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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR DIRICHLET PROBLEMS INVOLVING THE P(X)-LAPLACE OPERATOR

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Abstract. In this article, we study superlinear Dirichlet problems involving the $p(x)$-Laplace operator without using the Ambrosetti-Rabinowitz's superquadraticity condition. Using a variant Fountain theorem, but not including Palais-Smale type assumptions, we prove the existence and multiplicity of the solutions.

1. Introduction

We study the existence of infinitely many solutions for the Dirichlet boundary problems

$$\begin{gathered} -\Delta_{p(x)}u + |u|^{p(x)-2}u = f(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

and

$$\begin{gathered} -\Delta_{p(x)}u = f(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $p \in C(\overline{\Omega})$ such that $1 < p(x) < N$ for any $x \in \overline{\Omega}$ and f is a Carathéodory function.

The study of differential equations and variational problems involving the $p(x)$-Laplace operator $-\Delta_{p(x)}u := -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$, which is a natural generalization of the p-Laplace operator, have attracted a special interest in recent years. A lot of researchers have devoted their work to this area (see, e.g., [3, 4, 12, 14]) since there are some physical phenomena which can be modelled by such kind of equations. In particular, we may mention some applications related to the study of elastic mechanics and electrorheological fluids [1, 5, 11, 15, 17]. The appearance of such physical models was facilitated by the development of variable exponent Lebesgue $L^{p(x)}$ and Sobolev spaces $W^{1,p(x)}$.

Generally, to show the existence of solutions for Dirichlet problems which is superlinear, it is essential to assume the following superquadraticity condition, which is known as Ambrosetti-Rabinowitz's type condition [2]:

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(AR) There exists $M > 0$ and $\tau > p^+$ such that

$$0 < \tau F(x,s) \le f(x,s)s, \quad |s| \ge M,\ x \in \Omega,$$

where f is the nonlinear term in the equation with $F(x,t) = \int_0^t f(x,s)ds$ for $x \in \Omega$ and $t \in \mathbb{R}$.

There are many articles dealing with superlinear Dirichlet problems involving $p(x)$-Laplacian, in which (AR) is the main assumption to get the existence and multiplicity of solutions [9, 10]. However, there are many functions which are superlinear but not satisfy (AR).

It is well known that the main aim of using (AR) is to ensure the boundedness of the Palais-Smale type sequences of the corresponding functional. In the present paper we do not use (AR) and we know that without (AR) it becomes a very difficult task to get the boundedness. So, using a weaker assumption (G1) (see main results) instead of (AR), and some variant Fountain theorem, i.e., Theorem 2.5, we overcome these difficulties.

2. Abstract framework and preliminary results

We state some basic properties of the variable exponent Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain (for more details, see [6, 7, 8, 13]). Set

$$C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) : \inf p(x) > 1, \forall x \in \overline{\Omega}\}.$$

Let $p \in C_+(\overline{\Omega})$ and denote

$$p^- := \inf_{x\in\overline{\Omega}} p(x) \le p(x) \le p^+ := \sup_{x\in\overline{\Omega}} p(x) < \infty.$$

For any $p \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \{u : \Omega \to \mathbb{R} \text{ is measurable, } \int_\Omega |u(x)|^{p(x)}dx < \infty\}.$$

Then $L^{p(x)}(\Omega)$ endowed with the norm

$$|u|_{p(x)} = \inf\{\mu > 0 : \int_\Omega |\frac{u(x)}{\mu}|^{p(x)}dx \le 1\},$$

becomes a Banach space.

The modular of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho(u) = \int_\Omega |u(x)|^{p(x)}dx, \quad \forall u \in L^{p(x)}(\Omega).$$

Proposition 2.1 ([7, 13]). *If $u, u_n \in L^{p(x)}(\Omega)$ $(n = 1, 2, \dots)$, then we have*

(i) $|u|_{p(x)} < 1$ $(= 1, > 1)$ *if and only if* $\rho(u) < 1$ $(= 1, > 1)$*;*

(ii) $|u|_{p(x)} > 1$ *implies* $|u|^{p^-}_{p(x)} \le \rho(u) \le |u|^{p^+}_{p(x)}$, $|u|_{p(x)} < 1$ *implies* $|u|^{p^+}_{p(x)} \le \rho(u) \le |u|^{p^-}_{p(x)}$*;*

(iii) $\lim_{n\to\infty} |u_n|_{p(x)} = 0$ *if and only if* $\lim_{n\to\infty} \rho(u_n) = 0$*;* $\lim_{n\to\infty} |u_n|_{p(x)} = \infty$ *if and only if* $\lim_{n\to\infty} \rho(u_n) = \infty$.

