Big(\int_t^{a_1}e^{\lambda_i(t-s)}\|f_1(s)\|_Hds\Big)^2dt\\&\le\int_{-\infty}^{a_1}\Big(\int_t^{a_1}e^{\lambda_i(t-s)}ds\Big)\Big(\int_t^{a_1}e^{\lambda_i(t-s)}\|f_1(s)\|^2ds\Big)dt\\&=\frac{1}{\lambda_i}\int_{-\infty}^{a_1}\int_t^{a_1}e^{\lambda_i(t-s)}\|f_1(s)\|^2dsdt=\frac{1}{\lambda_i}\int_{-\infty}^{a_1}\Big(\int_{-\infty}^{s}e^{\lambda_i(t-s)}\|f_1(s)\|^2dt\Big)ds\\&=\frac{1}{\lambda_i}\int_{-\infty}^{a_1}\Big(\int_{-\infty}^{s}e^{\lambda_i(t-s)}dt\Big)\|f_1(s)\|^2ds\\&=\frac{1}{\lambda_i^2}\int_{-\infty}^{a_1}\|f_1(s)\|^2ds\\&=\frac{1}{\lambda_i^2}\|f_1\|^2_{L_2(H,(-\infty,a_1))}<\infty.\end{aligned}$$

Furthermore,

$$\begin{aligned}&\Big\|i\int_t^{\infty}e^{i(\tilde A_3-\lambda)(t-s)}f_3(s)ds\Big\|^2_{L_2(H,(a_3,+\infty))}\\&\le\int_{a_3}^{\infty}\Big(\int_t^{\infty}e^{\lambda_i(t-s)}\|f_3(s)\|_Hds\Big)^2dt\\&\le\int_{a_3}^{\infty}\Big(\int_t^{\infty}e^{\lambda_i(t-s)}ds\Big)\Big(\int_t^{\infty}e^{\lambda_i(t-s)}\|f_3(s)\|^2ds\Big)dt\\&=\frac{1}{\lambda_i}\int_{a_3}^{\infty}\Big(\int_t^{\infty}e^{\lambda_i(t-s)}\|f_3(s)\|^2ds\Big)dt\\&=\frac{1}{\lambda_i}\int_{a_3}^{\infty}\Big(\int_{a_3}^{s}e^{\lambda_i(t-s)}\|f_3(s)\|^2dt\Big)ds\\&=\frac{1}{\lambda_i}\int_{a_3}^{\infty}\Big(\int_{a_3}^{s}e^{\lambda_i(t-s)}dt\Big)\|f_3(s)\|^2ds\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\lambda_i^2}\int_{a_3}^{\infty}(1-e^{\lambda_i(a_3-s)})\|f_3(s)\|^2 ds\\
&\le \frac{1}{\lambda_i^2}\|f_3\|^2_{L_2(H,(a_3,+\infty))} < \infty.
\end{aligned}
$$

The above calculations imply that $u_1(\lambda;t) \in L_2(H,(-\infty,a_1))$, and that $u_3(\lambda;t) \in L_2(H,(a_3,+\infty))$ for $\lambda \in \mathbb{C}$, $\lambda_i = \operatorname{Im}\lambda > 0$. On the other hand, one can easily verify that $u(\lambda;t) = (u_1(\lambda;t), 0, u_3(\lambda;t))$ is a solution of boundary-value problem (3.1).

