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MULTIPLE SOLUTIONS FOR A QUASILINEAR (p, q)-ELLIPTIC SYSTEM

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Abstract. We prove the existence of three weak solutions of a quasilinear elliptic system involving a general (p, q)-elliptic operator in divergence form, with $1 < p \leqslant n$, $1 < q \leqslant n$. Our main tool is an adaptation of a three critical points theorem due to Ricceri.

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and $1 < p \leqslant n$, $1 < q \leqslant n$. In this article, we show the existence of multiple solutions for system of elliptic differential equations

$$\begin{gathered}
-\operatorname{div}(a_1(x, \nabla u)) = \lambda g_1(x, u) + \mu F_u(x, u, v) \quad \text{in } \Omega \\
-\operatorname{div}(a_2(x, \nabla v)) = \lambda g_2(x, v) + \mu F_v(x, u, v) \quad \text{in } \Omega \\
u = 0, \quad v = 0 \quad \text{on } \partial\Omega
\end{gathered} \tag{1.1}$$

where $1 < p, q \leqslant n$.

Many publication, such as [3, 7, 9], discuss quasilinear elliptic systems involving p-Laplacian operators and show the existence and multiplicity of solutions. Boccardo and Figueiredo [3] studied the existence of solutions for

$$\begin{gathered}
-\Delta_p u = F_u(x, u, v) \quad \text{in } \Omega \\
-\Delta_q u = F_v(x, u, v) \quad \text{in } \Omega \\
u = 0, \quad v = 0 \quad \text{on } \partial\Omega
\end{gathered}$$

where p, q are real numbers larger than 1.

Using the fibering method introduced by Pohozaev, Bozhkov and Mitidieri [7] proved the existence of multiple solutions for a quasilinear system involving a pair of (p,q)-Laplacian operators. In [9] the existence of three solutions for the eigenvalue problem

$$\begin{gathered}
-\Delta_p u = \lambda F_u(x, u, v) \quad \text{in } \Omega \\
-\Delta_q u = \lambda F_v(x, u, v) \quad \text{in } \Omega \\
u = 0, \quad v = 0 \quad \text{on } \partial\Omega
\end{gathered} \tag{1.2}$$

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where $p > n, q > n$ is ensured for suitable F.

Some other works [8, 12, 11, 10] studied mainly problems involving p-Laplacian type elliptic operators in divergence form and related eigenvalue problems

$$\begin{gathered} -\operatorname{div}(a(x, \nabla u)) = \lambda f(x, u) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{gathered}$$

These operators have p-Laplacian operator as a simple case; i.e., if $a(x, s) = |s|^{p-2}s$ then for $p \geqslant 2$ we have $\Delta_p u = \operatorname{div}(a(x, \nabla u))$ and moreover they have other important cases, such as the generalized mean curvature operator $\operatorname{div}\big((1+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u\big)$ which is generated by $a(x, s) = (1+|s|^2)^{\frac{p-2}{2}}s$ and is used in studying the geometric properties of manifolds especially minimal surfaces.

The existence of multiple solutions for this type of nonlinear differential equations was studied in [5, 12]. Many of these results are based on some three critical points theorems of Ricceri and Bonanno established in [13, 4]. In [15], Ricceri developed one of his results, [13, Theorem 1] by means of an abstract result, [14, Theorem 4].

In this article, we shall give a variant of Ricceri's three critical points theorem [15] which it seems its verification for some type of elliptic operators like $\operatorname{div}\big(a(x, \nabla u)\big)$ is easier. As an application, we study the existence of at least three weak solutions for (1.1). Our approach in dealing with (1.1) is very close to Ricceri's one in [15] but employs some calculations of [10] to adjust it to our problem.

2. Preliminaries

In the sequel, for any $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$ by $|\xi|$ we mean the usual Euclidean norm of ξ; that is, $|\xi| = \sqrt{\xi_1^2 + \xi_2^2 + \dots + \xi_n^2}$ which is produced by the inner product $\xi \cdot \eta = \sum_{i=1}^n \xi_i \eta_i$ in which $\xi, \eta \in \mathbb{R}^n$. Also for every $1 \leqslant p < \infty$ and open $\Omega \subset \mathbb{R}^n$ and measurable $u : \Omega \to \mathbb{R}$ we define

$$\|u\|_{L^p(\Omega)} = \Big(\int_\Omega |u|^p dx\Big)^{1/p}$$

and for $p > 1$ we assume the reflexive separable Sobolev space $W_0^{1,p}(\Omega)$ is endowed with the norm

$$\|u\|_p = \Big(\int_\Omega |\nabla u|^p dx\Big)^{1/p}$$

which is equivalent with its usual norm

$$\|u\|_{W_0^{1,p}(\Omega)} = \Big(\int_\Omega |u|^p + |\nabla u|^p dx\Big)^{1/p}.$$

By setting $p_1 = p$, $p_2 = q$, and inspired by De Nápoli and Mariani [10] and Deng and Pi [5], we assume that the $a_i : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}^n$, for $i = 1, 2$, satisfy the following conditions:

- (H1) There exists continuous function $A_i : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}$ such that $A_i(x, \xi)$ has $a_i(x, \xi)$ as its continuous derivative with respect to ξ at every $(x, \xi) \in \overline{\Omega} \times \mathbb{R}^n$ with the following additional properties:
  - (a) $A_i(x, 0) = 0, \quad \forall x \in \Omega.$
  - (b) There exists some constant $C_1 > 0$ such that a_i satisfies the growth condition

$$|a_i(x, \xi)| \leqslant C_1(1 + |\xi|^{p_i-1}), \quad \forall \xi \in \mathbb{R}^n. \tag{2.1}$$

(c) A_i is strictly convex: For every $t \in [0,1]$

$$A_i\big(x,(1-t)\xi + t\eta\big) \leqslant (1-t)A_i(x,\xi) + tA_i(x,\eta), \quad \forall x \in \Omega,\ \forall \xi,\eta \in \mathbb{R}^n \tag{2.2}$$

and this inequality is strict if $t \in (0,1)$.

(d) A_i satisfies the ellipticity condition: There exists a constant $C_2 > 0$ such that

$$A_i(x,\xi) \geqslant C_2|\xi|^{p_i}, \quad \forall x \in \Omega,\ \forall \xi \in \mathbb{R}^n. \tag{2.3}$$

Assumption (H1) has some consequences that will be helpful in this article. From the strict convexity and differentiability of $A_i(x,\xi)$ with respect to ξ, and assumption (H1)(c), we have

$$A_i(x,\eta) \geqslant A_i(x,\xi) + a_i(x,\xi)(\eta - \xi),$$

from which it follows that

$$\big(a_i(x,\xi) - a_i(x,\eta)\big) \cdot (\xi - \eta) \geqslant 0, \tag{2.4}$$

for every $x \in \Omega$ and $\xi, \eta \in \mathbb{R}^n$. Also, from (2.4) we obtain

$$a_i(x,\xi + t\eta)\eta \geqslant a_i(x,\xi)\eta \tag{2.5}$$

for every $t > 0$ and ξ, $\eta \in \mathbb{R}^n$.