Proposition 2.2 ([7, 13]). *If $u, u_n \in L^{p(x)}(\Omega)$ $(n = 1, 2, \dots)$, then the following statements are equivalent:*

(i) $\lim_{n\to\infty} |u_n - u|_{p(x)} = 0$;
(ii) $\lim_{n\to\infty} \rho(u_n - u) = 0$;
(iii) $u_n \to u$ *in measure in* Ω *and* $\lim_{n\to\infty} \rho(u_n) = \rho(u)$.

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

with the norm

$$\|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)},$$

or equivalently

$$\|u\|_{1,p(x)} = \inf \big\{\mu > 0 : \int_\Omega (|\frac{\nabla u(x)}{\mu}|^{p(x)} + |\frac{u(x)}{\mu}|^{p(x)})dx \le 1\big\}$$

for all $u \in W^{1,p(x)}(\Omega)$. The space $W_0^{1,p(x)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ with respect to the norm $\|u\|_{1,p(x)}$. For $u \in W_0^{1,p(x)}(\Omega)$, we define an equivalent norm

$$\|u\| = |\nabla u|_{p(x)},$$

since Poincaré inequality holds, i.e., there exists a positive constant c such that

$$|u|_{p(x)} \le c|\nabla u|_{p(x)},$$

for all $u \in W_0^{1,p(x)}(\Omega)$, see [9].

Proposition 2.3 ([7, 13])**.** *If* $1 < p^- \le p^+ < \infty$, *then* $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ *and* $W_0^{1,p(x)}(\Omega)$ *are separable and reflexive Banach spaces.*

Proposition 2.4 ([7, 13])**.** *Let* $q \in C_+(\overline{\Omega})$. *If* $q(x) < p^*(x)$ *for all* $x \in \overline{\Omega}$, *then the embedding* $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ *is compact and continuous, where*

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \ge N. \end{cases}$$

Let E be a Banach space with the norm $\|\cdot\|$ and $E = \overline{\oplus_{j\in\mathbb{N}} X_j}$ with $\dim X_j < \infty$ for any $j \in \mathbb{N}$. Set

$$Y_k = \oplus_{j=0}^k X_j, \quad Z_k = \overline{\oplus_{j=k}^\infty X_j},$$
$$B_k = \{u \in Y_k : \|u\| \le \rho_k\}, \quad N_k = \{u \in Z_k : \|u\| = r_k\} \quad \text{for } \rho_k > r_k > 0.$$

Let us consider the C^1-functional $I_\lambda : E \to \mathbb{R}$ defined by

$$I_\lambda(u) := A(u) - \lambda B(u), \quad \lambda \in [1,2].$$

Now we recall the following variant of the fountain theorem [18, Theorem 2.1], which is the main tool in the proof of the main results of this article. We will use the following assumptions:

(F1) I_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1,2]$. Moreover, $I_\lambda(-u) = I_\lambda(u)$ for all $(\lambda, u) \in [1,2] \times E$;
(F2) $B(u) \ge 0$ for all $u \in E$, and $A(u) \to \infty$ or $B(u) \to \infty$ as $\|u\| \to \infty$,
(F3) $B(u) \le 0$ for all $u \in E$; $B(u) \to -\infty$ as $\|u\| \to \infty$.