When $\lambda \in \mathbb{C}$, $\lambda_i = \operatorname{Im}\lambda < 0$ is true solution of the boundary-value problem

$$
\begin{gathered}
L_{W_1}u = \lambda u + f, \quad u = (u_1, 0, u_3), \quad f = (f_1, 0, f_3) \in L_2(1,0,1)\\
u_3(a_3) = W_1 u_1(a_1),
\end{gathered}
$$

where W_1 is a unitary operator in H, is in the form $u(\lambda;t) = (u_1(\lambda;t), 0, u_3(\lambda;t))$,

$$
\begin{gathered}
u_1(\lambda;t) = -i\int_{-\infty}^{t} e^{i(\tilde{A}_1-\lambda)(t-s)} f_1(s)ds, \quad t < a_1\\
u_3(\lambda;t) = e^{i(\tilde{A}_3-\lambda)(t-a_3)} f_3^* - i\int_{a_3}^{t} e^{i(\tilde{A}_3-\lambda)(t-s)} f_3(s)ds, \quad t > a_3,
\end{gathered}
$$

where

$$
f_3^* = W\Big(-i\int_{-\infty}^{a_1} e^{i(\tilde{A}_1-\lambda)(a_1-s)} f_1(s)ds\Big).
$$

First, we prove that $u(\lambda;t) \in L_2(1,0,1)$. In this case,

$$
\begin{aligned}
\|u_1(\lambda;t)\|^2_{L_2(H,(-\infty,a_1))} &= \int_{-\infty}^{a_1}\Big\| -i\int_{-\infty}^{t} e^{i(\tilde{A}_1-\lambda)(t-s)} f_1(s)ds\Big\|_H^2 dt\\
&\le \int_{-\infty}^{a_1}\Big(\int_{-\infty}^{t} e^{\lambda_i(t-s)}ds\Big)\Big(\int_{-\infty}^{t} e^{\lambda_i(t-s)}\|f_1(s)\|_H^2 ds\Big)dt\\
&= \frac{1}{|\lambda_i|}\int_{-\infty}^{a_1}\int_{-\infty}^{t} e^{\lambda_i(t-s)}\|f_1(s)\|_H^2\, ds\, dt\\
&= \frac{1}{|\lambda_i|}\int_{-\infty}^{a_1}\Big(\int_{s}^{a_1} e^{\lambda_i(t-s)}\|f_1(s)\|_H^2 dt\Big)ds\\
&= \frac{1}{|\lambda_i|}\int_{-\infty}^{a_1}\big(e^{\lambda_i(t-s)}\big)dt\|f_1(s)\|_H^2 ds\\
&= \frac{1}{|\lambda_i|^2}\int_{-\infty}^{a_1}(1-e^{\lambda_i(a_1-s)})\|f_1(s)\|_H^2 ds\\
&\le \frac{1}{|\lambda_i|^2}\|f_1\|^2_{L_2(H,(-\infty,a_1))} < \infty,
\end{aligned}
$$

$$
\begin{aligned}
\|f_3^*\|_H^2 &= \Big\|\int_{-\infty}^{a_1} e^{i(\tilde{A}_1-\lambda)(a_1-s)} f_1(s)ds\Big\|_H^2\\
&\le \Big(\int_{-\infty}^{a_1} e^{\lambda_i(a_1-s)}\|f_1(s)\|_H ds\Big)^2\\
&\le \Big(\int_{-\infty}^{a_1} e^{2\lambda_i(a_1-s)}ds\Big)\Big(\int_{-\infty}^{a_1}\|f_1(s)\|_H^2 ds\Big)\\
&= \frac{1}{2|\lambda_i|}\|f_1\|^2_{L_2(H,(-\infty,a_1))} < \infty,
\end{aligned}
$$

