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LOCAL WELL-POSEDNESS FOR DENSITY-DEPENDENT INCOMPRESSIBLE EULER EQUATIONS

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Abstract. In this article, we establish the local well-posedness for density-dependent incompressible Euler equations in critical Besov spaces.

1. Introduction

We consider the following density-dependent incompressible Euler equations in $\mathbb{R}^N$, $N \geq 2$,

$$\begin{gathered}
\partial_t \varrho + (v \cdot \nabla)\varrho = 0, \\
\partial_t(\varrho v) + (v \cdot \nabla)(\varrho v) + \nabla P = \varrho f, \\
\operatorname{div} v = 0, \\
(\varrho, v)|_{t=0} = (\varrho_0, v_0),
\end{gathered}$$

where $0 < m < \varrho_0(x) < M < \infty$ and $\lim_{x \to \pm\infty} \varrho_0(x) = \bar{\varrho}$, without loss of generality, assume that $\bar{\varrho} = 1$. Suppose that $f = 0$, just for simplicity, then one can rewrite the above equations as

$$\begin{gathered}
\partial_t \rho + (v \cdot \nabla)\rho = 0, \\
\partial_t v + (v \cdot \nabla)v + (1+\rho)\nabla P = 0, \\
\operatorname{div} v = 0, \\
\rho = \frac{1}{\varrho} - 1, \quad (\rho, v)|_{t=0} = (\rho_0, v_0).
\end{gathered} \tag{1.1}$$

If $\varrho_0(x) = \bar{\varrho} \equiv 1$, then (1.1) is the standard incompressible Euler equations. For this Euler model, we mention the following local well-posedness results. Given $v_0 \in H^m(\mathbb{R}^N)$, $m > N/2 + 1$, Kato [8] proved local existence and uniqueness for a solution belonging to $C([0,T]; H^m(\mathbb{R}^N))$ with $T = T(\|v_0\|_{H^m})$. Later on, many various function spaces (see [3, 9, 12, 13]) are used to establish the local existence and uniqueness for the incompressible Euler equations. For example, $W^{s,p}(\mathbb{R}^N)$ with $s > N/p + 1$, $1 < p < \infty$ is used in [9] and $F^s_{p,q}$ for $s > N/p + 1$, $1 < p < \infty$, $1 < q < \infty$ is used in [3].

For the density-dependent Euler equations, it is worth noting the following results. Beirão da Veiga and Valli [1] discussed the local existence and uniqueness for

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(1.1) in a bounded domain with various condition. In unbounded domain, Itoh [7] proved the local existence and uniqueness for (1.1) with initial data in $H^3(\mathbb{R}^3)$.

In this work we establish the local existence and uniqueness for the system (1.1) with initial data in the critical (with respect to the scaling invariance) Besov spaces.

Theorem 1.1. *Let $p \in (1,\infty)$. There exists a constant c depending only on N, such that for any given $\rho_0 \in B^{N/p+1}_{p,1}(\mathbb{R}^N)$ and $v_0 \in B^{N/p+1}_{p,1}(\mathbb{R}^N)$, $\operatorname{div} v_0 = 0$ with*

$$\|\rho_0\|_{B^{N/p+1}_{p,1}} \le c, \tag{1.2}$$

there exists a $T = T\big(p, \|\rho_0\|_{B^{N/p+1}_{p,1}}, \|v_0\|_{B^{N/p+1}_{p,1}}\big)$, the system (1.1) has a unique solution $(\rho, v, \nabla P)$ with $\rho \in C([0,T]; B^{N/p+1}_{p,1})$, $v \in C([0,T]; B^{N/p+1}_{p,1})$ and $\nabla P \in L^1(0,T; B^{N/p+1}_{p,1})$.

We remark that Theorem 1.1 gives a local existence and uniqueness theorem for (1.1) under a small perturbation of an initial constant density state. We wish to discuss the well-posedness for problem (1.1) without the restriction (1.2), in other words, a perturbation of any initial density in the future. It is worth to point out that local well-posedness is established in [14] for the periodic case without (1.2). For the case of supercritical Besov spaces, we refer to [15].

For the standard 2-D incompressible Euler equations in the critical (borderline) Besov spaces $B^{2/p+1}_{p,1}(\mathbb{R}^2)$, Vishik [12] showed the (global) well-posedness recently. Just as he said in his paper, it is of great interest to establish local well-posedness for high dimensional Euler equations. Obviously, Theorem 1.1 is true for the standard incompressible Euler equations ($\rho_0(x) \equiv 0$, (1.2) automatically holds). In other words, we recover the following local well-posedness theorem for incompressible Euler equations in the critical Besov space $B^{N/p+1}_{p,1}(\mathbb{R}^N)$ [13].

Corollary 1.2 ([13])**.** *Given any $v_0 \in B^{N/p+1}_{p,1}(\mathbb{R}^N)$, $1 < p < \infty$, there exists a $T = T(\|v_0\|_{B^{N/p+1}_{p,1}})$ and a unique solution $(v, \nabla P)$ to the system*

$$\begin{gathered}
\frac{\partial v}{\partial t} + (v\cdot\nabla)v + \nabla P = 0,\\
\operatorname{div} v = 0,\\
v(x,t=0) = v_0(x),
\end{gathered}$$

such that

$$v(x,t) \in C([0,T]; B^{N/p+1}_{p,1}) \quad \text{and} \quad \nabla P \in L^1(0,T; B^{N/p+1}_{p,1}).$$

In Theorem 1.1, if $p = 2$, then the smallness assumption on ρ_0 can be removed. More precisely, we have

Theorem 1.3. *Assume $\rho_0 \in B^{N/2+1}_{2,1}$ and $v_0 \in B^{N/2+1}_{2,1}$, $\operatorname{div} v_0 = 0$. Then there exists $T = T(\|\rho_0\|_{B^{N/2+1}_{2,1}}, \|v_0\|_{B^{N/2+1}_{2,1}})$ such that the system (1.1) has a uniqueness solution $(\rho, v, \nabla P)$ with $\rho \in C([0,T]; B^{N/2+1}_{2,1})$, $v \in C([0,T]; B^{N/2+1}_{2,1})$ and $\nabla P \in L^1(0,T; B^{N/2+1}_{2,1})$.*

Remark 1.4. After completing this article, the author was informed that Theorem 1.3 already was proved in [4]. However, the proof of Theorem 1.3 in Section 4 is different from proof in [4]. Another purpose is to investigate the difference of space

$B^{N/p+1}_{p,1}$ between general $p \neq 2$ and $p = 2$. We hope we can get ride of smallness restriction (1.2) in a future work.

2. Littlewood-Paley decomposition and Besov spaces

We start by recalling the Littlewood-Paley decomposition of temperate distributions. Let $\mathcal{S}$ be the class of Schwartz class of rapidly decreasing functions. Given $f \in \mathcal{S}$, the Fourier transform is defined as

$$\mathcal{F}(f) = \hat{f} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^N} e^{-ix\cdot\xi} f(x) dx.$$

One can extend $\mathcal{F}$ and $\mathcal{F}^{-1}$ to $\mathcal{S}'$ in the usual way, where $\mathcal{S}'$ denotes the set of all tempered distributions. Let $\phi \in \mathcal{S}$ satisfying

$$\operatorname{supp} \hat{\phi} \subset \big\{\xi : \frac{5}{6} \le |\xi| \le \frac{12}{5}\big\} \quad \text{and} \quad \sum_{j\in\mathbb{Z}} \hat{\phi}(2^{-j}\xi) = 1,$$

for $\xi \neq 0$. Setting $\hat{\phi}_j = \hat{\phi}(2^{-j}\xi)$, in other words, $\phi_j(x) = 2^{jN}\phi(2^j x)$, for any $f \in \mathcal{S}'$, we define

$$\Delta_j f = \phi_j * f \quad \text{and} \quad S_j f = \sum_{k\le j-1} \phi_k * f. \tag{2.1}$$

The homogeneous Besov semi-norm $\|f\|_{\dot{B}^s_{p,q}}$ and Triebel-Lizorkin semi-norm $\|f\|_{\dot{F}^s_{p,q}}$ are defined next.

Definition 2.1 ([10, 11]). For $-\infty < s < \infty$, $0 < p \le \infty$, $0 < q \le \infty$, set

$$\|f\|_{\dot{B}^s_{p,q}} = \begin{cases} \big(\sum_j 2^{jqs}\|\Delta_j f\|^q_{L^p}\big)^{1/q}, & \text{if } q \in (0,\infty), \\ \sup_j 2^{js}\|\Delta_j f\|_{L^p}, & \text{if } q = \infty. \end{cases}$$

$$\|f\|_{\dot{F}^s_{p,q}} = \begin{cases} \|\big(\sum_j 2^{jqs}|\Delta_j f|^q\big)^{1/q}\|_{L^p}, & \text{if } q \in (0,\infty), \\ \|\sup_j (2^{js}|\Delta_j f|)\|_{L^p}, & \text{if } q = \infty. \end{cases}$$

The spaces $\dot{B}^s_{p,q}$ and $\dot{F}^s_{p,q}$ are quasi-normed spaces with the above quasi-norm given by Definition 2.1. For $s > 0$, $(p,q) \in (1,\infty)\times[1,\infty]$, we define the inhomogeneous Besov space norm $\|f\|_{B^s_{p,q}}$ and inhomogeneous Triebel-Lizorkin space norm $\|f\|_{F^s_{p,q}}$ of $f \in \mathcal{S}'$ as

$$\|f\|_{B^s_{p,q}} = \|f\|_{L^p} + \|f\|_{\dot{B}^s_{p,q}}, \quad \|f\|_{F^s_{p,q}} = \|f\|_{L^p} + \|f\|_{\dot{F}^s_{p,q}}. \tag{2.2}$$

The inhomogeneous Besov and Triebel-Lizorkin spaces are Banach spaces equipped with the norm $\|f\|_{B^s_{p,q}}$ and $\|f\|_{F^s_{p,q}}$ respectively.