Theorem 2.5. *Assume the functional I_λ satisfies* (F1), *and either* (F2) *or* (F3). *For $k \geq 2$, let*

$$\Gamma_k := \{\psi \in C(B_k, E) : \psi \text{ is odd, } \psi\big|_{\partial B_k} = \mathrm{id}\},$$
$$c_k(\lambda) := \inf_{\psi \in \Gamma_k} \max_{u \in B_k} I_\lambda(\gamma(u)),$$
$$b_k(\lambda) := \inf_{u \in Z_k, \|u\| = r_k} I_\lambda(u),$$
$$a_k(\lambda) := \max_{u \in Y_k, \|u\| = \rho_k} I_\lambda(u).$$

If $b_k(\lambda) > a_k(\lambda)$ for all $\lambda \in [1,2]$, then $c_k(\lambda) \geq b_k(\lambda)$ for all $\lambda \in [1,2]$. Moreover, for a.e $\lambda \in [1,2]$, there exists a sequence $\{u_n^k(\lambda)\}_{n=1}^\infty$ such that $\sup_n \|u_n^k(\lambda)\| < \infty$, $I'_\lambda(u_n^k(\lambda)) \to 0$ and $I_\lambda(u_n^k(\lambda)) \to c_k(\lambda)$ as $n \to \infty$.

3. Main results

First, we study the Dirichlet boundary-value problem

$$\begin{gathered} -\Delta_{p(x)} u + |u|^{p(x)-2} u = f(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$.

We assume the following conditions:

- (S1) $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and $|f(x,t)| \leq c(1 + |t|^{q(x)-1})$ for a.e. $x \in \overline{\Omega}$ and all $t \in \mathbb{R}$, $f(x,t)t \geq 0$ for all $t > 0$, where $p, q \in C_+(\overline{\Omega})$ such that $p(x) < q(x) < p^*(x)$ for all $x \in \overline{\Omega}$;
- (S2) $\liminf_{|t| \to \infty} \frac{f(x,t)t}{|t|^\theta} \geq c > 0$ uniformly for $x \in \overline{\Omega}$, where $p^+ < \theta \leq q^-$;
- (S3) $\lim_{t \to 0} \frac{f(x,t)}{t^{p^- - 1}} = 0$ uniformly for $x \in \overline{\Omega}$, $\frac{f(x,u)}{u^{p^- - 1}}$ is an increasing function of $t \in \mathbb{R}$ for all $x \in \overline{\Omega}$.
- (S4) $f(x,-t) = -f(x,t)$ for all $x \in \overline{\Omega}$, $t \in \mathbb{R}$.
- (G1) There exists a constant $\xi \geq 1$, such that for any $s \in [0,1]$, $t \in \mathbb{R}$, and for each $G_\gamma \in \mathcal{F}$, and all $\eta \in [p^-, p^+]$, the inequality $\xi G_\gamma(x,t) \geq G_\eta(x, st)$ hold for a.e. $x \in \overline{\Omega}$, where
$$\mathcal{F} = \{G_\gamma : G_\gamma(x,t) = f(x,t)t - \gamma F(x,t), \gamma \in [p^-, p^+]\}.$$
Note that when $p(x) \equiv p$ a constant, $\mathcal{F} = \{f(x,t)t - pF(x,t)\}$ is consist of only one element.

Remark 3.1. It is not difficult to show that if $f(x,t)$ is increasing in t, then (AR) implies (G1) when t is large enough. However, in general, (AR) does not imply (G1); see [16, Remark 3.3].

Theorem 3.2. *Assume that* (S1)–(S4), (G1) *hold. Then problem* (3.1) *has infinitely many solutions $\{u_k\}$ satisfying*

$$J(u_k) = \int_\Omega \frac{1}{p(x)} (|\nabla u_k|^{p(x)} + |u_k|^{p(x)}) dx - \int_\Omega F(x, u_k) dx \to \infty \quad \text{as } k \to \infty,$$

where $J : W^{1,p(x)}(\Omega) \to \mathbb{R}$ is the functional corresponding to problem (3.1) *and $F(x,t) = \int_0^t f(x,s) ds$.*

Remark 3.3. Condition (S1) implies that the functional J is well defined and of class C^1. It is well known that the critical points of J are weak solutions of (3.1). Moreover, the derivative of J is given by

$$\langle J'(u), \upsilon\rangle = \int_\Omega (|\nabla u|^{p(x)-2}\nabla u\nabla \upsilon + |u|^{p(x)-2}u\upsilon)dx - \int_\Omega f(x,u)\upsilon dx,$$

for any $u, \upsilon \in W^{1,p(x)}(\Omega)$.