We say the mapping $F : X \to X^*$ satisfies the S_+ condition, if every sequence $\{x_n\}_{n=1}^\infty$ in X such that $x_n \rightharpoonup x$ and $\limsup_{n\to\infty}\langle F(x_n), x_n - xt\rangle \leqslant 0$ has a convergent subsequence $\{x_{n_k}\}_{k=1}^\infty$ such that $x_{n_k} \to x$.

Proposition 2.1. *Let X be a reflexive Banach space and $F, J : X \to \mathbb{R}$ two C^1 functionals on X. If the mapping $F' : X \to X^*$ satisfies S_+ condition and $J' : X \to X^*$ is compact and $F + J : X \to \mathbb{R}$ is coercive then $F + J$ satisfies the Palais-Smale condition.*

Proof. If $\{x_n\}_{n=1}^\infty$ is a sequence in X such that $|F(x_n) + J(x_n)| < M$ for some $M > 0$ and any $n \in \mathbb{N}$ and $\|F'(x_n) + J'(x_n)\| \to 0$ then coercivity of $F + J$ implies boundedness of $\{x_n\}_{n=1}^\infty$ and since X is reflexive, there exists a subsequence $\{x_{n_k}\}_{k=1}^\infty$ of $\{x_n\}_{n=1}^\infty$ and $x \in X$ such that $x_{n_k} \rightharpoonup x$. Now compactness of $J' : X \to X^*$ implies there exists $x^* \in X^*$ such that $J(x_{n_k}) \to x^*$ up to a subsequence. Then since

$$\langle J'(x_{n_k}), x_{n_k} - x\rangle = \langle J'(x_{n_k}) - x^*, x_{n_k} - x\rangle + \langle x^*, x_{n_k} - x\rangle$$

and $\{x_{n_k}\}_{k=1}^\infty$ is bounded and $x_{n_k} \rightharpoonup x$, we have $\langle J'(x_{n_k}), x_{n_k} - x\rangle \to 0$. Therefore,

$$\begin{aligned}
&\limsup_{n\to\infty}\langle F'(x_{n_k}), x_{n_k} - x\rangle \\
&\leqslant \limsup_{n\to\infty}\langle F'(x_{n_k}) + J'(x_{n_k}), x_{n_k} - x\rangle - \lim_{n\to\infty}\langle J'(x_{n_k}), x_{n_k} - x\rangle \\
&\leqslant \limsup_{n\to\infty}\|F'(x_{n_k}) + J'(x_{n_k})\|\,\|x_{n_k} - x\| = 0.
\end{aligned}$$

Hence, by S_+ condition of F', for a subsequence of $\{x_{n_k}\}_{k=1}^\infty$ without relabeling $x_{n_k} \to x$. $\square$

3. Main results

First we give a theorem that is a variant of [15, Theorem 1].

Theorem 3.1. *Let X be a separable and reflexive real Banach space; $I \subset \mathbb{R}$ an interval; $\Phi : X \to \mathbb{R}$ a weakly sequentially lower semicontinuous C^1 functional, bounded on each bounded subset of X and has unique global minimum at $x_0 \in X$ and further the mapping $\Phi' : X \to X^*$ satisfies S_+ condition and for every bounded $E \subset X$ there exist constants $C > 0$ and $\nu > 0$ such that for every $x \in E$*

$$\Phi(x) - \Phi(x_0) \geqslant C\|x - x_0\|^{\nu}.$$

Also suppose $J : X \to \mathbb{R}$ be a C^1 functional with compact derivative such that for each $\lambda \in I$, the functional $\Phi - \lambda J$ is coercive and has a strict local not global minimum at x_0.

Then for each compact interval $[a, b] \subset I$, there exists $r > 0$ with the following property: for every $\lambda \in [a, b]$ and every C^1 functional $\Psi : X \to \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the equation

$$\Phi'(x) = \lambda J'(x) + \mu \Psi'(x)$$

has at least three solutions whose norms are less than r.

To prove the above theorem, we need the following lemma which is a variant of [15, Theorem C].

Lemma 3.2. *Let X be a separable and reflexive real Banach space, $\Phi : X \to \mathbb{R}$ a functional that has unique global minimum at $x_0 \in X$ and furthermore for every bounded $E \subset X$ there exist constants $C > 0$ and $\nu > 0$ such that for every $x \in E$*

$$\Phi(x) - \Phi(x_0) \geqslant C\|x - x_0\|^{\nu}. \tag{3.1}$$

Let $J : X \to \mathbb{R}$ be a weakly sequentially lower semicontinuous functional. Assume that $\Phi + J$ has a local strict minimum at x_0 in the strong topology of X and

$$\lim_{\|x\| \to \infty} \big(\Phi(x) + J(x)\big) = \infty.$$

Then x_0 is a strict local minimum of $\Phi + J$ in the weak topology of X.

Proof. The main part of the proof is the same as that of [15, Theorem C]. We show x_0 must be a strict local minimum in the weak topology of X. If not, by assumption there exists $\rho > 0$ such that

$$\Phi(x_0) + J(x_0) < \Phi(x) + J(x)$$

for every $x \in X$ satisfying $\|x\| > \rho$. Set

$$B = \{x \in X \ : \|x\| \leqslant \rho\}.$$

Since X is separable and reflexive, the set B is metrizable in its weak topology which we denote its metric by σ. Since we suppose x_0 is not a strict local minimum in weak topology of X, there exists a sequence $\{x_n\}$ in X such that for every $n \in \mathbb{N}$,

$$\sigma(x_0, x_n) < \frac{1}{n}, \quad \Phi(x_n) + J(x_n) \leqslant \Phi(x_0) + J(x_0). \tag{3.2}$$

So, $x_n \in B$ and $x_n \rightharpoonup x_0$. Then weakly sequentially lower semicontinuity of J implies

$$\liminf_{n\to\infty} \Phi(x_n) + J(x_0) \leqslant \liminf_{n\to\infty} \Phi(x_n) + \liminf_{n\to\infty} J(x_n)$$

$$\leqslant \liminf_{n\to\infty} \big(\Phi(x_n) + J(x_n)\big) \leqslant \Phi(x_0) + J(x_0).$$

and therefore,

$$\liminf_{n\to\infty} \Phi(x_n) \leqslant \Phi(x_0).$$

But $\Phi(x_0)$ is the global minimum of $\Phi(x)$ so, for a suitable convergent subsequence of $\Phi(x_n)$ we have

$$\lim_{n\to\infty} \Phi(x_n) = \Phi(x_0)$$

then by (3.1) we have $x_n \to x_0$ which contradicts strict local minimality of $\Phi(x_0) + J(x_0)$ in the strong topology of X by (3.2). $\square$