Let us now state some classical results.

Lemma 2.2 (Bernstein's Lemma [10, 11]). *Assume that $k \in \mathbb{Z}^+$, $f \in L^p$, $1 \le p \le \infty$, and $\operatorname{supp} \hat{f} \subset \{2^{j-2} \le |\xi| < 2^j\}$, then there exists a constant $C(k)$ such that the following inequality holds.*

$$C(k)^{-1} 2^{jk}\|f\|_{L^p} \le \|D^k f\|_{L^p} \le C(k) 2^{jk}\|f\|_{L^p}.$$

For any $k \in \mathbb{Z}^+$, there exists a constant $C(k)$ such that the following inequalities are true:

$$C(k)^{-1}\|D^k f\|_{\dot{B}^s_{p,q}} \le \|f\|_{\dot{B}^{s+k}_{p,q}} \le C(k)\|f\|_{\dot{B}^s_{p,q}}, \tag{2.3}$$

$$C(k)^{-1}\|D^k f\|_{\dot{F}^s_{p,q}} \le \|f\|_{\dot{F}^{s+k}_{p,q}} \le C(k)\|f\|_{\dot{F}^s_{p,q}}. \tag{2.4}$$

Lemma 2.3 (Embeddings [10, 11]). *(I) Let $s \in \mathbb{R}$, $p \in (1,\infty)$, $\epsilon > 0$ and $q_1, q_2 \in [1,\infty]$, $q_1 < q_2$, then*

$$\dot{B}^s_{p,1} \hookrightarrow \dot{F}^s_{p,2} \hookrightarrow \dot{B}^s_{p,\infty}, \quad \dot{B}^{s+\epsilon}_{p,q_1} \hookrightarrow \dot{B}^s_{p,q_2}.$$

(II) Let $p \in (1,\infty)$, then

$$\dot{B}^{N/p}_{p,1} \hookrightarrow L^\infty, \quad B^{N/p}_{p,1} \hookrightarrow L^\infty.$$

Proposition 2.4 (Product). *If $s \ge N/p$, and suppose $f, g \in \dot{B}^{N/p}_{p,1} \cap \dot{B}^s_{p,1}$, then $fg \in \dot{B}^s_{p,1}$ and*

$$\|fg\|_{\dot{B}^s_{p,1}} \le C\Big(\|f\|_{\dot{B}^s_{p,1}}\|g\|_{\dot{B}^{N/p}_{p,1}} + \|g\|_{\dot{B}^s_{p,1}}\|f\|_{\dot{B}^{N/p}_{p,1}}\Big). \tag{2.5}$$

We will prove this proposition in the appendix.

Lemma 2.5 (Commutator [5]). *Suppose that $s \in (-N/p-1, N/p]$. Then for $f \in \dot{B}^{N/p+1}_{p,1}$ and $g \in \dot{B}^s_{p,1}$, we have*

$$\|[f,\Delta_j]g\|_{L^p} \le C_j 2^{-j(s+1)}\|f\|_{\dot{B}^{N/p+1}_{p,1}}\|g\|_{\dot{B}^s_{p,1}},$$

with $\sum_j C_j \le 1$.

Lemma 2.6 (Interpolation [10, 11]). *Let $1 \le p_1, q_1, p_2 \le \infty$ and $1 \le q_2 < \infty$. Then*

$$\|f\|_{\dot{B}^s_{p,q}} \le \|f\|^\theta_{\dot{B}^{s_1}_{p_1,q_1}}\|f\|^{1-\theta}_{\dot{B}^{s_2}_{p_2,q_2}},$$

holds for all $f \in \dot{B}^{s_1}_{p_1,q_1} \cap \dot{B}^{s_2}_{p_2,q_2}$ provided that

$$s = \theta s_1 + (1-\theta)s_2, \quad \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}.$$

3. Proof of Theorem 1.1

In this section, we establish the existence and uniqueness of the solutions to (1.1) (Theorem 1.1). In the sequel, C denotes a absolute constant, which maybe different from line to line.

Consider the linear system

$$\begin{gathered} \partial_t v + (w\cdot\nabla)v + \nabla P = f, \\ \operatorname{div} v = 0, \\ v(x,t=0) = v_0(x). \end{gathered} \tag{3.1}$$

We have easily the existence of a local solution for (3.1).

Proposition 3.1. *Assume that* $\operatorname{div} w = 0$, *$w \in L^\infty(0,T; B^{N/p+1}_{p,1})$, $f \in L^1(0,T; B^{N/p+1}_{p,1})$, for some $T > 0$. Then for any $v_0 \in B^{N/p+1}_{p,1}$,* $\operatorname{div} v_0 = 0$, *there exists a unique solution $v \in C(0,T; B^{N/p+1}_{p,1})$ to (3.1), and then ∇P can be determined uniquely.*

The above proposition will be showed in the appendix. To prove the existence, we consider the following approximate linear iteration system for (1.1),

$$\begin{gathered}
\partial_t \rho^{n+1} + v^n \cdot \nabla \rho^{n+1} = 0, \\
\partial_t v^{n+1} + v^n \cdot \nabla v^{n+1} + \nabla P^{n+1} = -\rho^n \nabla P^n, \\
\operatorname{div} v^{n+1} = \operatorname{div} v^n = 0, \\
(\rho^{n+1}, v^{n+1})|_{t=0} = (\rho^{n+1}(0), v^{n+1}(0)) = (S_{n+1}\rho_0, S_{n+1}v_0),
\end{gathered} \tag{3.2}$$

where $(\rho^0, v^0, P^0) = (0,0,0)$. If we have the uniform estimate for the sequence $(\rho^n, v^n, \nabla P^n)$ by induction, which satisfies the conditions in Proposition 3.1, then the second equation of (3.2) can be solved with v^{n+1} and ∇P^{n+1}. While ρ^{n+1} can be obtained easily by solving the linear transport equation. So we establish uniform estimates first.

Uniform estimates. For the first equation of (3.2), thanks to the divergence free of v^n, it follows that for any $1 < p \leq \infty$,

$$\|\rho^{n+1}(.,t)\|_{L^p} \leq \|\rho^{n+1}(0)\|_{L^p}, \quad \text{for } t \geq 0. \tag{3.3}$$

Applying the operator Δ_j on the both sides of the linear transport equation, we obtain

$$\partial_t \Delta_j \rho^{n+1} + (v^n \cdot \nabla)\Delta_j \rho^{n+1} = [v^n, \Delta_j]\nabla \rho^{n+1}. \tag{3.4}$$

Multiply (3.4) by $|\Delta_j \rho^{n+1}|^{p-2}\Delta_j \rho^{n+1}$, and integrate over $\mathbb{R}^N$, due to the divergence free of v^n, then one has

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\|\Delta_j \rho^{n+1}\|_{L^p}^p &\leq \|[v^n, \Delta_j]\nabla \rho^{n+1}\|_{L^p}\|\Delta_j \rho^{n+1}\|_{L^p}^{p-1} \\
&\leq CC_j 2^{-js}\|v^n\|_{\dot{B}^{N/p+1}_{p,1}}\|\rho^{n+1}\|_{\dot{B}^s_{p,1}}\|\Delta_j \rho^{n+1}\|_{L^p}^{p-1},
\end{aligned}$$

where we used the commutator estimate for $s \leq \frac{N}{p} + 1$ and Hölder's inequality. Thus

$$\frac{d}{dt}\|\Delta_j \rho^{n+1}\|_{L^p} \leq CC_j 2^{-js}\|v^n\|_{\dot{B}^{N/p+1}_{p,1}}\|\rho^{n+1}\|_{\dot{B}^s_{p,1}}. \tag{3.5}$$

Multiplying (3.5) by 2^{js} and taking summation over j, we have

$$\frac{d}{dt}\|\rho^{n+1}\|_{\dot{B}^s_{p,1}} \leq C\|v^n\|_{\dot{B}^{N/p+1}_{p,1}}\|\rho^{n+1}\|_{\dot{B}^s_{p,1}}. \tag{3.6}$$

By Gronwall inequality and (3.3), we have the estimate for ρ^{n+1},

$$\sup_{0\leq t\leq T}\|\rho^{n+1}(.,t)\|_{B^s_{p,1}} \leq \|\rho^{n+1}(0)\|_{B^s_{p,1}} \exp\Big(\int_0^T C\|v^n(.,t)\|_{\dot{B}^{N/p+1}_{p,1}}dt\Big). \tag{3.7}$$

Multiplying each coordinate in second equation of (3.2) by $|v_l^{n+1}|^{p-2}v_l^{n+1}$, where v_l^{n+1} is the l-th coordinate of the vector field v^{n+1}, thanks to Hölder's inequality, we have

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\|v_l^{n+1}\|_{L^p}^p &\leq \|\rho^n \nabla P^n\|_{L^p}\|v_l^{n+1}\|_{L^p}^{p-1} + \|\nabla P^{n+1}\|_{L^p}\|v_l^{n+1}\|_{L^p}^{p-1} \\
&\leq C\|\rho^n\|_{\dot{B}^{N/p}_{p,1}}\|\nabla P^n\|_{L^p}\|v_l^{n+1}\|_{L^p}^{p-1} + \|\nabla P^{n+1}\|_{L^p}\|v_l^{n+1}\|_{L^p}^{p-1}.
\end{aligned}$$