Second, we consider the Dirichlet boundary problem

$$\begin{gathered} -\Delta_{p(x)}u = f(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.2}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$. We will use the following assumptions:

- (E1) $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and $|f(x,t)| \le c(1 + |t|^{q(x)-1})$ a.e. $x \in \overline{\Omega}$ and all $t \in \mathbb{R}$, where $p, q \in C_+(\overline{\Omega})$ such that $p(x) < q(x) < p^*(x)$ for $x \in \overline{\Omega}$.
- (E2) $\frac{f(x,t)}{t^{p^- -1}}$ is increasing in $t \in \mathbb{R}$ for t large enough.

Theorem 3.4. *Assume that* (S2), (S4), (G1), (E1)-(E2) *hold. Then problem* (3.2) *has infinitely many solutions* $\{u_k\}$ *satisfying*

$$\Psi(u_k) = \int_\Omega \frac{1}{p(x)}|\nabla u_k|^{p(x)}dx - \int_\Omega F(x,u_k)dx \to \infty \quad \text{as } k \to \infty,$$

where $\Psi : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ *is the functional corresponding to problem* (3.2).

Since the proof of Theorem 3.4 is very similar to the proof of Theorem 3.2, we only prove Theorem 3.2 and omit the other proof.

We say that $u \in W^{1,p(x)}(\Omega)$ is a weak solution of (3.1) if

$$\int_\Omega (|\nabla u|^{p(x)-2}\nabla u\nabla \upsilon + |u|^{p(x)-2}u\upsilon)dx = \int_\Omega f(x,u)\upsilon dx,$$

for any $\upsilon \in W^{1,p(x)}(\Omega)$.

Let us choose an orthonormal basis $\{e_j\} \subset W^{1,p(x)}(\Omega)$ and define $X_j := \mathbb{R}e_j$. Then the spaces Y_k and Z_k can be defined as in Section 2. Let us consider the C^1-functional $J_\lambda : W^{1,p(x)}(\Omega) \to \mathbb{R}$ defined by

$$J_\lambda(u) = \int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx - \lambda \int_\Omega F(x,u)dx := A(u) - \lambda B(u), \quad \lambda \in [1,2].$$

Then $B(u) \ge 0$, $A(u) \to \infty$ as $\|u\|_{1,p(x)} \to \infty$, and $J_\lambda(-u) = J_\lambda(u)$ for all $\lambda \in [1,2]$, $u \in W^{1,p(x)}(\Omega)$.

In the view of Theorem 2.5, we can get the proof of Theorem 3.2 by help of the following two lemmas.

Lemma 3.5. *Under the assumptions of Theorem 3.2 there exist a sequence* $\lambda_n \to 1$, *as* $n \to \infty$, $\bar{c}_k > \bar{b}_k > 0$, *and* $\{z_n\}_{n=1}^\infty \subset W^{1,p(x)}(\Omega)$, *such that*

$$J'_\lambda(z_n) = 0, \quad J_\lambda(z_n) \in [\bar{b}_k, \bar{c}_k].$$

Proof. It is easy to prove that, for some $\rho_k > 0$ large enough, we have $a_k(\lambda) := \max_{u \in Y_k, \|u\| = p_k} J_\lambda(u) \le 0$ uniformly for $\lambda \in [1,2]$. Indeed, by the conditions (S1)–(S3), for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that $f(x,u)u \ge C_\varepsilon|u|^\theta - \varepsilon|u|^{p^-}$.