Proof of Theorem 3.1. Following the arguments in [15, Theorem 1], since any C^1 functional with compact derivative on X is weakly sequentially continuous [17, Corollary 41.9], and in particular, it is bounded on each bounded subset of X, so for any compact $[a,b] \subset I$ and $\sigma > \sup_{\lambda\in[a,b]} \big(\Phi(x_0) - \lambda J(x_0)\big)$,

$$\begin{aligned} &\cup_{\lambda\in[a,b]} \{x \in X : \Phi(x) - \lambda J(x) < \sigma\} \\ &\subset \{x \in X : \Phi(x) - aJ(x) < \sigma\} \cup \{x \in X : \Phi(x) - bJ(x) < \sigma\}. \end{aligned}$$

By the coercivity assumption, the set on the right is bounded and there exists $\eta > 0$ such that

$$\cup_{\lambda\in[a,b]} \{x \in X : \Phi(x) - \lambda J(x) < \sigma\} \subset B_\eta \tag{3.3}$$

where $B_\eta = \{x \in X : \|x\| < \eta\}$. Now, set

$$c^* = \sup_{B_\eta} \Phi + \max\{|a|, |b|\} \sup_{B_\eta} |J|$$

and choose $r > \eta$ so that

$$\cup_{\lambda\in[a,b]} \{x \in X : \Phi(x) - \lambda J(x) < c^* + 2\} \subset B_r\,. \tag{3.4}$$

Now, for any C^1 functional $\Psi : X \to \mathbb{R}$ with compact derivative, choose a bounded C^1 function $g : \mathbb{R} \to \mathbb{R}$ with bounded derivative such that $g(t) = t$ for every $-\sup_{B_r} |\Psi| \leqslant t \leqslant \sup_{B_r} |\Psi|$. Then $\tilde\Psi : X \to \mathbb{R}$ defined by $\tilde\Psi(x) = g \circ \Psi(x)$ is a C^1 functional on X such that $\tilde\Psi(x) = \Psi(x)$ for all $x \in B_r$. On the other hand, for every $E \subset X$

$$\tilde\Psi'(E) \subset g'\big(\Psi(E)\big)\Psi'(E)$$

and therefore $\tilde\Psi' : X \to X^*$ is compact. In addition, by Lemma 3.2 the functional $\Phi - \lambda J$ has a strict local minimum at x_0 in the weak topology of X, for any $\lambda \in [a,b]$. So, by applying [14, Theorem 4] to the functionals $-\tilde\Psi$ and $\Phi - \lambda J$ by taking τ as the weak topology of X and considering (3.3) and the fact that the topology $\tau_{\Phi-\lambda J}$ is weaker than the strong one, the existence of some $\gamma > 0$ is deduced such that for each $\mu \in [0,\gamma]$ the functional $\Phi - \lambda J - \mu\tilde\Psi$ has at least two local minimum in B_η, say x_1, x_2. Now, If

$$\delta = \min\{\gamma, \frac{1}{\sup_{\mathbb{R}} |g|}\}$$

then for every $\mu \in [0,\delta]$ the functional $\Phi - \lambda J - \mu\tilde\Psi$ is coercive by assumption and satisfies Palais-Smale condition, by Proposition 2.1. Set

$$\mathcal{S} = \{u \in C([0,1], X) : u(0) = x_1,\ u(1) = x_2\},$$

$$c_{\lambda,\mu} = \inf_{u\in\mathcal{S}} \sup_{t\in[0,1]} \big(\Phi(u(t)) - \lambda J(u(t)) - \mu\tilde\Psi(u(t))\big)$$

then by the Mountain Pass Theorem [1, Theorem 8.2]), there exists $x_3 \in X$ distinct from x_1 and x_2 such that

$$\Phi'(x_3) - \lambda J'(x_3) - \mu \tilde{\Psi}'(x_3) = 0, \quad \Phi(x_3) - \lambda J(x_3) - \mu \tilde{\Psi}(x_3) = c_{\lambda,\mu}.$$

Now since

$$\begin{aligned} c_{\lambda,\mu} &\leqslant \sup_{t\in[0,1]} \Phi(x_1 + t(x_2 - x_1)) - \lambda J(x_1 + t(x_2 - x_1)) - \mu \tilde{\Psi}(x_1 + t(x_2 - x_1)) \\ &\leqslant \sup_{B_\eta} \Phi + \max\{|a|, |b|\} \sup_{B_\eta} J + \delta \sup_{\mathbb{R}} |g| \leqslant c^* + 1, \end{aligned}$$

we have $\Phi(x_3) - \lambda J(x_3) < c^* + 2$ and therefore $x_3 \in B_r$ by (3.4). Since $\Psi(x) = \tilde{\Psi}(x)$ for every $x \in B_r$ so $\Psi'(x_i) = \tilde{\Psi}'(x_i)$ for $i = 1, 2, 3$. Thus x_1, x_2, x_3 are three solutions of $\Phi'(x) = \lambda J'(x) + \mu \Psi'(x)$ in B_r □

Our main tool in studying (1.1) is the following Theorem, which in fact is a restatement of [15, Theorem 2]. It adopts it to our situation and its proof is the same as that of [15, Theorem 2], except that we use Theorem 3.1 instead of [15, Theorem 1], and remove the phrase $\hat{x}_\lambda = x_0$. Therefore we omit its proof.

Theorem 3.3. *Let X be a separable and reflexive real Banach space; $I \subset \mathbb{R}$ an interval; $\Phi : X \to \mathbb{R}$ a weakly sequentially lower semicontinuous C^1 functional that has unique global minimum at $x_0 \in X$ and for every bounded $E \subset X$ there exist some constants $C > 0$ and $\nu > 0$ such that for every $x \in E$*

$$\Phi(x) - \Phi(x_0) \geqslant C\|x - x_0\|^\nu.$$

Let $J : X \to \mathbb{R}$ be a C^1 functional with compact derivative. Finally, setting

$$\alpha = \max\big\{0, \limsup_{\|x\|\to\infty} \frac{J(x)}{\Phi(x)}, \limsup_{x\to x_0} \frac{J(x)}{\Phi(x)}\big\}, \quad \beta = \sup\big\{\frac{J(x)}{\Phi(x)} : x \in \Phi^{-1}(]0,\infty[)\big\},$$

assume that $\alpha < \beta$. Then, for each compact interval $[a, b] \subset]\frac{1}{\beta}, \frac{1}{\alpha}[$ (with the conventions $\frac{1}{0} = \infty$, $\frac{1}{\infty} = 0$) there exists $r > 0$ with the following property: for every $\lambda \in [a, b]$ and every C^1 functional $\Psi : X \to \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [a, b]$, the equation

$$\Phi'(x) = \lambda J'(x) + \mu \Psi'(x)$$

has at least three solutions whose norms are less than r.