So

$$\begin{aligned}\sup_{0\le t\le T}\|v^{n+1}\|_{L^p} &\le \|v^{n+1}(0)\|_{L^p} + C\int_0^T \|\rho^n\|_{\dot B^{N/p}_{p,1}}\|\nabla P^n(.,t)\|_{L^p}dt \\ &\quad + \int_0^T \|\nabla P^{n+1}(.,t)\|_{L^p}dt.\end{aligned} \tag{3.8}$$

Now taking Δ_j on the second equation of (3.2), we obtain

$$\partial_t\Delta_j v^{n+1} + v^n\cdot\nabla\Delta_j v^{n+1} + \nabla\Delta_j P^{n+1} = [v^n,\Delta_j]\nabla v^{n+1} - \Delta_j(\rho^n\nabla P^n). \tag{3.9}$$

Multiplying (3.9) coordinate by coordinate with $|\Delta_j v_l^{n+1}|^{p-2}\Delta_j v_l^{n+1}$, and integrating over $\mathbb{R}^N$, we have

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\|\Delta_j v_l^{n+1}\|_{L^p}^p \\ &\le C\left\|[v^n,\Delta_j]\nabla v_l^{n+1}\right\|_{L^p}\|\Delta_j v_l^{n+1}\|_{L^p}^{p-1} \\ &\quad + \|\Delta_j(\rho^n\nabla P^n)\|_{L^p}\|\Delta_j v_l^{n+1}\|_{L^p}^{p-1} + \|\Delta_j\nabla P^{n+1}\|_{L^p}\|\Delta_j v_l^{n+1}\|_{L^p}^{p-1} \\ &\le CC_j 2^{-j(N/p+1)}\|v^n\|_{\dot B^{N/p+1}_{p,1}}\|v^{n+1}\|_{\dot B^{N/p+1}_{p,1}}\|\Delta_j v_l^{n+1}\|_{L^p}^{p-1} \\ &\quad + \|\Delta_j(\rho^n\nabla P^n)\|_{L^p}\|\Delta_j v_l^{n+1}\|_{L^p}^{p-1} + \|\Delta_j\nabla P^{n+1}\|_{L^p}\|\Delta_j v_l^{n+1}\|_{L^p}^{p-1}.\end{aligned} \tag{3.10}$$

Then applying $2^{j(N/p+1)}$ on (3.10) and taking summation yields

$$\begin{aligned}\frac{d}{dt}\|v^{n+1}\|_{\dot B^{\frac{N}{p}+1}_{p,1}} &\le C\|v^n\|_{\dot B^{\frac{N}{p}+1}_{p,1}}\|v^{n+1}\|_{\dot B^{\frac{N}{p}+1}_{p,1}} + C\|\rho^n\nabla P^n\|_{\dot B^{\frac{N}{p}+1}_{p,1}} + \|\nabla P^{n+1}\|_{\dot B^{\frac{N}{p}+1}_{p,1}} \\ &\le C\|v^n\|_{\dot B^{N/p+1}_{p,1}}\|v^{n+1}\|_{\dot B^{N/p+1}_{p,1}} + C\|\rho^n\|_{\dot B^{N/p}_{p,1}}\|\nabla P^n\|_{\dot B^{N/p+1}_{p,1}} \\ &\quad + C\|\rho^n\|_{\dot B^{N/p+1}_{p,1}}\|\nabla P^n\|_{\dot B^{N/p}_{p,1}} + \|\nabla P^{n+1}\|_{\dot B^{\frac{N}{p}+1}_{p,1}}.\end{aligned}$$

By Gronwall's inequality and (3.8), for all $0\le t\le T$, we have

$$\begin{aligned}\|v^{n+1}\|_{B^{N/p+1}_{p,1}} &\le \|v^{n+1}(0)\|_{B^{N/p+1}_{p,1}}\exp\Big(C\int_0^T\|v^n(.,t)\|_{\dot B^{N/p+1}_{p,1}}dt\Big) \\ &\quad + \int_0^T A_n(t)\exp\Big(C\int_t^T\|v^n(.,\tau)\|_{\dot B^{N/p+1}_{p,1}}d\tau\Big)dt \\ &\quad + C\int_0^T\|\rho^n\|_{\dot B^{N/p}_{p,1}}\|\nabla P^n(.,t)\|_{L^p}dt,\end{aligned} \tag{3.11}$$

where

$$A_n(t) = C\|\rho^n\|_{\dot B^{\frac{N}{p}}_{p,1}}\|\nabla P^n\|_{\dot B^{\frac{N}{p}+1}_{p,1}} + C\|\rho^n\|_{\dot B^{\frac{N}{p}+1}_{p,1}}\|\nabla P^n\|_{\dot B^{\frac{N}{p}}_{p,1}} + \|\nabla P^{n+1}\|_{B^{\frac{N}{p}+1}_{p,1}}. \tag{3.12}$$

Now we give estimates for the pressure. Taking divergence on both sides of the second equation of (3.2), we have

$$-\Delta P^{n+1} = \operatorname{div}(v^n\cdot\nabla v^{n+1}) + \operatorname{div}(\rho^n\nabla P^n);$$

thus

$$\partial_i\partial_j P^{n+1} = R_iR_j\operatorname{div}(v^n\cdot\nabla v^n) + R_iR_j\operatorname{div}(\rho^n\nabla P^n). \tag{3.13}$$

For $1 < p < \infty$, in [10, 11], it was proved that $\dot F^0_{p,2} = L^p$ and R_i is bounded from $\dot F^s_{p,q}$ into itself [6]. Due to Bernstein's lemma, we have

$$\begin{aligned}
\|\nabla P^{n+1}\|_{L^p} &= \|\nabla P^{n+1}\|_{\dot F^0_{p,2}} \le C\sum_{i,j=1}^N \|\partial_i\partial_j P^{n+1}\|_{\dot F^{-1}_{p,2}}\\
&\le C\|\operatorname{div}(v^n\cdot\nabla v^{n+1})\|_{\dot F^{-1}_{p,2}} + C\|\operatorname{div}(\rho^n\nabla P^n)\|_{\dot F^{-1}_{p,2}}\\
&\le C\|v^n\cdot\nabla v^{n+1}\|_{L^p} + C\|\rho^n\nabla P^n\|_{L^p}\\
&\le C\|v^n\|_{\dot B^{N/p}_{p,1}}\|v^{n+1}\|_{\dot B^1_{p,1}} + C\|\rho^n\|_{\dot B^{N/p}_{p,1}}\|\nabla P^n\|_{L^p},
\end{aligned} \tag{3.14}$$

where we used the embedding Lemma 2.3. From (3.13) it follows that

$$\begin{aligned}
\|\nabla P^{n+1}\|_{\dot B^{N/p+1}_{p,1}} &\le C\sum_{i,j=1}^N \|\partial_i\partial_j P^{n+1}\|_{\dot B^{N/p}_{p,1}}\\
&\le C\sum_{i,j,k,l=1}^N \|R_iR_j\partial_k v^n_l\partial_l v^{n+1}_k\|_{\dot B^{N/p}_{p,1}} + C\|\rho^n\nabla P^n\|_{\dot B^{N/p+1}_{p,1}}\\
&\le C\|v^n\|_{\dot B^{N/p+1}_{p,1}}\|v^{n+1}\|_{\dot B^{N/p+1}_{p,1}} + C\|\rho^n\|_{\dot B^{N/p}_{p,1}}\|\nabla P^n\|_{\dot B^{N/p+1}_{p,1}}\\
&\quad + C\|\rho^n\|_{\dot B^{N/p+1}_{p,1}}\|\nabla P^n\|_{\dot B^{N/p}_{p,1}}.
\end{aligned} \tag{3.15}$$

Combining (3.14) and (3.15), one has

$$\begin{aligned}
&\int_0^T \|\nabla P^{n+1}(.,t)\|_{B^{N/p+1}_{P,1}}\\
&\le C\|\rho^n\|_{L^\infty(0,T;B^{N/p+1}_{p,1})}\int_0^T \|\nabla P^n(.,t)\|_{B^{N/p+1}_{p,1}}dt\\
&\quad + CT\|v^n\|_{L^\infty(0,T;B^{\frac{N}{p}+1}_{p,1})}\|v^{n+1}\|_{L^\infty(0,T;B^{\frac{N}{p}+1}_{p,1})}.
\end{aligned} \tag{3.16}$$

Note that although the above constants C maybe depend on N, m, M and p, it is nothing to do with n, therefore we can obtain uniform estimates by induction.

In fact, suppose that initial data ρ_0 and v_0 satisfies

$$\|\rho_0\|_{B^{N/p+1}_{p,1}} \le \frac{C_1}{2}, \quad \|v_0\|_{B^{N/p+1}_{p,1}} \le \frac{C_2}{2},$$

for some $C_1, C_2 > 0$ and C_1 is sufficiently small. Then the following inequalities hold

$$\begin{gathered}
\|\rho^{n+1}\|_{L^\infty(0,T_*;B^{N/p+1}_{p,1})} \le C_1,\\
\|v^{n+1}\|_{L^\infty(0,T_*;B^{N/p+1}_{p,1})} \le C_2,\\
\|\nabla P^{n+1}\|_{L^1(0,T_*;B^{N/p+1}_{p,1})} \le C_3,
\end{gathered} \tag{3.17}$$

for all $n \ge 0$ and some $C_3 < C_2/(8CC_1)$, provided that T_* (independent on n) is sufficiently small.