Further, on the finite dimensional subspace Y_k, we can find some constants $c > 0$ such that

$$|u|_\theta \geq c\|u\|_{1,p(x)}, \quad |u|_{p^-} \leq c\|u\|_{1,p(x)}, \quad \forall u \in Y_k.$$

By Propositions 2.1 and 2.4, we have

$$\begin{aligned} J_\lambda(u) &\leq \frac{1}{p^-}\int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})dx - \frac{\lambda C_\varepsilon}{\theta}\int_\Omega |u|^\theta dx + \frac{\lambda\varepsilon}{p^-}\int_\Omega |u|^{p^-}dx \\ &\leq \frac{1}{p^-}\|u\|_{1,p(x)}^{p^+} - \frac{\lambda C_\varepsilon c^\theta}{\theta}\|u\|_{1,p(x)}^\theta + \frac{\lambda c^{p^-}}{p^-}\|u\|_{1,p(x)}^{p^-}. \end{aligned}$$

Since $\theta > p^+$, it follows that

$$a_k(\lambda) := \max_{u\in Y_k,\, \|u\|_{1,p(x)}=\rho_k} J_\lambda(u) \to -\infty \quad \text{as } \|u\|_{1,p(x)} \to +\infty$$

uniformly for $\lambda \in [1, 2]$ and for all $u \in Y_k$.

On the other hand, by conditions (S1) and (S3), for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that $|f(x, u)| \leq \varepsilon|u|^{p^- -1} + C_\varepsilon|u|^{q(x)-1}$. Let

$$\beta_k := \sup_{u\in Z_k,\|u\|_{1,p(x)}=1} |u|_{q(x)}, \quad \vartheta_k := \sup_{u\in Z_k,\|u\|_{1,p(x)}=1} |u|_{p^-}.$$

Then $\beta_k \to 0$ and $\vartheta_k \to 0$ as $k \to \infty$ (see [10]). Therefore, when $u \in Z_k$ and $\|u\|_{1,p(x)} > 1$, we have

$$\begin{aligned} J_\lambda(u) &\geq \frac{1}{p^+}\int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})dx - \lambda\varepsilon\int_\Omega |u|^{p^-}dx - \lambda C_\varepsilon\int_\Omega |u|^{q(x)}dx \\ &\geq \frac{1}{p^+}\|u\|_{1,p(x)}^{p^-} - c|u|_{p^-}^{p^-} - c|u|_{q(x)}^{q^+} \\ &\geq \frac{1}{p^+}\|u\|_{1,p(x)}^{p^-} - c\vartheta_k^{p^-}\|u\|_{1,p(x)}^{p^-} - c\beta_k^{q^+}\|u\|_{1,p(x)}^{q^+}, \end{aligned}$$

where $c = \max\{2\varepsilon, 2C_\varepsilon\}$. For sufficiently large k, we have $c\vartheta_k^{p^-} < \frac{1}{2p^+}$. Then, it follows

$$J_\lambda(u) \geq \frac{1}{2p^+}\|u\|_{1,p(x)}^{p^-} - c\beta_k^{q^+}\|u\|_{1,p(x)}^{q^+}.$$

If we choose $r_k := (2cq^+\beta_k^{q^+})^{\frac{1}{p^- - q^+}}$, then for $u \in Z_k$ with $\|u\|_{1,p(x)} = r_k$, we obtain

$$\begin{aligned} J_\lambda(u) &\geq \frac{1}{2p^+}(2cq^+\beta_k^{q^+})^{\frac{p^-}{p^- - q^+}} - c\beta_k^{q^+}(2cq^+\beta_k^{q^+})^{\frac{q^+}{p^- - q^+}} \\ &\geq \frac{q^+ - p^+}{2p^+q^+}(2cq^+\beta_k^{q^+})^{\frac{p^-}{p^- - q^+}} := \bar{b}_k, \end{aligned}$$

which implies

$$b_k(\lambda) := \inf_{u\in Z_k,\|u\|_{1,p(x)}=r_k} J_\lambda(u) \to \infty \quad \text{as } k \to \infty$$

uniformly for λ. So, by Theorem 2.5, for a.e. $\lambda \in [1, 2]$, there exists a sequence $\{u_n^k(\lambda)\}_{n=1}^\infty$ such that

$$\begin{gathered} \sup_n \|u_n^k(\lambda)\|_{1,p(x)} < \infty, \quad J'_\lambda(u_n^k(\lambda)) \to 0, \\ J_\lambda(u_n^k(\lambda)) \to c_k(\lambda) \geq b_k(\lambda) \geq \bar{b}_k \quad \text{as } n \to \infty. \end{gathered}$$