Hereafter we denote by X the product real Banach space $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ in which $p, q > 1$ and equip it with the norm

$$\|(u, v)\| = \|u\|_p + \|v\|_q = (\int_\Omega |\nabla u|^p dx)^{1/p} + (\int_\Omega |\nabla v|^q dx)^{\frac{1}{q}}.$$

At every $(u, v) \in X$, define

$$\Phi(u, v) = \int_\Omega A_1(x, \nabla u)\, dx + \int_\Omega A_2(x, \nabla v)\, dx, \quad \Psi(u, v) = \int_\Omega F\big(x, u(x), v(x)\big)\, dx,$$

$$J(u, v) = \int_\Omega \int_0^{u(x)} g_1(x, s)\, ds\, dx + \int_\Omega \int_0^{v(x)} g_2(x, s)\, ds\, dx$$

in which g_1, g_2 satisfy the following inequalities for some constant $C > 0$,

$$|g_1(x, \xi)| \leqslant C(1 + |\xi|^{\tau-1}), \quad |g_2(x, \xi)| \leqslant C(1 + |\xi|^{\kappa-1}), \tag{3.5}$$

for a.e. $x \in \Omega$ where $1 < \tau < p$ and $1 < \kappa < q$.

Before stating and proving our main result for (1.1), i.e., Theorem 3.8, we establish some lemmas which are useful in proving this theorem. In fact, we gathered needed hypotheses of Theorem 3.8 in these lemmas.

Lemma 3.4. *Let $\Phi : X \to \mathbb{R}$ be defined as above. If the functions A_i for $i = 1, 2$ satisfy* (H1)*, then $\Phi \in C^1(X; \mathbb{R})$. In particular $\Phi' : X \to X^*$ is continuous.*

Proof. At $(u, v) \in X$ for every $(\xi, \mu) \in X$ and $0 < |t| < 1$, by applying the Mean Value Theorem for A_i's we obtain

$$\begin{aligned}
&\langle \Phi'(u, v), (\xi, \mu)\rangle \\
&= \lim_{t\to 0} \frac{\Phi(u + t\xi, v + t\mu) - \Phi(u, v)}{t} \\
&= \lim_{t\to 0} \Big(\int_\Omega a_1(x, \nabla u + t\theta_1(x)\nabla\xi) \cdot \nabla\xi \, dx + \int_\Omega a_2(x, \nabla v + t\theta_2(x)\nabla\mu) \cdot \nabla\mu \, dx\Big)
\end{aligned}$$

in which $0 < \theta_1(x), \theta_2(x) < 1$ for every $x \in \Omega$. Now by the Cauchy-Schwarz inequality and (2.1),

$$\begin{aligned}
\big|a_1(x, \nabla u + t\theta_1(x)\nabla\xi) \cdot \nabla\xi\big| &\leqslant C\big(1 + |\nabla u + t\theta_1(x)\nabla\xi|^{p-1}\big)\, |\nabla\xi| \\
&\leqslant C(1 + 2^{p-1}\big(|\nabla u|^{p-1} + |\nabla\xi|^{p-1}\big))|\nabla\xi|,
\end{aligned}$$

and since

$$\int_\Omega (1 + 2^{p-1}\big(|\nabla u|^{p-1} + |\nabla\xi|^{p-1}\big))|\nabla\xi| \, dx \leqslant C\Big(m(\Omega) + \|u\|_p^p + \|\xi\|_p^p\Big)^{1/p'} \|\xi\|_p$$

where C denotes a constant and $m(\Omega)$ is the Lebesgue measure of Ω and $p' = \frac{p}{p-1}$ is the Hölder conjugate of p, then the Dominated Convergence Theorem implies

$$\lim_{t\to 0} \int_\Omega a_1(x, \nabla u + t\theta_1(x)\nabla\xi) \cdot \nabla\xi \, dx = \int_\Omega a_1(x, \nabla u) \cdot \nabla\xi \, dx.$$

Similarly,

$$\lim_{t\to 0} \int_\Omega a_2(x, \nabla v + t\theta_2(x)\nabla\mu) \cdot \nabla\mu \, dx = \int_\Omega a_2(x, \nabla v) \cdot \nabla\mu \, dx,$$

and the functional Φ is Gâteaux differentiable at every $(u, v) \in X$ and

$$\langle \Phi'(u, v), (\xi, \mu)\rangle = \int_\Omega a_1(x, \nabla u) \cdot \nabla\xi + a_2(x, \nabla v) \cdot \nabla\mu \, dx. \quad \forall (\xi, \eta) \in X$$

Now we prove $\Phi' : X \to X^*$ is continuous. Suppose $(u_n, v_n) \to (u, v)$ in X then by the Hölder inequality for every $(\xi, \eta) \in X$ we have

$$\begin{aligned}
&\big|\langle \Phi'(u_n, v_n) - \Phi'(u, v), (\xi, \mu)\rangle\big| \\
&\leqslant \int_\Omega \Big|\Big(a_1(x, \nabla u_n) - a_1(x, \nabla u)\Big) \cdot \nabla\xi\Big| + \Big|\Big(a_2(x, \nabla v_n) - a_2(x, \nabla v)\Big) \cdot \nabla\mu\Big| \, dx \\
&\leqslant \|a_1(x, \nabla u_n) - a_1(x, \nabla u)\|_{L^{p'}(\Omega)} \|\xi\|_p + \|a_2(x, \nabla v_n) - a_2(x, \nabla v)\|_{L^{q'}(\Omega)} \|\mu\|_q \\
&\leqslant \Big(\|a_1(x, \nabla u_n) - a_1(x, \nabla u)\|_{L^{p'}(\Omega)} + \|a_2(x, \nabla v_n) - a_2(x, \nabla v)\|_{L^{q'}(\Omega)}\Big)\|(\xi, \mu)\|,
\end{aligned}$$

where $q' = \frac{q}{q-1}$ is the Hölder conjugate of q. Hence, it is sufficient to show that

$$\lim_{n\to\infty} \|a_1(x, \nabla u_n) - a_1(x, \nabla u)\|_{L^{p'}(\Omega)} + \|a_2(x, \nabla v_n) - a_2(x, \nabla v)\|_{L^{q'}(\Omega)} = 0.$$