We show (3.17) by mathematical induction. Note that (3.17) holds obviously for $n = 0$. Suppose (3.17) is true for n, we want to prove (3.17) holds for $n+1$. From (3.7), (3.11), (3.12) and (3.16), we have

$$\|\rho^{n+1}\|_{L^\infty(0,T;B^{N/p+1}_{p,1})} \le \frac{C_1}{2}\exp(TC_2),$$

$$
\begin{aligned}
&\|v^{n+1}\|_{L^\infty(0,T;B^{N/p+1}_{p,1})}\\
&\le \frac{C_2}{2}\exp(CTC_2)+CC_1C_3+C(C_1C_3+C_2\|v^{n+1}\|_{L^\infty(0,T;B^{N/p+1}_{p,1})}T)\exp(CTC_2),\\
&\qquad \|\nabla P^{n+1}\|_{L^1(0,T;B^{N/p+1}_{p,1})}\le CC_1C_3+CTC_2\|v^{n+1}\|_{L^\infty(0,T;B^{N/p+1}_{p,1})},
\end{aligned}
$$

So one can choose T_* sufficient small, such that

$$
CC_2T_*\exp(CT_*C_2)\le \frac{1}{4}.
$$

Moreover, T_* satisfies

$$
\begin{gathered}
\exp(T_*C_2)\le 2,\\
2C_2\exp(CT_*C_2)+4CC_1C_3\exp(CT_*C_2)\le 5C_2,\\
CT_*C_2^2\le \frac{C_3}{2},
\end{gathered}
$$

provided that $C_1<C/2$. Then by induction, (3.17) holds for $n+1$-th step. Hence we get the uniform estimate for each n.

Convergence.

To prove the convergence, it is sufficient to estimate the difference of the iteration. Take the difference between the equation (3.2) for the $(n+1)$-th step and the n-th step, and set

$$
w^{n+1}=\rho^{n+1}-\rho^n,\quad u^{n+1}=v^{n+1}-v^n,,\quad \Pi^{n+1}=P^{n+1}-P^n,
$$

then we obtain the equation

$$
\begin{gathered}
\partial_t w^{n+1}+v^n\cdot\nabla w^{n+1}+u^n\cdot\nabla\rho^n=0,\\
\partial_t u^{n+1}+v^n\cdot\nabla u^{n+1}+u^n\cdot\nabla v^n+\nabla\Pi^{n+1}=-w^n\nabla P^n-\rho^{n-1}\nabla\Pi^n,\\
\operatorname{div}u^{n+1}=\operatorname{div}v^n=0,\\
(w^{n+1},u^{n+1})|_{t=0}=(w^{n+1}(0),u^{n+1}(0))=(\Delta_n\rho_0,\Delta_n v_0),
\end{gathered}
\tag{3.18}
$$

First, we do the estimate for w^{n+1}. Multiplying $|w^{n+1}|^{p-2}w^{n+1}$ on both sides of the first equation of (3.18) and integrating over $\mathbb{R}^N$, we have

$$
\begin{aligned}
\frac{d}{dt}\|w^{n+1}\|_{L^p}&\le \|u^n\cdot\nabla\rho^n\|_{L^p}\le \|u^n\|_{L^\infty}\|\nabla\rho^n\|_{L^p}\\
&\le C\|u^n\|_{\dot B^{N/p}_{p,1}}\|\nabla\rho^n\|_{\dot F^0_{p,2}}\\
&\le \|u^n\|_{\dot B^{N/p}_{p,1}}\|\rho^n\|_{\dot B^1_{p,1}}.
\end{aligned}
\tag{3.19}
$$

Applying Δ_j on both sides of the first equation of (3.18), we have

$$
\partial_t\Delta_j w^{n+1}+v^n\cdot\Delta_j w^{n+1}+\Delta_j(u^n\cdot\nabla\rho^n)=[v^n,\Delta_j]\nabla w^{n+1}.
\tag{3.20}
$$

Multiplying (3.20) by $|\Delta_j w^{n+1}|^{p-2}\Delta_j w^{n+1}$ and integrating over $\mathbb{R}^N$, we have

$$
\begin{aligned}
\frac{1}{p}\frac{d}{dt}\|\Delta_j w^{n+1}\|^p_{L^p}&\le \|\Delta_j(u^n\cdot\nabla\rho^n)\|_{L^p}\|\Delta_j w^{n+1}\|^{p-1}_{L^p}\\
&\quad+C_j2^{-jN/p}\|v^n\|_{\dot B^{N/p+1}_{p,1}}\|\nabla w^{n+1}\|_{\dot B^{N/p-1}_{p,1}}\|\Delta_j w^{n+1}\|^{p-1}_{L^p}.
\end{aligned}
$$

Then applying $2^{jN/p}$ and taking summation, we obtain

$$
\frac{d}{dt}\|w^{n+1}\|_{\dot B^{N/p}_{p,1}}\le C\|u^n\|_{\dot B^{N/p}_{p,1}}\|\rho^n\|_{\dot B^{N/p+1}_{p,1}}+C\|v^n\|_{\dot B^{N/p+1}_{p,1}}\|w^{n+1}\|_{\dot B^{N/p}_{p,1}}.
\tag{3.21}
$$

Combining (3.19) and (3.21), it follows that

$$\frac{d}{dt}\|w^{n+1}\|_{B^{N/p}_{p,1}} \le C\|\rho^n\|_{B^{N/p+1}_{p,1}}\|u^n\|_{B^{N/p}_{p,1}} + C\|v^n\|_{B^{N/p+1}_{p,1}}\|w^{n+1}\|_{B^{N/p}_{p,1}}. \tag{3.22}$$

Just as what done for v^{n+1}, multiplying the second equation of (3.18) coordinate by coordinate with $|u_l^{n+1}|^{p-2}u_l^{n+1}$, where u_l^{n+1} is the l-th coordinate of the vector field u^{n+1}. Thanks to Hölder's inequality, we have

$$\begin{aligned}\frac{d}{dt}\|u^{n+1}\|_{L^p} &\le \|u^n\cdot\nabla v^n\|_{L^p} + \|w^n\nabla P^n\|_{L^p}\\ &\quad + \|\rho^{n-1}\nabla\Pi^n\|_{L^p} + \|\nabla\Pi^{n+1}\|_{L^p}\\ &\le C\|u^n\|_{\dot B^{N/p}_{p,1}}\|v^n\|_{\dot B^{1}_{p,1}} + C\|w^n\|_{\dot B^{N/p}_{p,1}}\|\nabla P^n\|_{L^p}\\ &\quad + C\|\rho^{n-1}\|_{\dot B^{N/p}_{p,1}}\|\nabla\Pi^n\|_{L^p} + \|\nabla\Pi^{n+1}\|_{L^p}.\end{aligned} \tag{3.23}$$

Applying Δ_j on the second equation of (3.18), we obtain

$$\begin{aligned}&\partial_t\Delta_j u^{n+1} + v^n\cdot\nabla\Delta_j u^{n+1} + \nabla\Delta_j\Pi^{n+1}\\ &= [v^n,\Delta_j]\nabla u^{n+1} - \Delta_j(u^n\cdot\nabla v^n) - \Delta_j(w^n\nabla P^n + \rho^{n-1}\nabla\Pi^n).\end{aligned} \tag{3.24}$$

Multiplying each coordinate with $|\Delta_j u_l^{n+1}|^{p-2}\Delta_j u_l^{n+1}$, and integrating over $\mathbb{R}^N$, we have

$$\begin{aligned}&\frac{d}{dt}\|\Delta_j u^{n+1}\|_{L^p}\\ &\le CC_j2^{-jN/p}\|v^n\|_{\dot B^{N/p+1}_{p,1}}\|u^{n+1}\|_{\dot B^{N/p}_{p,1}} + \|\Delta_j(u^n\cdot\nabla v^n)\|_{L^p}\\ &\quad + \|\Delta_j(w^n\nabla P^n)\|_{L^p} + \|\Delta_j(\rho^{n-1}\nabla\Pi^n)\|_{L^p} + \|\Delta_j\nabla\Pi^{n+1}\|_{L^p}.\end{aligned} \tag{3.25}$$

Then applying $2^{jN/p}$ on (3.25) and taking summation yields

$$\begin{aligned}&\frac{d}{dt}\|u^{n+1}\|_{\dot B^{N/p}_{p,1}}\\ &\le C\|v^n\|_{\dot B^{N/p+1}_{p,1}}\|u^{n+1}\|_{\dot B^{N/p}_{p,1}} + C\|u^n\cdot\nabla v^n\|_{\dot B^{N/p}_{p,1}}\\ &\quad + \|w^n\nabla P^n\|_{\dot B^{N/p}_{p,1}} + \|\rho^{n-1}\nabla\Pi^n\|_{\dot B^{N/p}_{p,1}} + \|\nabla\Pi^{n+1}\|_{\dot B^{N/p}_{p,1}}\\ &\le C\|v^n\|_{\dot B^{N/p+1}_{p,1}}\|u^{n+1}\|_{\dot B^{N/p}_{p,1}} + C\|u^n\|_{\dot B^{N/p}_{p,1}}\|v^n\|_{\dot B^{N/p+1}_{p,1}}\\ &\quad + C\|w^n\|_{\dot B^{N/p}_{p,1}}\|\nabla P^n\|_{\dot B^{N/p}_{p,1}} + C\|\rho^{n-1}\|_{\dot B^{N/p}_{p,1}}\|\nabla\Pi^n\|_{\dot B^{N/p}_{p,1}} + \|\nabla\Pi^{n+1}\|_{\dot B^{N/p}_{p,1}}.\end{aligned} \tag{3.26}$$