Moreover, since $c_k(\lambda) \le \sup_{u\in B_k} J_\lambda(u) := \bar{c}_k$ and $W^{1,p(x)}(\Omega)$ is embedded compactly to $L^{q(x)}(\Omega)$, and thanks to the standard arguments, $\{u_n^k(\lambda)\}_{n=1}^\infty$ has a convergent subsequence. Hence, there exists $z^k(\lambda)$ such that $J'_\lambda(z^k(\lambda)) = 0$ and $J_\lambda(z^k(\lambda)) \in [\bar{b}_k, \bar{c}_k]$. Consequently, we can find $\lambda_n \to 1$ and $\{z_n\}$ desired as the claim. $\square$

Lemma 3.6. *$\{z_n\}_{n=1}^\infty$ is bounded in $W^{1,p(x)}(\Omega)$.*

Proof. We argue by contradiction. Passing to a subsequence if necessary, still denoted by $\{z_n\}$, we may assume that $\|z_n\|_{1,p(x)} \to \infty$ as $n \to \infty$. Let $\{\omega_n\} \subset W^{1,p(x)}(\Omega)$ and put $\omega_n := \frac{z_n}{\|z_n\|_{1,p(x)}}$. Since $\|\omega_n\|_{1,p(x)} = 1$, up to subsequences, we obtain

$$\begin{gathered}
\omega_n \rightharpoonup \omega \quad \text{in } W^{1,p(x)}(\Omega),\\
\omega_n \to \omega \quad \text{in } L^{\gamma(x)}(\Omega),\\
\omega_n(x) \to \omega(x) \quad \text{a.e. } x \in \Omega.
\end{gathered}$$

Then, the main concern is whether $\{\omega_n\} \subset W^{1,p(x)}(\Omega)$ vanish or not. We shall prove that none of these alternatives can occur and this contradiction will prove that $\{\omega_n\} \subset W^{1,p(x)}(\Omega)$ is bounded.

If $\omega = 0$, we can define a sequence $\{t_n\} \subset \mathbb{R}$, as argued in [16], such that

$$J_{\lambda_n}(t_n z_n) := \max_{t\in[0,1]} J_{\lambda_n}(t z_n). \tag{3.3}$$

Let $\overline{\omega}_n := (2p^+c)^{\frac{1}{p^-}}\omega_n$ with $c > 0$. Then for n is large enough, we have

$$\begin{aligned}
J_{\lambda_n}(t_n z_n) &\ge J_{\lambda_n}(\overline{\omega}_n) \ge A((2p^+c)^{\frac{1}{p^-}}\omega_n) - \lambda_n B(\overline{\omega}_n)\\
&\ge \frac{1}{p^+}(2p^+c)A(\omega_n) - \lambda_n B(\overline{\omega}_n) \ge 2c - \lambda_n B(\overline{\omega}_n) \ge c,
\end{aligned} \tag{3.4}$$

which implies that $\lim_{n\to\infty} J_{\lambda_n}(t_n z_n) = \infty$ by the fact $c > 0$ can be large arbitrarily. Noting that $J_{\lambda_n}(0) = 0$ and $J_{\lambda_n}(z_n) \to c$, then $0 < t_n < 1$, when n is large enough. Hence, we obtain

$$\langle J'_{\lambda_n}(t_n z_n), t_n z_n\rangle = A'(t_n z_n) - \lambda_n B'(t_n z_n) = 0. \tag{3.5}$$

Thus, from (3.4) and (3.5), we can write

$$\lambda_n\big(\frac{1}{\overline{p}_{t_n}} B'(t_n z_n) - B(t_n z_n)\big) = \frac{1}{\overline{p}_{t_n}} A'(t_n z_n) - \lambda_n B(t_n z_n) = J_{\lambda_n}(t_n z_n) \to \infty$$

as $n \to \infty$, where $\overline{p}_{t_n} = \frac{A'(t_n z_n)}{A(t_n z_n)}$.