If not, we have

$$\limsup_{n\to\infty} \|a_1(x,\nabla u_n) - a_1(x,\nabla u)\|_{L^{p'}(\Omega)} + \|a_2(x,\nabla v_n) - a_2(x,\nabla v)\|_{L^{q'}(\Omega)} > 0,$$

then there exists a subsequence of $\{(u_n, v_n)\}$ which we denote it by the same notation $\{(u_n, v_n)\}$ for which

$$\lim_{n\to\infty} \|a_1(x,\nabla u_n) - a_1(x,\nabla u)\|_{L^{p'}(\Omega)} + \|a_2(x,\nabla v_n) - a_2(x,\nabla v)\|_{L^{q'}(\Omega)} > 0. \quad (3.6)$$

Since $(u_n, v_n) \to (u, v)$ in X, we have $u_n \to u$ and $v_n \to v$ in $W_0^{1,p}(\Omega)$ and $W_0^{1,q}(\Omega)$ respectively. So there exist subsequences $\{u_{n_k}\}$ and $\{v_{n_k}\}$ of $\{u_n\}$ and $\{v_n\}$ respectively and some functions $g \in L^p(\Omega)$ and $h \in L^q(\Omega)$ such that $|\nabla u_{n_k}(x)| \leqslant g(x)$ and $\nabla u_{n_k} \to \nabla u$ a.e. and $|\nabla v_{n_k}(x)| \leqslant h(x)$ and $\nabla v_{n_k} \to \nabla v$ a.e. as well. Thus for some constant C and a.e. $x \in \Omega$ we have

$$|a_1(x,\nabla u_{n_k}) - a_1(x,\nabla u)| \leqslant C(2 + |\nabla u_{n_k}|^{p-1} + |\nabla u|^{p-1}) \leqslant 2C(1 + g^{p-1})$$

and by a similar argument

$$|a_2(x,\nabla v_{n_k}) - a_1(x,\nabla v)| \leqslant 2C(1 + h^{p-1}).$$

Now by the Dominated Convergence Theorem

$$\lim_{k\to\infty} \|a_1(x,\nabla u_{n_k}) - a_1(x,\nabla u)\|_{L^{p'}(\Omega)} + \|a_2(x,\nabla v_{n_k}) - a_2(x,\nabla v)\|_{L^{q'}(\Omega)} = 0,$$

which contradicts (3.6). Therefore $\Phi' : X \to X^*$ is continuous and *a priori* $\Phi \in C^1(X;\mathbb{R})$. $\square$

Lemma 3.5. *Let $\Phi : X \to \mathbb{R}$ be defined as previously. Then $\Phi' : X \to X^*$ satisfies S_+ condition*

Proof. If $(u_n, v_n) \rightharpoonup (u, v)$ in X and

$$\limsup_{n\to\infty} \langle \Phi'(u_n, v_n), (u_n - u, v_n - v)\rangle \leqslant 0 \quad (3.7)$$

then since $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$ in $W_0^{1,p}(\Omega)$ and $W_0^{1,q}(\Omega)$ respectively

$$\begin{aligned}
&\limsup_{n\to\infty} \langle \Phi'(u_n, v_n), (u_n - u, v_n - v)\rangle \\
&= \limsup_{n\to\infty} \Big(\int_\Omega \big(a_1(x,\nabla u_n) - a_1(x,\nabla u)\big)(\nabla u_n - \nabla u)\,dx \\
&\quad + \int_\Omega \big(a_2(x,\nabla v_n) - a_2(x,\nabla v)\big)(\nabla v_n - \nabla v)\,dx \Big)
\end{aligned}$$

and by (2.4) and (3.7),

$$\lim_{n\to\infty} \langle \Phi'(u_n, v_n), (u_n - u, v_n - v)\rangle = 0,$$

and obviously

$$\lim_{n\to\infty} \int_\Omega \big(a_1(x,\nabla u_n) - a_1(x,\nabla u)\big)(\nabla u_n - \nabla u)\,dx = 0, \quad (3.8)$$

$$\lim_{n\to\infty} \int_\Omega \big(a_2(x,\nabla v_n) - a_2(x,\nabla v)\big)(\nabla v_n - \nabla v)\,dx = 0. \quad (3.9)$$

We shall prove $u_n \to u$ as a consequence of (3.8), and in a similar way (3.9) implies $v_n \to v$. By imitating the proof of [5, Lemma 2.3], put

$$P_n(x) = \big(a_1(x,\nabla u_n) - a_1(x,\nabla u)\big) \cdot (\nabla u_n - \nabla u).$$

Then (2.4) implies $P_n(x) \geqslant 0$ and because (3.8), there exists a subsequence of $\{u_n\}$ still denoted by $\{u_n\}$ for which $\lim_{n\to\infty} P_n(x) = 0$ a.e. in Ω. Let

$$E = \cap_{n\in\mathbb{N}}\{x \in \Omega : \lim_{n\to\infty} P_n(x) = 0,\ |\nabla u_n(x)| < \infty,\ |\nabla u(x)| < \infty\}.$$

Then $m(\Omega - E) = 0$, $\lim_{n\to\infty} P_n(x) = 0$ in E.

If $x_0 \in E$ then by the Mean Value Theorem and inequality (2.3),

$$\begin{aligned}
&|\nabla u_n(x_0)|^p \\
&\leqslant C_2^{-1} A_1\big(x_0, \nabla u_n(x_0)\big) = C_2^{-1} a_1\big(x_0, t_n \nabla u_n(x_0)\big) \cdot \nabla u_n(x_0) \quad \text{for some } t_n \in (0,1) \\
&\leqslant C_2^{-1} a_1\big(x_0, \nabla u_n(x_0)\big) \cdot \nabla u_n(x_0) \quad \text{by (2.5)} \\
&\leqslant C_2^{-1}[P_n(x_0) + a_1(x_0, \nabla u_n(x_0))\nabla u(x_0) + a_1(x_0, \nabla u(x_0)) \cdot (\nabla u_n(x_0) - \nabla u(x_0))] \\
&\leqslant C_2^{-1}[P_n(x_0) + C_1(1 + |\nabla u_n(x_0)|^{p-1})|\nabla u(x_0)| + C_1(1 + |\nabla u(x_0)|^{p-1})|\nabla u_n(x_0)| \\
&\quad + a_1\big(x_0, \nabla u(x_0)\big) \cdot \nabla u(x_0)] \quad \text{by (2.1)}
\end{aligned}$$

which implies $|\nabla u_n(x_0)| \leqslant C$ for some constant $C > 0$. Because by our assumption $\lim_{n\to\infty} P_n(x_0) = 0$, for any polynomial $q(t) = t^p + kt^{p-1} + mt + c$ with $p > 1$,

$$\lim_{t\to\infty} q(t) = \infty.$$

Now, if $\nabla u_n(x_0) \nrightarrow \nabla u(x_0)$, then $\{\nabla u_n(x_0)\}$ has a convergent subsequence which is denoted by the same notation $\{\nabla u_n(x_0)\}$ and converges to a vector $v_0 \neq \nabla u(x_0)$. Hence

$$\lim_{n\to\infty} P_n(x_0) = (a_1(x_0, v_0) - a_1\big(x_0, \nabla u(x_0)\big)) \cdot (v_0 - \nabla u(x_0)) > 0,$$

which contradicts the assumption $x_0 \in E$. Therefore, $\nabla u_n(x) \to \nabla u(x)$ for every $x \in E$.