Combining (3.23) and (3.26), we have

$$\begin{aligned}&\frac{d}{dt}\|u^{n+1}\|_{B^{N/p}_{p,1}}\\ &\le C\|v^n\|_{B^{N/p+1}_{p,1}}\|u^{n+1}\|_{B^{N/p}_{p,1}} + C\|v^n\|_{B^{N/p+1}_{p,1}}\|u^n\|_{B^{N/p}_{p,1}}\\ &\quad + C\|\nabla P^n\|_{B^{N/p}_{p,1}}\|w^n\|_{B^{N/p}_{p,1}} + C\|\rho^{n-1}\|_{B^{N/p}_{p,1}}\|\nabla\Pi^n\|_{B^{N/p}_{p,1}} + \|\nabla\Pi^{n+1}\|_{B^{N/p}_{p,1}}.\end{aligned} \tag{3.27}$$

Now we give estimates for $\nabla\Pi^{n+1}$. Applying the operator div on both sides of the second equation of (3.18), we have

$$-\Delta\Pi^{n+1} = \operatorname{div}(v^n\cdot\nabla u^{n+1}) + \operatorname{div}(u^n\cdot\nabla v^n) + \operatorname{div}(w^n\nabla P^n + \rho^{n-1}\nabla\Pi^n);$$

thus

$$\begin{aligned}\partial_i\partial_j\Pi^{n+1} &= R_iR_j\operatorname{div}(v^n\cdot\nabla u^{n+1}) + R_iR_j\operatorname{div}(u^n\cdot\nabla v^n)\\ &\quad + R_iR_j\operatorname{div}(w^n\nabla P^n) + R_iR_j\operatorname{div}(\rho^{n-1}\nabla\Pi^n).\end{aligned}$$

Thanks to the divergence free of v^n, we have

$$\operatorname{div}(v^n\cdot\nabla u^{n+1}) = \sum_{k,l=1}^N \partial_k(v_l^n\partial_l u_k^{n+1}) = \sum_{k,l=1}^N \partial_k\partial_l(v_l^n u_k^{n+1}) = \sum_{k,l=1}^N \partial_l(\partial_k v_l^n u_k^{n+1}).$$

Due to Bernstein's lemma, we have

$$\begin{aligned}\|\nabla\Pi^{n+1}\|_{L^p} &= \|\nabla\Pi^{n+1}\|_{\dot F^0_{p,2}} \le C\sum_{i,j=1}^N \|\partial_i\partial_j\Pi^{n+1}\|_{\dot F^{-1}_{p,2}}\\ &\le C\|\operatorname{div}(v^n\cdot\nabla u^{n+1})\|_{\dot F^{-1}_{p,2}} + C\|\operatorname{div}(u^n\cdot\nabla v^n)\|_{\dot F^{-1}_{p,2}}\\ &\quad + C\|\operatorname{div}(w^n\nabla P^n)\|_{\dot F^{-1}_{p,2}} + C\|\operatorname{div}(\rho^{n-1}\nabla\Pi^n)\|_{\dot F^{-1}_{p,2}}\\ &\le C\sum_{k,l=1}^N \|\partial_k v_l^n u_k^{n+1}\|_{L^p} + C\|u^n\cdot\nabla v^n\|_{L^p}\\ &\quad + C\|w^n\nabla P^n\|_{L^p} + C\|\rho^{n-1}\nabla\Pi^n\|_{L^p}\\ &\le C\|v^n\|_{\dot B^{N/p+1}_{p,1}}\|u^{n+1}\|_{L^p} + C\|u^n\|_{L^p}\|v^n\|_{\dot B^{N/p+1}_{p,1}}\\ &\quad + C\|w^n\|_{\dot B^{N/p}_{p,1}}\|\nabla P^n\|_{L^p} + C\|\rho^{n-1}\|_{\dot B^{N/p}_{p,1}}\|\nabla\Pi^n\|_{L^p},\end{aligned}\tag{3.28}$$

where we used the embedding in Lemma 2.3 and the product estimate. Similarly,

$$\begin{aligned}\|\nabla\Pi^{n+1}\|_{\dot B^{N/p}_{p,1}} &\le C\sum_{i,j=1}^N \|\partial_i\partial_j\Pi^{n+1}\|_{\dot B^{N/p-1}_{p,1}}\\ &\le C\sum_{i,j,k,l=1}^N \|R_iR_j\partial_l(\partial_k v_l^n u_k^{n+1})\|_{\dot B^{N/p-1}_{p,1}} + C\|w^n\nabla P^n\|_{\dot B^{N/p}_{p,1}}\\ &\quad + C\|u^n\cdot\nabla v^n\|_{\dot B^{N/p}_{p,1}} + C\|\rho^{n-1}\nabla\Pi^n\|_{\dot B^{N/p}_{p,1}}\\ &\le C\sum_{k,l=1}^N \|\partial_k v_l^n u_k^{n+1}\|_{\dot B^{N/p}_{p,1}} + C\|w^n\nabla P^n\|_{\dot B^{N/p}_{p,1}}\\ &\quad + C\|u^n\cdot\nabla v^n\|_{\dot B^{N/p}_{p,1}} + C\|\rho^{n-1}\nabla\Pi^n\|_{\dot B^{N/p}_{p,1}}\\ &\le C\|v^n\|_{\dot B^{N/p+1}_{p,1}}\|u^{n+1}\|_{\dot B^{N/p}_{p,1}} + C\|w^n\|_{\dot B^{N/p}_{p,1}}\|\nabla P^n\|_{\dot B^{N/p}_{p,1}}\\ &\quad + C\|v^n\|_{\dot B^{N/p+1}_{p,1}}\|u^n\|_{\dot B^{N/p}_{p,1}} + C\|\rho^{n-1}\|_{\dot B^{N/p}_{p,1}}\|\nabla\Pi^n\|_{\dot B^{N/p}_{p,1}}.\end{aligned}\tag{3.29}$$

Combining (3.28) and (3.29), it follows that

$$\begin{aligned}\|\nabla\Pi^{n+1}\|_{B^{N/p}_{p,1}} &\le C\|v^n\|_{B^{N/p+1}_{p,1}}\|u^{n+1}\|_{B^{N/p}_{p,1}} + C\|\nabla P^n\|_{B^{N/p}_{p,1}}\|w^n\|_{B^{N/p}_{p,1}}\\ &\quad + C\|v^n\|_{B^{N/p+1}_{p,1}}\|u^n\|_{B^{N/p}_{p,1}} + C\|\rho^{n-1}\|_{B^{N/p}_{p,1}}\|\nabla\Pi^n\|_{B^{N/p}_{p,1}}.\end{aligned}\tag{3.30}$$

Therefore, if we add (3.22), (3.27) and (3.30) together, then we obtain,

$$
\begin{aligned}
&\frac{d}{dt}\|w^{n+1}\|_{B^{\frac{N}{p}}_{p,1}} + \frac{d}{dt}\|u^{n+1}\|_{B^{\frac{N}{p}}_{p,1}} + \|\nabla\Pi^{n+1}\|_{B^{\frac{N}{p}}_{p,1}} \\
&\leq C_4\Big(\|w^{n+1}\|_{B^{\frac{N}{p}}_{p,1}} + \|u^{n+1}\|_{B^{\frac{N}{p}}_{p,1}} + \|u^{n}\|_{B^{\frac{N}{p}}_{p,1}}\Big) \\
&\quad + C\|\nabla P^n\|_{B^{N/p}_{p,1}}\|w^n\|_{B^{N/p}_{p,1}} + C\|\rho^{n-1}\|_{B^{N/p}_{p,1}}\|\nabla\Pi^n\|_{B^{N/p}_{p,1}},
\end{aligned} \tag{3.31}
$$

where C_4 is a constant depending on the uniform bounds of $\|\rho^n\|_{L^\infty(0,T_*;B^{N/p+1}_{p,1})}$, $\|v^{n-1}\|_{L^\infty(0,T_*;B^{N/p+1}_{p,1})}$ and $\|v^n\|_{L^\infty(0,T_*;B^{N/p+1}_{p,1})}$. Then integrate (3.31) on the time interval $(0,T_1)\subset[0,T_*]$, T_1 sufficiently small, such that

$$
C_4T_1 \leq \frac{1}{4}, \quad C\|\nabla P^n\|_{L^1(0,T_1;B^{N/p}_{p,1})} \leq \frac{1}{4}.
$$

Then (3.31) yields

$$
\begin{aligned}
&\|w^{n+1}\|_{L^\infty(0,T_1;B^{N/p}_{p,1})} + \|u^{n+1}\|_{L^\infty(0,T_1;B^{N/p}_{p,1})} + \|\nabla\Pi^{n+1}\|_{L^1(0,T_1;B^{N/p}_{p,1})} \\
&\leq \frac{4}{3}\Big(\|w^{n+1}(0)\|_{B^{N/p}_{p,1}} + \|u^{n+1}(0)\|_{B^{N/p}_{p,1}}\Big) + \frac{1}{3}\|w^n\|_{L^\infty(0,T_1;B^{N/p}_{p,1})} \\
&\quad + \frac{1}{3}\|u^n\|_{L^\infty(0,T_1;B^{N/p}_{p,1})} + \frac{4}{3}C\|\rho^{n-1}\|_{L^\infty(0,T_1;B^{N/p}_{p,1})}\|\nabla\Pi^n\|_{L^1(0,T_1;B^{N/p}_{p,1})}.
\end{aligned} \tag{3.32}
$$

Due to the smallness of C_1, say $CC_1 \leq 1/4$, from (3.32) it follows that

$$
\|w^{n+1}\|_{L^\infty(0,T_1;B^{N/p}_{p,1})} + \|u^{n+1}\|_{L^\infty(0,T_*;B^{N/p}_{p,1})} + \|\nabla\Pi^{n+1}\|_{L^1(0,T_1;B^{N/p}_{p,1})} \to 0,
$$

as n tends to infinity.