Let $\gamma_{z_n} = \overline{p}_n = \frac{A'(z_n)}{A(z_n)}$, $\gamma_{t_n z_n} = \overline{p}_{t_n}$, then $\gamma_{z_n}, \gamma_{t_n z_n} \in [p^-, p^+]$. Thus, $G_{\gamma_{z_n}}$, $G_{\gamma_{t_n z_n}} \in \mathcal{F}$. Using condition (G1) and the fact $\inf_n \frac{\overline{p}_{t_n}}{\overline{p}_n \xi} > 0$, we have

$$\begin{aligned}
\big(\frac{1}{\overline{p}_n} B'(z_n) - B(z_n)\big) &= \frac{1}{\overline{p}_n}\int_\Omega G_{\gamma_{z_n}}(x, z_n)dx \ge \frac{1}{\overline{p}_n \xi}\int_\Omega G_{\gamma_{t_n z_n}}(x, t_n z_n)dx\\
&= \frac{\overline{p}_{t_n}}{\overline{p}_n \xi}\big(\frac{1}{\overline{p}_{t_n}} B'(t_n z_n) - B(t_n z_n)\big) \to +\infty.
\end{aligned}$$

This contradicts the following result of Lemma 3.5,

$$\lambda_n\big(\frac{1}{\overline{p}_n} B'(z_n) - B(z_n)\big) = J_{\lambda_n}(z_n) - \frac{1}{\overline{p}_n}\langle J'_{\lambda_n}(z_n), z_n\rangle = J_{\lambda_n}(z_n) \in [\bar{b}_k, \bar{c}_k].$$

If $\omega \neq 0$, since $J'_{\lambda_n}(z_n) = 0$, we have, by Proposition 2.1,

$$\begin{aligned} 1 - o(1) &= \int_\Omega \frac{f(x,z_n)z_n}{\varphi(z_n)}dx \ \geq \int_\Omega \frac{f(x,z_n)z_n}{\|z_n\|^{p^+}_{1,p(x)}}dx \\ &\geq \int_\Omega \frac{f(x,z_n)z_n}{\|z_n\|^{\theta}_{1,p(x)}}dx = \int_\Omega \frac{f(x,z_n)z_n}{|z_n|^\theta}|\omega_n|^\theta dx, \end{aligned} \tag{3.6}$$

where $\varphi(z_n) := \int_\Omega (|\nabla z_n|^{p(x)} + |z_n|^{p(x)})dx$.

Define the set $\Omega_0 = \{x \in \Omega : \omega(x) = 0\}$. Then for $x \in \Omega \backslash \Omega_0 = \{x \in \Omega : \omega(x) \neq 0\}$, we have $|z_n(x)| \to +\infty$ as $n \to \infty$. Hence, by $(\mathbf{S}_1)$ and $(\mathbf{S}_2)$, we have

$$\frac{f(x,z_n)z_n}{|z_n|^\theta}|\omega_n|^\theta dx \to +\infty \quad \text{as } n \to \infty.$$

Using Fatou's lemma and that $|\Omega\backslash\Omega_0| > 0$, we obtain

$$\int_{\Omega\backslash\Omega_0} \frac{f(x,z_n)z_n}{|z_n|^\theta}|\omega_n|^\theta dx \to +\infty \quad \text{as } n \to \infty. \tag{3.7}$$

On the other hand, by condition (S2), there exists $c > -\infty$ such that $\frac{f(x,t)t}{t^\theta} \geq c$ for $t \in \mathbb{R}$ and a.e. $x \in \overline{\Omega}$. Moreover, we have $\int_{\Omega_0} |\omega_n|^\theta dx \to 0$ as $n \to \infty$. Thus, there exists $\Lambda > -\infty$ such that

$$\int_{\Omega_0} \frac{f(x,z_n)z_n}{|z_n|^\theta}|\omega_n|^\theta dx \geq c\int_{\Omega_0} |\omega_n|^\theta dx \geq \Lambda > -\infty. \tag{3.8}$$

Combining (3.6), (3.7) and (3.8), we obtain a contradiction. Therefore, $\{z_n\}_{n=1}^\infty$ is bounded, and the proof is complete. □

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