As a consequence, $P_n(x) \to 0$ a.e. in Ω and if

$$g_n(x) = P_n(x) + \big(a_1(x, \nabla u_n) - a_1(x, \nabla u)\big) \cdot \nabla u + a_1(x, \nabla u) \cdot (\nabla u_n - \nabla u) + a_1(x, \nabla u) \cdot \nabla u$$

then above calculations show that

$$|\nabla u_n(x)|^p \leqslant C_2^{-1} g_n(x); \tag{3.10}$$

furthermore,

$$g_n(x) \to a_1(x, \nabla u) \cdot \nabla u \tag{3.11}$$

a.e. in Ω. By Lemma 3.4, the hypothesis $(u_n, v_n) \rightharpoonup (u, v)$ implies

$$\begin{gathered}
\lim_{n\to\infty} \int_\Omega \big(a_1(x, \nabla u_n) - a_1(x, \nabla u)\big) \cdot \nabla u \, dx = \lim_{n\to\infty} \langle \Phi'(u_n, v_n) - \Phi'(u, v), (u, 0)\rangle = 0, \\
\lim_{n\to\infty} \int_\Omega a_1(x, \nabla u) \cdot (\nabla u_n - \nabla u) \, dx = \lim_{n\to\infty} \langle \Phi'(u, v), (u_n - u, 0)\rangle = 0.
\end{gathered}$$

On the other hand, (3.8) gives

$$\lim_{n\to\infty} \int_\Omega P_n(x) \, dx = 0,$$

and hence

$$\lim_{n\to\infty} \int_\Omega g_n(x) = \int_\Omega a_1(x, \nabla u) \cdot \nabla u. \tag{3.12}$$

By (3.10), we obtain

$$|\nabla u_n(x) - \nabla u(x)|^p \leqslant 2^{p-1}(|\nabla u_n(x)|^p + |\nabla u(x)|^p) \leqslant 2^{p-1}(C_2^{-1} g_n(x) + |\nabla u(x)|^p)$$

and since $\nabla u_n(x) \to \nabla u(x)$ a.e. in Ω, so (3.11) implies

$$\lim_{n\to\infty} C_2^{-1} g_n(x) + |\nabla u(x)|^p = C_2^{-1} a_1(x, \nabla u) \cdot \nabla u + |\nabla u(x)|^p,$$

a.e. in Ω. By (3.12) we find

$$\begin{aligned}\lim_{n\to\infty} \int_\Omega C_2^{-1} g_n(x) + |\nabla u(x)|^p\,dx &= \int_\Omega C_2^{-1} a_1(x, \nabla u) \cdot \nabla u + |\nabla u(x)|^p\,dx \\ &\leqslant C_2^{-1} \|a_1(x, \nabla u)\|_{L^{p'}(\Omega)} \|u\|_p + \|u\|_p^p.\end{aligned}$$

by the Hölder inequality in which $p' = \frac{p}{p-1}$. Therefore, the Dominated Convergence Theorem implies

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n(x) - \nabla u(x)|^p\,dx = 0,$$

and therefore $u_n \to u$ in $W_0^{1,p}(\Omega)$. Similarly we have $v_n \to v$ in $W_0^{1,q}(\Omega)$ and finally $(u_n, v_n) \to (u, v)$ in X. $\square$

Lemma 3.6. *The functional $\Phi : X \to \mathbb{R}$ is weakly sequentially lower semicontinuous and the functional $J : X \to \mathbb{R}$ is C^1 with compact derivative and $\Phi - \lambda J$ is weakly sequentially lower semicontinuous and coercive for each $\lambda \in \mathbb{R}$.*

Proof. If $(u_n, v_n) \rightharpoonup (u, v)$ in X and $\liminf_{n\to\infty} \Phi(u_n, v_n) < \Phi(u, v)$ then there exists a subsequence of $\{(u_n, v_n)\}$ denote it by $\{(u_{n_k}, v_{n_k})\}$ such that $\{\Phi(u_{n_k}, v_{n_k})\}$ converges and $\lim_{n\to\infty} \Phi(u_{n_k}, v_{n_k}) < \Phi(u, v)$.

Since $\Phi \in C^1(X; \mathbb{R})$ by Lemma 3.4, the Mean Value Theorem implies the existence of $t_n \in (0, 1)$ for every $n \in \mathbb{N}$ such that

$$\Phi(u_n, v_n) - \Phi(u, v) = \langle \Phi'\big(u + t_n(u_n - u), v + t_n(v_n - v)\big), (u_n - u, v_n - v)\rangle.$$

On the other hand, (2.5) implies

$$\langle \Phi'(u, v), (\xi, \eta)\rangle \leqslant \langle \Phi'(u + t\xi, v + t\eta), (\xi, \eta)\rangle \tag{3.13}$$

for any $t \geqslant 0$ and $(\xi, \eta) \in X$. Therefore,

$$\langle \Phi'(u, v), (u_n - u, v_n - v)\rangle \leqslant \langle \Phi'\big(u + t_n(u_n - u), v + t_n(v_n - v)\big), (u_n - u, v_n - v)\rangle$$

and as a consequence,

$$\begin{aligned}&\limsup_{k\to\infty} \langle \Phi'(u, v), (u_{n_k} - u, v_{n_k} - v)\rangle \\ &\leqslant \lim_{k\to\infty} \langle \Phi'\big(u + t_{n_k}(u_{n_k} - u), v + t_{n_k}(v_{n_k} - v)\big), (u_{n_k} - u, v_{n_k} - v)\rangle < 0\end{aligned}$$

which contradicts $(u_n, v_n) \rightharpoonup (u, v)$ since $\Phi'(u, v) \in X^*$ by Lemma 3.4. Thus $\liminf_{n\to\infty} \Phi(u_n, v_n) \geqslant \Phi(u, v)$ and $\Phi : X \to \mathbb{R}$ is weakly sequentially lower semicontinuous.