Therefore, from the uniform estimates, we find that there exists a limit $(\rho,v,\nabla P)$ belonging to $C(0,T;B^{N/p+1}_{p,1})\times C(0,T;B^{N/p+1}_{p,1})\times L^1(0,T;B^{N/p}_{p,1})$, which is the solution to (1.1), for sufficient small T.

This complete the proof of local existence theorem. Next we turn our attention to the uniqueness of solutions.

Uniqueness. Suppose $(\rho_1,v_1,\nabla P_1)$ and $(\rho_2,v_2,\nabla P_2)$ are two solutions to (1.1) with the same initial data. If we set $\rho=\rho_1-\rho_2$, $v=v_1-v_2$ and $P=P_1-P_2$, then we get a similar system as (3.18) as

$$
\begin{gathered}
\partial_t\rho + v_1\cdot\nabla\rho + v\cdot\nabla\rho_2 = 0, \\
\partial_t v + v_1\cdot\nabla v + v\cdot\nabla v_2 + \nabla P = -\rho_1\nabla P - \rho\nabla P_2, \\
\operatorname{div} v_1 = \operatorname{div} v_2 = 0, \\
(\rho,v)|_{t=0} = (0,0).
\end{gathered} \tag{3.33}
$$

Just as in the convergence part for the sequences, we can treat (3.33) as (3.18), and obtain

$$
\begin{aligned}
&\|\rho\|_{L^\infty(0,T_1;B^{N/p}_{p,1})} + \|v\|_{L^\infty(0,T_1;B^{N/p}_{p,1})} + \|\nabla P\|_{L^1(0,T_1;B^{N/p}_{p,1})} \\
&\leq \frac{1}{4}\Big(\|\rho\|_{L^\infty(0,T_1;B^{N/p}_{p,1})} + \|v\|_{L^\infty(0,T_1;B^{N/p}_{p,1})} + \|\nabla P\|_{L^1(0,T_1;B^{N/p}_{p,1})}\Big),
\end{aligned}
$$

provided that T is sufficiently small and $\|\rho_0\|_{B^{N/p+1}_{p,1}}$ is sufficiently small. This implies the uniqueness.

4. Proof of Theorem 1.3

Now we use the following iteration system

$$
\begin{gathered}
\partial_t \rho^{n+1} + v^n \cdot \nabla \rho^{n+1} = 0, \\
\partial_t v^{n+1} + v^n \cdot \nabla v^{n+1} + (1+\rho^n)\nabla P^{n+1} = 0, \\
\operatorname{div} v^{n+1} = \operatorname{div} v^n = 0, \\
(\rho^{n+1}, v^{n+1})|_{t=0} = (\rho^{n+1}(0), v^{n+1}(0)) = (S_{n+1}\rho_0, S_{n+1}v_0),
\end{gathered} \tag{4.1}
$$

with the corresponding linear system

$$
\begin{gathered}
\partial_t v + w \cdot \nabla v + (1+\rho)\nabla P = 0, \\
\operatorname{div} v = 0, \\
v(x, t=0) = v_0(x).
\end{gathered} \tag{4.2}
$$

First, we have the following existence and uniqueness result, which will be proved in the appendix.

Proposition 4.1. *Assume that* $\operatorname{div} w = 0$, $w \in L^\infty(0,T; B^{N/2+1}_{2,1})$, $\rho \in L^\infty(0,T; B^{N/2+1}_{2,1})$, *for some* $T > 0$. *Then for any* $v_0 \in B^{N/2+1}_{2,1}$, $\operatorname{div} v_0 = 0$, *there exists a unique solution* $v \in C(0,T; B^{N/2+1}_{2,1})$ *to* (4.2). *Consequently,* ∇P *can be uniquely determined.*

Now, we go to the proof for Theorem 1.3.

Uniform estimates.

As in (3.7), for ρ^{n+1}, we have the estimate

$$
\sup_{0\le t\le T} \|\rho^{n+1}(.,t)\|_{B^s_{p,1}} \le \|\rho^{n+1}(0)\|_{B^s_{p,1}} \exp\Big(\int_0^T C\|v^n(.,t)\|_{\dot B^{N/p+1}_{p,1}} dt\Big). \tag{4.3}
$$

Multiplying the second equation of (4.1) by v^{n+1} and integrating over $\mathbb{R}^N$, we obtain

$$
\frac{d}{dt}\|v^{n+1}(.,t)\|_{L^2} \le \|1+\rho^n\|_{L^\infty}\|\nabla P^{n+1}\|_{L^2} \le C(1+\|\rho^n\|_{\dot B^{N/2}_{2,1}})\|\nabla P^{n+1}\|_{L^2}. \tag{4.4}
$$

Applying Δ_j on the second equation of (3.2), we obtain

$$
\partial_t \Delta_j v^{n+1} + v^n \cdot \nabla \Delta_j v^{n+1} = [v^n, \Delta_j]\nabla v^{n+1} - \Delta_j\big((1+\rho^n)\nabla P^n\big). \tag{4.5}
$$

Multiplying (4.5) by $\Delta_j v^{n+1}$ and taking the divergence free property of v^n into account, we have

$$
\begin{aligned}
\frac{d}{dt}\|\Delta_j v^{n+1}\|_{L^2} &\le CC_j 2^{-j(N/2+1)}\|v^n\|_{\dot B^{N/2+1}_{2,1}}\|v^{n+1}\|_{\dot B^{N/2+1}_{2,1}} \\
&\quad + \|\Delta_j\big((1+\rho^n)\nabla P^{n+1}\big)\|_{L^2}.
\end{aligned} \tag{4.6}
$$

Applying $2^{j(N/2+1)}$ on (4.6) and taking summation, and using the product estimate, we have

$$
\begin{aligned}
\frac{d}{dt}\|v^{n+1}\|_{\dot B^{N/2+1}_{2,1}} &\le C\|v^n\|_{\dot B^{N/2+1}_{2,1}}\|v^{n+1}\|_{\dot B^{N/2+1}_{2,1}} + C\|(1+\rho^n)\nabla P^{n+1}\|_{\dot B^{N/2+1}_{2,1}} \\
&\le C\|v^n\|_{\dot B^{N/2+1}_{2,1}}\|v^{n+1}\|_{\dot B^{N/2+1}_{2,1}} + C(1+\|\rho^n\|_{\dot B^{N/2}_{2,1}})\|\nabla P^{n+1}\|_{\dot B^{N/2+1}_{2,1}} \\
&\quad + C(1+\|\rho^n\|_{\dot B^{N/2+1}_{2,1}})\|\nabla P^{n+1}\|_{\dot B^{N/2}_{2,1}}.
\end{aligned} \tag{4.7}
$$

So the remaining thing is to give an estimate for the pressure. For this purpose, we apply the operator div on both sides of the second equation of the system (4.1), and get

$$\operatorname{div}\left((1+\rho^n)\nabla P^{n+1}\right) = -\operatorname{div}(v^n\cdot\nabla v^{n+1}). \tag{4.8}$$

Since $1+\rho=\frac{1}{\varrho}$ is bounded away from 0, we can assume that $1+\rho^n$ bounded away from 0, without loss of generality (otherwise, we take $\rho^n(0)=S_{n+m}\rho_0$, such that $1+\rho^n$ bounded away from 0 for sufficiently large integer m).

Multiplying (4.8) by P^{n+1} and integrating by parts, one has

$$\|\nabla P^{n+1}\|_{L^2}\le C\|v^n\cdot\nabla v^{n+1}\|_{L^2}\le C\|v^n\|_{L^2}\|v^{n+1}\|_{\dot B^{N/2+1}_{2,1}}. \tag{4.9}$$

Applying Δ_j on (4.8), we have

$$-\operatorname{div}\left((1+\rho^n)\nabla\Delta_j P^{n+1}\right) = \operatorname{div}\left([\Delta_j,(1+\rho^n)]\nabla P^{n+1}\right)+\Delta_j\operatorname{div}\left(v^n\cdot\nabla v^{n+1}\right). \tag{4.10}$$

Multiplying (4.10) by $\Delta_j P^{n+1}$ and integrating over $\mathbb{R}^N$, due to the bounds of $1+\rho^n$, we obtain that

$$\begin{aligned}\|\Delta_j\nabla P^{n+1}\|_{L^2}^2 &\le C\|[\Delta_j,(1+\rho^n)]\nabla P^{n+1}\|_{L^2}\|\Delta_j\nabla P^{n+1}\|_{L^2}\\ &\quad + C2^{-j}\|\Delta_j(v^n\cdot\nabla v^{n+1})\|_{L^2}\|\Delta_j\nabla P^{n+1}\|_{L^2},\end{aligned} \tag{4.11}$$