$$
\begin{aligned}
\|e^{i(\tilde{A}_3-\lambda)(t-a_3)}f_3^*\|^2_{L_2(H,(a_3,+\infty))} &\le \int_{a_3}^{\infty} e^{2\lambda_i(t-a_3)}dt\|f_3^*\|^2_H \\
&= \frac{1}{2|\lambda_i|}\|f_3^*\|^2_H \\
&\le \frac{1}{4|\lambda_i|^2}\|f\|^2_{L_2(H,(a_3,+\infty))} < \infty
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big\|\int_{a_3}^{t} e^{i(\tilde{A}_3-\lambda)(t-s)}f_3(s)ds\Big\|^2_{L_2(H,(a_3,+\infty))} \\
&\le \int_{a_3}^{\infty}\Big(\int_{a_3}^{t} e^{\lambda_i(t-s)}\|f_3(s)\|_H ds\Big)^2 dt \\
&\le \int_{a_3}^{\infty}\Big(\int_{a_3}^{t} e^{\lambda_i(t-s)}ds\Big)\Big(\int_{a_3}^{t} e^{\lambda_i(t-s)}\|f_3(s)\|^2_H ds\Big)dt \\
&= \int_{a_3}^{\infty}\Big(\frac{1}{\lambda_i}(1-e^{\lambda_i(t-a_3)})\Big)\Big(\int_{a_3}^{t} e^{\lambda_i(t-s)}\|f_3(s)\|^2_H ds\Big)dt \\
&\le \frac{1}{|\lambda_i|}\int_{a_3}^{\infty}\Big(\int_{a_3}^{t} e^{\lambda_i(t-a_3)}\|f_3(s)\|^2_H ds\Big)dt \\
&= \frac{1}{|\lambda_i|}\int_{a_3}^{\infty}\Big(\int_{s}^{\infty} e^{\lambda_i(t-s)}\|f_3(s)\|^2_H dt\Big)ds \\
&= \frac{1}{|\lambda_i|}\int_{a_3}^{\infty}\Big(\int_{s}^{a_3} e^{\lambda_i(t-s)}dt\Big)\|f_3(s)\|^2_H ds \\
&= \frac{1}{|\lambda_i|^2}\|f_3\|^2_{L_2(H,(a_3,+\infty))} < \infty.
\end{aligned}
$$

The above calculations show that $u_1(\lambda;\cdot) \in L_2(H,(-\infty,a_1))$, and that $u_3(\lambda;\cdot) \in L_2(H,(a_3,+\infty))$; i.e., $u(\lambda;\cdot) = (u_1(\lambda;\cdot),0,u_3(\lambda,\cdot)) \in L_2(1,0,1)$ in case $\lambda \in \mathbb{C}$, $\lambda_i = \operatorname{Im}\lambda < 0$. On the other hand it can be verified that the function $u(\lambda;\cdot)$ satisfies the equation $L_{W_1}u = \lambda u(\lambda;\cdot) + f$ and $u_3(a_3) = W_1 u_1(a_1)$.

Therefore, the following result has been proved that for the resolvent set $\rho(L_{W_1})$

$$
\rho(L_{W_1}) \supset \{\lambda \in \mathbb{C} : \operatorname{Im}\lambda \neq 0\}.
$$

Now, we will study continuous spectrum $\sigma_c(L_{W_1})$ of the extension L_{W_1}. For $\lambda \in \mathbb{C}$, $\lambda_i = \operatorname{Im}\lambda > 0$, norm of the resolvent operator $R_\lambda(L_{W_1})$ of the L_{W_1} is of the form

$$
\begin{aligned}
\|R_\lambda(L_{W_1})f(t)\|^2_{L_2} = \Big\|&e^{i(\tilde{A}_1-\lambda)(t-a_1)}f_1^* + i\int_t^{a_1} e^{i(\tilde{A}_1-\lambda)(t-s)}f_1(s)ds\Big\|^2_{L_2(H,(-\infty,a_1))} \\
&+ \Big\|i\int_t^{\infty} e^{i(\tilde{A}_3-\lambda)(t-s)}f_3(s)ds\Big\|^2_{L_2(H,(a_3,+\infty))},
\end{aligned}
$$

where $f = (f_1,0,f_3) \in L_2(1,0,1)$. Then, it is clear that for any $f = (f_1,0,f_3)$ in $L_2(1,0,1)$ the following inequality is true.

$$
\|R_\lambda(L_{W_1})f(t)\|^2_{L_2} \ge \Big\|i\int_t^{\infty} e^{i(\tilde{A}_3-\lambda)(t-s)}f_3(s)ds\Big\|^2_{L_2(H,(a_3,+\infty))}.
$$