It can be shown easily that J is a C^1 functional [2, Theorem 2.9] and

$$\langle J'(u, v), (\xi, \eta)\rangle = \int_\Omega g_1(x, u)\xi + g_2(x, v)\eta\,dx.$$

If $\{(u_n, v_n)\}$ is a bounded sequence in X then it has a weakly convergent subsequence by reflexivity of X which we also denote it by $\{(u_n, v_n)\}$ and assume $(u_n, v_n) \rightharpoonup (u, v)$. Since $1 < p, q \leqslant n$, the embedding $X \hookrightarrow L^p(\Omega) \times L^q(\Omega)$ is compact, up to a subsequence $(u_n, v_n) \to (u, v)$ and by [16, Proposition 26.6], the

Nemytski operators $g_1 : L^p(\Omega) \to L^{p'}(\Omega)$ and $g_2 : L^q(\Omega) \to L^{q'}(\Omega)$ are continuous and bounded where $p' = \frac{p}{p-1}$ and $q' = \frac{q}{q-1}$. Then

$$\begin{aligned}
&\big|\langle J'(u_n, v_n) - J'(u, v), (\xi, \mu)\rangle\big| \\
&\leqslant \Big| \int_\Omega \big(g_1(x, u_n) - g_1(x, u)\big)\xi + \big(g_2(x, v_n) - g_2(x, v)\big)\mu \, dx\Big| \\
&\leqslant \|g_1(x, u_n) - g_1(x, u)\|_{L^{p'}(\Omega)} \|\xi\|_{L^p(\Omega)} + \|g_2(x, v_n) - g_2(x, v)\|_{L^{q'}(\Omega)} \|\mu\|_{L^q(\Omega)} \\
&\leqslant \max\big\{\|g_1(x, u_n) - g_1(x, u)\|_{L^{p'}(\Omega)}, \|g_2(x, v_n) - g_2(x, v)\|_{L^{q'}(\Omega)}\big\}\|(\xi, \mu)\|
\end{aligned}$$

hence

$$\begin{aligned}
&\|J'(u_n, v_n) - J'(u, v)\| \\
&\leqslant \max\big\{\|g_1(x, u_n) - g_1(x, u)\|_{L^{p'}(\Omega)}, \|g_2(x, v_n) - g_2(x, v)\|_{L^{q'}(\Omega)}\big\},
\end{aligned}$$

for any $n \in \mathbb{N}$ and $(\xi, \mu) \in X$. Therefore, $J' : X \to X^*$ is compact and $J : X \to \mathbb{R}$ is weakly sequentially continuous by Corollary 41.9 [17]. Hence $\Phi - \lambda J$ are weakly sequentially lower semicontinuous functionals on X for every $\lambda \in \mathbb{R}$.

By (2.3) we obtain

$$\Phi(u, v) = \int_\Omega A_1(x, \nabla u) + A_2(x, \nabla v) \, dx \geqslant C_2\big(\|u\|_p^p + \|v\|_q^q\big)$$

and since according to (3.5),

$$\begin{aligned}
J(u, v) &\leqslant \int_\Omega \Big| \int_0^u g_1(x, s) \, ds\Big| + \Big| \int_0^v g_2(x, s) \, ds\Big| \, dx \\
&\leqslant C \int_\Omega (|u| + |u|^\tau + |v| + |v|^\kappa) \, dx \\
&\leqslant C(\|u\|_p^\tau + \|v\|_q^\kappa),
\end{aligned} \tag{3.14}$$

we have

$$\Phi(u, v) - \lambda J(u, v) \geqslant C_2\big(\|u\|_p^p + \|v\|_q^q\big) - C|\lambda|\big(\|u\|_p^\tau + \|v\|_q^\kappa\big).$$

Then for every $\lambda \in \mathbb{R}$,

$$\liminf_{\|(u,v)\| \to \infty} \Phi(u, v) - \lambda J(u, v) = \infty$$

and hence $\mathcal{E}_\lambda = \Phi - \lambda J$ is coercive. $\square$

Now we consider the properties of Ψ that we need in this article.

Lemma 3.7. *Let $F : \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ be a Carathédory function such that $F(x, 0, 0) \in L^1(\Omega)$ and $F(x, u, v)$ has continuous partial derivatives with respect to u and v in every $x \in \Omega$ and for some constant $C > 0$*

$$|F_u(x, u, v)| \leqslant C(1 + |u|^{p-1} + |v|^{q\frac{p-1}{p}}), \quad |F_v(x, u, v)| \leqslant C(1 + |u|^{p\frac{q-1}{q}} + |v|^{q-1})$$

for every $x \in \Omega$ and $u, v \in \mathbb{R}$. Then $\Psi \in C^1(X; \mathbb{R})$ and its derivative $\Psi' : X \to X^*$ is compact.

Proof. Since $F(x, u, v)$ is C^1 with respect to u, v, then for every $x \in \Omega$ there exist $\gamma(x), \theta(x)$ in $(0, 1)$ such that

$$\begin{aligned}
|F(x, u, v) - F(x, 0, 0)| &\leqslant |F(x, u, v) - F(x, u, 0)| + |F(x, u, 0) - F(x, 0, 0)| \\
&\leqslant |F_u(x, \gamma(x)u, 0)||u| + |F_v(x, u, \theta(x)v)||v|
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C(1+|u|^{p-1})|u| + C(1+|u|^{p\frac{q-1}{q}} + |v|^{q-1})|v| \\
&\leqslant C(1+|u|^p + |v|^q)
\end{aligned}
$$

hence $\Psi(u,v) \in \mathbb{R}$. Also for every $(u,v),(\xi,\mu)$ in X and $t \in \mathbb{R}-\{0\}$, by the Mean Value Theorem,

$$
\begin{aligned}
&\lim_{t\to 0} \frac{\Psi(u+t\xi, v+t\mu) - \Psi(u,v)}{t} \\
&= \lim_{t\to 0} \frac{1}{t} \int_\Omega F\big(x, u(x)+t\xi(x), v(x)+t\mu(x)\big) - F\big(x,u(x),v(x)\big)\,dx \\
&= \lim_{t\to 0} \Big\{ \int_\Omega F_u\big(x, u(x)+t\theta(x)\xi(x), v(x)+t\mu(x)\big)\xi(x)\,dx \\
&\quad + \int_\Omega F_v\Big(x, u(x), v(x)+t\gamma(x)\mu(x)\Big)\mu(x)\,dx\Big\},
\end{aligned}
$$

in which $0<\theta(x),\gamma(x)<1$ for any $x\in\Omega$. But F_u is continuous and

$$
F_u\big(x,u(x)+t\theta(x)\xi(x), v(x)+t\mu(x)\big) \to F_u\big(x,u(x),v(x)\big) \quad \text{as } t\to 0
$$