Multiplying by $2^{jN/2}$ on (4.11) and taking summation, we have

$$\begin{aligned}&\|\nabla P^{n+1}\|_{\dot B^{N/2}_{2,1}}\\ &\le C(1+\|\rho^n\|_{\dot B^{N/2+1}_{2,1}})\|\nabla P^{n+1}\|_{\dot B^{N/2-1}_{2,1}}+C\|v^n\|_{\dot B^{N/2}_{2,1}}\|v^{n+1}\|_{\dot B^{N/2+1}_{2,1}}\\ &\le C(1+\|\rho^n\|_{\dot B^{N/2+1}_{2,1}})\|\nabla P^{n+1}\|_{\dot B^{N/2}_{2,1}}^{(N-1)/(N+1)}\|\nabla P^{n+1}\|_{\dot B^{-1/2}_{2,1}}^{2/(N+1)}\\ &\quad + C\|v^n\|_{\dot B^{N/2}_{2,1}}\|v^{n+1}\|_{\dot B^{N/2+1}_{2,1}}\\ &\le C(1+\|\rho^n\|_{\dot B^{N/2+1}_{2,1}})\|\nabla P^{n+1}\|_{\dot B^{N/2}_{2,1}}^{(N-1)/(N+1)}\|\nabla P^{n+1}\|_{L^2}^{2/(N+1)}\\ &\quad + C\|v^n\|_{\dot B^{N/2}_{2,1}}\|v^{n+1}\|_{\dot B^{N/2+1}_{2,1}},\end{aligned} \tag{4.12}$$

where we used the interpolation and embedding lemmas, which listed in Section 2. So thanks to Young's inequality and (4.9), it follows from (4.12) that

$$\begin{aligned}\|\nabla P^{n+1}\|_{\dot B^{N/2}_{2,1}} &\le C(1+\|\rho^n\|_{\dot B^{N/2+1}_{2,1}}^{(N+1)/2})\|v^n\|_{L^2}\|v^{n+1}\|_{\dot B^{N/2+1}_{2,1}}\\ &\quad + C\|v^n\|_{\dot B^{N/2}_{2,1}}\|v^{n+1}\|_{\dot B^{N/2+1}_{2,1}}.\end{aligned} \tag{4.13}$$

If we apply $2^{j(N/2+1)}$ on (4.11), similarly, we obtain

$$\begin{aligned}\|\nabla P^{n+1}\|_{\dot B^{N/2+1}_{2,1}} &\le C(1+\|\rho^n\|_{\dot B^{N/2+1}_{2,1}})\|\nabla P^{n+1}\|_{\dot B^{N/2}_{2,1}}+C\|\operatorname{div}(v^n\cdot\nabla v^{n+1})\|_{\dot B^{N/2}_{2,1}}\\ &\le C(1+\|\rho^n\|_{\dot B^{N/2+1}_{2,1}})\|\nabla P^{n+1}\|_{\dot B^{N/2+1}_{2,1}}^{(N+1)/(N+3)}\|\nabla P^{n+1}\|_{\dot B^{-1/2}_{2,1}}^{2/(N+3)}\\ &\quad + C\|\partial_k v_l^n\partial_l v_k^{n+1}\|_{\dot B^{N/2}_{2,1}}\\ &\le C(1+\|\rho^n\|_{\dot B^{N/2+1}_{2,1}})\|\nabla P^{n+1}\|_{\dot B^{N/2+1}_{2,1}}^{(N+1)/(N+3)}\|\nabla P^{n+1}\|_{L^2}^{2/(N+3)}\\ &\quad + C\|v^n\|_{\dot B^{N/2+1}_{2,1}}\|v^{n+1}\|_{\dot B^{N/2+1}_{2,1}},\end{aligned}$$

where we used interpolation, embedding and product lemmas. Hence

$$\begin{aligned}\|\nabla P^{n+1}\|_{\dot B^{N/2+1}_{2,1}} &\le C(1+\|\rho^n\|^{(N+3)/2}_{\dot B^{N/2+1}_{2,1}})\|v^n\|_{L^2}\|v^{n+1}\|_{\dot B^{N/2+1}_{2,1}}\\ &\quad + C\|v^n\|_{\dot B^{N/2+1}_{2,1}}\|v^{n+1}\|_{\dot B^{N/2+1}_{2,1}}.\end{aligned}\tag{4.14}$$

Now combining (4.4), (4.7) (4.13) and (4.14), we obtain

$$\begin{aligned}&\frac{d}{dt}\|v^{n+1}\|_{B^{\frac N2+1}_{2,1}}\\ &\le C\|v^n\|_{L^2}(1+\|\rho^n\|_{\dot B^{\frac N2}_{2,1}})(1+\|\rho^n\|^{\frac{N+1}{2}}_{\dot B^{\frac N2+1}_{2,1}}+\|\rho^n\|^{\frac{N+3}{2}}_{\dot B^{\frac N2+1}_{2,1}})\|v^{n+1}\|_{B^{\frac N2+1}_{2,1}}\\ &\quad + C(1+\|\rho^n\|_{\dot B^{N/2}_{2,1}})\|v^n\|_{\dot B^{N/2+1}_{2,1}}\|v^{n+1}\|_{B^{N/2+1}_{2,1}}\\ &\quad + C(1+\|\rho^n\|_{\dot B^{N/2+1}_{2,1}})\|v^n\|_{\dot B^{N/2}_{2,1}}\|v^{n+1}\|_{B^{N/2+1}_{2,1}}.\end{aligned}\tag{4.15}$$

Apply Gronwall's inequality on (4.15), we obtain

$$\sup_{0\le t\le T}\|v^{n+1}\|_{B^{N/2+1}_{2,1}} \le \|v^{n+1}(0)\|_{\dot B^{N/2+1}_{2,1}}\exp\Big(\int_0^T B_n(t)dt\Big),\tag{4.16}$$

where $B_n(t)$ is the coefficient of $\|v^{n+1}\|_{B^{N/2+1}_{2,1}}$ in (4.15); i.e.,

$$\begin{aligned}B_n(t) &= C(1+\|\rho^n\|_{\dot B^{N/2}_{2,1}})\Big(\|v^n\|_{\dot B^{N/2+1}_{2,1}}+\|v\|_{L^2}(1+\|\rho^n\|^{\frac{N+1}{2}}_{\dot B^{\frac N2+1}_{2,1}}\\ &\quad +\|\rho^n\|^{\frac{N+3}{2}}_{\dot B^{\frac N2+1}_{2,1}})\Big)+C(1+\|\rho^n\|_{\dot B^{N/2+1}_{2,1}})\|v^n\|_{\dot B^{N/2}_{2,1}}.\end{aligned}\tag{4.17}$$

It is clear that the uniform estimates follows from (4.3), (4.16) and (4.17).
Convergence. Let w^{n+1}, u^{n+1} and Π^{n+1} be the same sequences as those in Section 3. The system reads

$$\begin{gathered}\partial_t w^{n+1}+v^n\cdot\nabla w^{n+1}+u^n\cdot\nabla\rho^n=0,\\ \partial_t u^{n+1}+v^n\cdot\nabla u^{n+1}+u^n\cdot\nabla v^n+(1+\rho^n)\nabla\Pi^{n+1}+w^n\nabla P^n=0,\\ \operatorname{div} w^{n+1}=\operatorname{div} v^n=0,\\ (w^{n+1},u^{n+1})|_{t=0}=(w^{n+1}(0),u^{n+1}(0))=(\Delta_n\rho_0,\Delta_n v_0).\end{gathered}\tag{4.18}$$

The estimates for w^{n+1}, u^{n+1} and Π^{n+1} are similar to those in Section 3, so we just write down the estimates directly.

$$\frac{d}{dt}\|w^{n+1}\|_{B^{N/2}_{2,1}} \le C\|\rho^n\|_{B^{N/2+1}_{2,1}}\|u^n\|_{B^{N/2}_{2,1}}+C\|v^n\|_{B^{N/2+1}_{2,1}}\|w^{n+1}\|_{B^{N/2}_{2,1}}.\tag{4.19}$$

$$\begin{aligned}\frac{d}{dt}\|u^{n+1}\|_{L^2} &\le C\|u^n\|_{\dot B^{N/2}_{2,1}}\|v^n\|_{\dot B^1_{2,1}}+C(1+\|\rho^n\|_{\dot B^{N/2}_{2,1}})\|\nabla\Pi^n\|_{L^2}\\ &\quad + C\|w^n\|_{\dot B^{N/2}_{2,1}}\|\nabla P^n\|_{L^2}.\end{aligned}\tag{4.20}$$

$$\begin{aligned}\frac{d}{dt}\|u^{n+1}\|_{\dot B^{N/2}_{2,1}} &\le C\|v^n\|_{\dot B^{N/2+1}_{2,1}}\|u^{n+1}\|_{\dot B^{N/2}_{2,1}}+C(1+\|\rho^n\|_{\dot B^{N/2}_{2,1}})\|\nabla\Pi^n\|_{\dot B^{N/2}_{2,1}}\\ &\quad + C\|w^n\|_{\dot B^{N/2}_{2,1}}\|\nabla P^n\|_{\dot B^{N/2}_{2,1}}+C\|u^n\|_{\dot B^{N/2}_{2,1}}\|v^n\|_{\dot B^{N/2+1}_{2,1}}.\end{aligned}\tag{4.21}$$

$$\begin{aligned}\|\nabla\Pi^{n+1}\|_{L^2} &\le C\|v^n\|_{\dot B^{N/2}_{2,1}}\|u^{n+1}\|_{\dot B^1_{2,1}}+C\|u^n\|_{\dot B^{N/2}_{2,1}}\|v^n\|_{\dot B^1_{2,1}}\\ &\quad + C\|w^n\|_{\dot B^{N/2}_{2,1}}\|\nabla P^n\|_{L^2}.\end{aligned}\tag{4.22}$$

$$\begin{aligned}\|\nabla\Pi^{n+1}\|_{\dot B^{N/2}_{2,1}} &\le C(1+\|\rho^n\|_{\dot B^{N/2+1}_{2,1}})\|\nabla\Pi^{n+1}\|_{L^2}+C\|v^n\|_{\dot B^{N/2+1}_{2,1}}\|u^{n+1}\|_{\dot B^{N/2}_{2,1}}\\ &\quad + C\|w^n\|_{\dot B^{N/2}_{2,1}}\|\nabla P^n\|_{\dot B^{N/2}_{2,1}}+C\|v^n\|_{\dot B^{N/2+1}_{2,1}}\|u^n\|_{\dot B^{N/2}_{2,1}}.\end{aligned}\tag{4.23}$$

It follows from (4.9) and (4.13) that the estimate for $\nabla\Pi^{n+1}$ can be represented in term of ρ^{n-1}, v^{n-1} and v^n. Due to the uniform estimates and (4.19)-(4.23), then the convergence follows from the same argument as that in Section 3.