The vector functions $f^*(\lambda;t)$ which is of the form $f^*(\lambda;t) = (0,0,e^{i(\tilde{A}_3-\bar{\lambda})t}f_3)$, $\lambda \in \mathbb{C}$, $\lambda_i = \operatorname{Im}\lambda > 0$, $f_3 \in H$ belong to $L_2(1,0,1)$. Indeed,

$$\begin{aligned}
\|f^*(\lambda;t)\|^2_{L_2} &= \int_{a_3}^{\infty} \|e^{i(\tilde{A}_3-\bar{\lambda})t}f_3\|^2_H dt = \int_{a_3}^{\infty} e^{-2\lambda_i t}dt\|f_3\|^2_H \\
&= \frac{1}{2\lambda_i}e^{-2\lambda_i a_3}\|f_3\|^2_H < \infty.
\end{aligned}$$

For such functions $f^*(\lambda;\cdot)$, we have

$$\begin{aligned}
&\|R_\lambda(L_{W_1})f^*(\lambda;t)\|^2_{L_2(H,(a_3+\infty))} \\
&\geq \Big\|i\int_t^{\infty} e^{i(\tilde{A}_3-\lambda)(t-s)}e^{i(\tilde{A}_3-\bar{\lambda})s}f_3 ds\Big\|^2_{L_2(H,(a_3,+\infty))} \\
&= \Big\|\int_t^{\infty} e^{-i\lambda t}e^{-2\lambda_i s}e^{i\tilde{A}_3 t}f_3 ds\Big\|^2_{L_2(H,(a_3,+\infty))} \\
&= \Big\|e^{-i\lambda t}e^{i\tilde{A}_3 t}\int_t^{\infty} e^{-2\lambda_i s}f_3 ds\Big\|^2_{L_2(H,(a_3,+\infty))} \\
&= \Big\|e^{-i\lambda t}\int_t^{\infty} e^{-2\lambda_i s}ds\Big\|^2_{L_2(H,(a_3,+\infty))}\|f_3\|^2_H \\
&= \frac{1}{4\lambda_i^2}\int_{a_3}^{\infty} e^{-2\lambda_i t}dt\|f_3\|^2_H \\
&= \frac{1}{8\lambda_i^3}e^{-2\lambda_i a_3}\|f_3\|^2_H.
\end{aligned}$$

From this we obtain

$$\|R_\lambda(L_{W_1})f^*(\lambda;\cdot)\|_{L_2} \geq \frac{e^{-\lambda_i a_3}}{2\sqrt{2}\lambda_i\sqrt{\lambda_i}}\|f\|_H = \frac{1}{2\lambda_i}\|f^*(\lambda;\cdot)\|_{L_2};$$

i.e., for $\lambda_i = \operatorname{Im}\lambda > 0$ and $f \neq 0$,

$$\frac{\|R_\lambda(L_{W_1})f^*(\lambda;\cdot)\|_{L_2}}{\|f^*(\lambda;\cdot)\|_{L_2}} \geq \frac{1}{2\lambda_i}.$$

is valid. On the other hand, it is clear that

$$\|R_\lambda(L_{W_1})\| \geq \frac{\|R_\lambda(L_{W_1})f^*(\lambda;\cdot)\|_{L_2}}{\|f^*(\lambda;\cdot)\|_{L_2}}, \quad f_3 \neq 0.$$

Consequently,

$$\|R_\lambda(L_{W_1})\| \geq \frac{1}{2\lambda_i} \quad \text{for } \lambda \in \mathbb{C}, \quad \lambda_i = \operatorname{Im}\lambda > 0.$$

□

The spectrum of selfadjoint extensions of the minimal operator $L_0(0,1,0)$ will be investigated next.