and for $|t|<1$,

$$
\begin{aligned}
&\big|F_u\big(x,u(x)+t\theta(x)\xi(x), v(x)+t\mu(x)\big)\xi(x)\big| \\
&\leqslant C\Big(1+(|u(x)|+|\xi(x)|)^{p-1} + (|v(x)|+|\mu(x)|)^{q\frac{p-1}{p}}\Big)|\xi(x)|
\end{aligned}
$$

therefore, the Dominated Convergence Theorem implies

$$
\lim_{t\to 0}\int_\Omega F_u\big(x,u(x)+t\theta(x)\xi(x), v(x)+t\mu(x)\big)\xi(x)\,dx = \int_\Omega F_u\big(x,u(x),v(x)\big)\xi(x)\,dx
$$

and similarly

$$
\lim_{t\to 0}\int_\Omega F_v\Big(x,u(x),v(x)+t\gamma(x)\mu(x)\Big)\mu(x)\,dx = \int_\Omega F_v\Big(x,u(x),v(x)\Big)\mu(x)\,dx.
$$

Therefore,

$$
\begin{aligned}
\langle \Psi'(u,v),(\xi,\mu)\rangle &= \lim_{t\to 0}\frac{\Psi(u+t\xi,v+t\mu)-\Psi(u,v)}{t} \\
&= \int_\Omega F_u(x,u,v)\xi + F_v(x,u,v)\mu\,dx
\end{aligned}
$$

and Ψ is Gâteaux differentiable at any $(u,v)\in X$ and for every $(\xi,\mu)\in X$

$$
\langle \Psi'(u,v),(\xi,\mu)\rangle = \int_\Omega F_u(x,u,v)\xi + F_v(x,u,v)\mu\,dx.
$$

The continuity and compactness of Ψ' can be proved like the continuity of Φ' and the compactness of J' respectively. $\square$

Now we are ready to prove our next main result which deals with the existence of three weak solutions for (1.1), by introducing some controls on the behaviour of antiderivatives of g_1 and g_2 at zero.

Theorem 3.8. *Let g_1, g_2 satisfy* (3.5) *and suppose*

$$
\max\Big\{\limsup_{\xi\to 0}\frac{\sup_{x\in\Omega}G_1(x,\xi)}{|\xi|^p}, \limsup_{\xi\to 0}\frac{\sup_{x\in\Omega}G_2(x,\xi)}{|\xi|^q}\Big\} \leqslant 0, \tag{3.15}
$$

where
$$G_1(x,\xi)=\int_0^\xi g_1(x,s)\,ds,\quad G_2(x,\xi)=\int_0^\xi g_2(x,s)\,ds$$
for any $(x,\xi)\in\Omega\times\mathbb{R}$. *Also, suppose the function* $F:\overline{\Omega}\times\mathbb{R}^2\to\mathbb{R}$ *satisfies all hypotheses of Lemma 3.7 and in addition*
$$\sup\big\{J(u,v):(u,v)\in X\big\}>0.$$
Then, if we set
$$\gamma=\inf\big\{\frac{\Phi(u,v)}{J(u,v)}:(u,v)\in X,\ J(u,v)>0,\ \Phi(u,v)>0\big\}$$
for each compact interval $[a,b]\subset]\gamma,\infty[$ *there exists* $r>0$ *such that for every* $\lambda\in[a,b]$, *there exists* $\delta>0$ *such that for every* $\mu\in[0,\delta]$, *the problem* (1.1) *has at least three weak solutions whose norms in* X *are less than* r.

Proof. First note that if $p\leqslant q$ then for every bounded $E\subset X$ there exists some constant $C>0$ such that
$$\Phi(u,v)-\Phi(0,0)\geqslant C_2\big(\|u\|_p^p+\|v\|_q^q\big)\geqslant C\big(\|u\|_p+\|v\|_q\big)^p=C\|(u,v)\|^p$$
for every $(u,v)\in E$, and if $p>q$ then
$$\Phi(u,v)-\Phi(0,0)\geqslant C\|(u,v)\|^q.$$
Furthermore every weak solution of (1.1) is a solution of $\Phi'(x)=\lambda J'(x)+\mu\Psi'(x)$. Since $1<\tau<p,\ 1<\kappa<q$
$$\limsup_{\|(u,v)\|\to\infty}\frac{J(u,v)}{\Phi(u,v)}\leqslant\limsup_{\|(u,v)\|\to\infty}\frac{C(\|u\|_p^\tau+\|v\|_q^\kappa)}{\|u\|_p^p+\|v\|_q^q}=0$$
and (3.5) in conjunction with (3.15) implies, there exist ρ_1,ρ_2 so that $0<\rho_1<\rho_2$ and
$$G_1(x,\xi)+G_2(x,\eta)\leqslant\epsilon(|\xi|^p+|\eta|^q)$$
for every $x\in\Omega$, every ξ,η in $\mathbb{R}-([-\rho_2,-\rho_1]\cup[\rho_1,\rho_2])$. Since $G_1(x,\xi)$, $G_2(x,\eta)$ are bounded on $\Omega\times([-\rho_2,-\rho_1]\cup[\rho_1,\rho_2])$, we can choose $C'>0$ and $p<m<\frac{pn}{n-p}$ and $q<\ell<\frac{qn}{n-q}$ such that
$$G_1(x,\xi)+G_2(x,\eta)\leqslant\epsilon(|\xi|^p+|\eta|^q)+C'(|\xi|^m+|\eta|^\ell)$$
for all $(x,\xi)\in\Omega\times\mathbb{R}$. Now the continuity of the Sobolev embedding implies for some constant C, independent of ϵ
$$J(u,v)\leqslant C\big(\epsilon(\|u\|_p^p+\|v\|_q^q)+C'(\|u\|_p^m+\|v\|_q^\ell)\big)$$
for every $(u,v)\in X$. On the other hand, (2.3) implies $\Phi(u,v)\geqslant C_2(\|u\|_p^p+\|v\|_q^q)$ and since $p<m,\ q<\ell$
$$\limsup_{(u,v)\to(0,0)}\frac{J(u,v)}{\Phi(u,v)}\leqslant\frac{C\epsilon}{C_2}.\tag{3.16}$$
Since $\epsilon>0$ is arbitrary
$$\limsup_{(u,v)\to(0,0)}\frac{J(u,v)}{\Phi(u,v)}=0.$$
Hence, by (3.16) we have $\alpha=0$ in Theorem 3.3 and since all other hypotheses of Theorem 3.3 for the functionals Φ and J and the point $x_0=(0,0)\in X$ are established in Lemmas 3.4, 3.5 and 3.6 and the functional Ψ has needed properties by Lemma 3.7, therefore the result is proved. □

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