The uniqueness follows from the analogous argument and estimates as that in Section 3 and (4.19)-(4.23).

5. Appendix

To prove the second part of Proposition 2.4, we show the following lemma, which clearly implies Proposition 2.4.

Lemma 5.1. *Let $s>0$, $1<p<\infty$. If f and g belong to $\dot B^s_{p,1}\cap L^\infty$, then fg is in $\dot B^s_{p,1}$, and*

$$\|fg\|_{\dot B^s_{p,1}} \le C\Big(\|f\|_{L^\infty}\|g\|_{\dot B^s_{p,1}}+\|g\|_{L^\infty}\|f\|_{\dot B^s_{p,1}}\Big).\tag{5.1}$$

Proof. We use Bony's decomposition [2] to represent the product as

$$fg=T_fg+T_gf+R(f,g),$$

where

$$T_fg=\sum_j S_{j-1}f\Delta_jg,\quad R(f,g)=\sum_j\Delta_jf(\Delta_{j-1}+\Delta_j+\Delta_{j+1})g.$$

By compactness of the supports of the series of Fourier transform, for any u,v,

$$\Delta_k\Delta_lu\equiv 0,\quad |k-l|\ge 2,\quad \Delta_k(S_{q-1}u\Delta_qv)=0,\quad \text{if } |k-q|\ge 5.$$

It follows that

$$\begin{aligned}\sum_{j\in\mathbb Z}2^{js}\|\Delta_jT_fg\|_{L^p} &= \sum_{j\in\mathbb Z}2^{js}\sum_{|j-j'|\le 4}\|\Delta_j(S_{j'-1}f\Delta_{j'}g)\|_{L^p}\\ &\le C\sup_q\|S_qf\|_{L^\infty}\sum_{j'\in\mathbb Z}2^{js}\|\Delta_{j'}g\|_{L^p}\\ &\le C\|g\|_{L^\infty}\|f\|_{\dot B^s_{p,1}}.\end{aligned}\tag{5.2}$$

Similarly,

$$\|T_gf\|_{\dot B^s_{p,1}}\le C\|f\|_{L^\infty}\|g\|_{\dot B^s_{p,1}}.\tag{5.3}$$

It follows from Bony's formula that

$$\begin{aligned}\Delta_jR(f,g) &= \sum_{\max\{i',j'\}\ge j-3,|i'-j'|\le 1}\Delta_j(\Delta_{i'}f\Delta_{j'}g)\\ &= \sum_{j'\ge j-4}\sum_{|i'-j'|\le 1}\Delta_j(\Delta_{i'}f\Delta_{j'}g).\end{aligned}$$

Therefore, by Minkowski inequality, we have

$$\begin{aligned}\sum_{j\in\mathbb{Z}} 2^{js} &\le \sum_{k\ge -4}\sum_{m=-1}^{1}\sum_{j'\in\mathbb{Z}} 2^{(j'-k)s}\|\Delta_{j'-k}(\Delta_{j'-m}f\Delta_{j'}g)\|_{L^p}\\ &\le C\sum_{k\ge -4}2^{-ks}\sum_{m=-1}^{1}\sum_{j'\in\mathbb{Z}}2^{j's}\|\Delta_{j'-m}f\Delta_{j'}g\|_{L^p}\\ &\le C\sup_q\|\Delta_q f\|_{L^\infty}\sum_{j'\in\mathbb{Z}}2^{j's}\|\Delta_{j'}g\|_{L^p}\\ &\le C\|f\|_{L^\infty}\|g\|_{\dot B^s_{p,1}}.\end{aligned} \tag{5.4}$$

Then (5.1) follows from (5.2), (5.3) and (5.4). □

Remark 5.2. Actually, we can prove the Moser type inequality

$$\|fg\|_{\dot B^s_{p,q}} \le C\Big(\|f\|_{L^{p_1}}\|g\|_{\dot B^s_{p_2,q}} + \|g\|_{L^{r_1}}\|f\|_{\dot B^s_{r_2,q}}\Big),$$

provided that $f\in L^{p_1}\cap \dot B^s_{r_2,q}$, $s>0$, $1\le p,q,p_1,r_2\le\infty$, $g\in L^{r_1}\cap\dot B^s_{p_2,q}$, $1\le r_1,p_2\le\infty$ and

$$\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{r_1}+\frac{1}{r_2}.$$

Proof of Proposition 3.1. The idea is to approximate (3.1) by linear transport equations. First we find that (3.1) is equivalent to the system

$$\begin{gathered}\partial_t v + w\cdot\nabla v+\nabla P = f,\\ -\Delta P = \operatorname{div}(w\cdot\nabla v) - \operatorname{div} f,\\ v(x,t=0)=v_0(x),\quad \operatorname{div} v_0 = 0.\end{gathered} \tag{5.5}$$

So we approximate (5.5) by the linear transport equations

$$\begin{gathered}\partial_t v^{n+1} + w\cdot\nabla v^{n+1}+\nabla P^n = f,\\ -\Delta P^n = \operatorname{div}(w\cdot\nabla v^n) - \operatorname{div} f,\\ v^{n+1}(x,t=0)=S_{n+1}v_0(x).\end{gathered} \tag{5.6}$$

The existence theorem for (5.6) is well-known for each n. Just as the proof of Theorem 1.1, we should give a uniform estimates for the sequence v^{n+1} and the convergence of the corresponding sequence. In order to do so, we only need to do a priori estimates for the equivalent system (5.5). First, we have

$$\frac{d}{dt}\|v(.,t)\|_{B^{N/p+1}_{p,1}} \le C\|w\|_{\dot B^{N/p+1}_{p,1}}\|v\|_{B^{N/p+1}_{p,1}} + \|f\|_{B^{N/p+1}_{p,1}} + \|\nabla P\|_{B^{N/p+1}_{p,1}}. \tag{5.7}$$

The estimate for the pressure is easy now, it reads

$$\|\nabla P\|_{B^{N/p+1}_{p,1}} \le C\|w\|_{\dot B^{N/p+1}_{p,1}}\|v\|_{\dot B^{N/p+1}_{p,1}} + C\|f\|_{B^{N/p+1}_{p,1}}.$$

Therefore, from (5.7) it follows that

$$\frac{d}{dt}\|v(.,t)\|_{B^{N/p+1}_{p,1}} \le C\|w\|_{\dot B^{N/p+1}_{p,1}}\|v\|_{B^{N/p+1}_{p,1}} + C\|f\|_{B^{N/p+1}_{p,1}}. \tag{5.8}$$

Apply Gronwall inequality on (5.8),

$$\begin{aligned} \|v(.,t)\|_{B^{N/p+1}_{p,1}} &\leq \|v_0\|_{B^{N/p+1}_{p,1}} \exp\Big(\int_0^t C\|w(.,s)\|_{\dot B^{N/p+1}_{p,1}} ds\Big) \\ &\quad + \int_0^t \|f(.,\tau)\|_{B^{\frac{N}{p}+1}_{p,1}} \exp\Big(\int_\tau^t C\|w(.,s)\|_{\dot B^{\frac{N}{p}+1}_{p,1}} ds\Big) d\tau. \end{aligned} \tag{5.9}$$

Since we have the a priori estimate (5.9), the existence and uniqueness of solutions for the system (5.5) can be obtained by the approximate sequence v^{n+1}, solutions to (5.6). This completes the proof. □

Proof of Proposition 4.1. Just as for Proposition 3.1, note that (4.3) is equivalent to the linear system

$$\begin{gathered} \partial_t v + w\cdot\nabla v + (1+\rho)\nabla P = 0, \\ -\operatorname{div}\left((1+\rho)\nabla P\right) = \operatorname{div}(w\cdot\nabla v), \\ v(x,t=0) = v_0(x), \quad \operatorname{div} v_0 = 0. \end{gathered} \tag{5.10}$$

The linear transport approximate system is

$$\begin{gathered} \partial_t v^{n+1} + w\cdot\nabla v^{n+1} + (1+\rho)\nabla P^n = 0, \\ -\operatorname{div}\left((1+\rho)\nabla P^n\right) = \operatorname{div}(w\cdot\nabla v^n), \\ v^{n+1}(x,t=0) = S_{n+1}v_0(x). \end{gathered} \tag{5.11}$$

It is easy to establish a priori estimates for the system (5.10), then we can prove the existence and uniqueness of the solution, which is a limit of the iteration sequence. We would like to skip the details of the proof, for conciseness. □

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After publication, the author received the following comments.

The smallness assumption on initial data was removed in: Raphaël Danchin; On the well-posedness of the incompressible density-dependent Euler equations in the L^p framework. J. Differential Equations 248 (2010), 8, 2130–2170.

The case $p = \infty$ was treated in: Raphaël Danchin, Francesco Fanelli; The well-posedness issue for the density-dependent Euler equations in endpoint Besov spaces. J. Math. Pures Appl. (9) 96 (2011), 3, 253–278.

The author wants to thank the anonymous reader for sending this information.

It was also commented that a result similar to Theorem 1.1 was obtained in: Young Zhou; Local well-posedness and regularity criterion for the density dependent incompressible Euler equations. Nonlinear Anal. 73 (2010), no. 3, 750–766.

Our article studies the critical case $s = p/n + 1$, in the space $B^{p/n+1}_{p,1}$; while the above reference studies the super-critical case $s > p/n + 1$, in the space $B^s_{p,q}$.

End of addendum.

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