Theorem 3.3. *The spectrum of the selfadjoint extension L_{W_2} of the minimal operator $L_0(0,1,0)$ in the Hilbert space $L_2(0,1,0)$ is of the form*

$$\begin{aligned}
\sigma(L_{W_2}) = \Big\{\lambda \in \mathbb{R}\colon \lambda = \frac{1}{b_2-a_2}\arg\mu + \frac{2n\pi}{b_2-a_2}, \quad n \in \mathbb{Z}, \\
\mu \in \sigma(W_2^* e^{i\tilde{A}_2(b_2-a_2)}),\ 0 \leq \arg\mu < 2\pi\Big\}
\end{aligned}$$

Proof. The general solution of the following problem to spectrum of the selfadjoint extension L_{W_2},

$$\tilde{l}_2(u_2) = iu_2' + \tilde{A}_2 u_2 = \lambda u_2 + f_2, \quad u_2, f_2 \in L_2(H, (a_2, b_2))$$
$$u_2(b_2) = W_2 u_2(a_2), \quad \lambda \in \mathbb{R}$$

is of the form

$$u_2(t) = e^{i(\tilde{A}_2 - \lambda)(t-a_2)} f_2^* + \int_{a_2}^{t} e^{i(\tilde{A}_2-\lambda)(t-s)} \; f_2(s)ds,$$
$$a_2 < t < b_2,$$
$$(e^{i\lambda(b_2-a_2)} - W_2^* e^{i\tilde{A}_2(b_2-a_2)}) f_2^* = W_2^* e^{i\lambda(b_2-a_2)} \int_{a_2}^{b_2} e^{i(\tilde{A}_2-\lambda)(b_2-s)} f_2(s)ds$$

This implies that $\lambda \in \sigma(L_{W_2})$ if and only if λ is a solution of the equation $e^{i\lambda(b_2-a_2)} = \mu$, where $\mu \in \sigma(W_2^* e^{i\tilde{A}_2(b_2-a_2)})$. We obtain that

$$\lambda = \frac{1}{b_2 - a_2} \arg \mu + \frac{2n\pi}{b_2 - a_2}, \quad n \in \mathbb{Z},\ \mu \in \sigma(W_2^* e^{i\tilde{A}_2(b_2-a_2)}).$$

□

Theorem 3.4. *Spectrum $\sigma(L_{W_1 W_2})$ of any selfadjoint extension $L_{W_1 W_2} = L_{W_1} \oplus L_{W_2}$ coincides with $\mathbb{R}$.*

Proof. Validity of this assertion is a simple result of the following claim that a proof of which it is clear. If S_1 and S_2 are linear closed operators in any Hilbert spaces H_1 and H_2 respectively, then we have

$$\sigma_p(S_1 \oplus S_2) = \sigma_p(S_1) \cup \sigma_p(S_2),$$
$$\sigma_c(S_1 \oplus S_2) = (\sigma_p(S_1) \cup \sigma_p(S_2))^c \cap (\sigma_r(S_1) \cup \sigma_r(S_2))^c \cap (\sigma_c(S_1) \cup \sigma_c(S_2)).$$

□

Note that for the singular differential operators for n-th order in scalar case in the finite interval has been studied in [10].

Example 3.5. By the last theorem the spectrum of following boundary-value problem

$$i\frac{\partial u(t,x)}{\partial t} + \operatorname{sgn} t \frac{\partial^2 u(t,x)}{\partial x^2} = f(t,x), \quad |t| > 1,\ x \in [0,1],$$
$$i\frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} = f(t,x), \quad |t| < 1/2,\ x \in [0,1],$$
$$u(1/2, x) = e^{i\psi} u(-1/2, x), \quad \psi \in [0, 2\pi),$$
$$u(1,x) = e^{i\varphi} u(-1,x), \quad \varphi \in [0, 2\pi),$$
$$u_x(t,0) = u_x(t,1) = 0, \quad |t| > 1,\ |t| < 1/2$$

in the space $L_2((-\infty,-1) \times (0,1)) \oplus L_2((-1/2,1/2) \times (0,1)) \oplus L_2((1,\infty) \times (0,1))$ coincides with $\mathbb{R}$.

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