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SOLVABILITY IN THE SENSE OF SEQUENCES TO SOME NON-FREDHOLM OPERATORS

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Abstract. We study the solvability of certain linear nonhomogeneous elliptic problems and show that under reasonable technical conditions the convergence in $L^2(\mathbb{R}^d)$ of their right sides implies the existence and the convergence in $H^2(\mathbb{R}^d)$ of the solutions. The equations involve second order differential operators without Fredholm property and we use the methods of spectral and scattering theory for Schrödinger type operators analogously to our preceding work [17].

1. Introduction

Consider the equation

$$-\Delta u + V(x)u - au = f, \tag{1.1}$$

where $u \in E = H^2(\mathbb{R}^d)$ and $f \in F = L^2(\mathbb{R}^d)$, $d \in \mathbb{N}$, a is a constant and $V(x)$ is a function converging to 0 at infinity. If $a \geq 0$, then the essential spectrum of the operator $A : E \to F$ corresponding to the left-hand side of equation (1.1) contains the origin. As a consequence, the operator does not satisfy the Fredholm property. Its image is not closed, for $d > 1$ the dimensions of its kernel and the codimension of its image are not finite. In this work we will study some properties of such operators. Let us note that elliptic problems involving non-Fredholm operators were studied extensively in recent years (see [16, 17, 18, 19, 20, 21, 22, 23, 5]) along with their potential applications to the theory of reaction-diffusion equations (see [7, 8]). In the particular case where $a = 0$ the operator A satisfies the Fredholm property in some properly chosen weighted spaces [1, 2, 3, 4, 5]. However, the case with $a \neq 0$ is essentially different and the approach developed in these works cannot be applied.

One of the important questions about equations with non-Fredholm operators concerns their solvability. We will study it in the following setting. Let f_n be a sequence of functions in the image of the operator A, such that $f_n \to f$ in $L^2(\mathbb{R}^d)$ as $n \to \infty$. Denote by u_n a sequence of functions from $H^2(\mathbb{R}^d)$ such that

$$Au_n = f_n, \quad n \in \mathbb{N}.$$

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Since the operator A does not satisfy the Fredholm property, then the sequence u_n may not be convergent. We will call a sequence u_n such that $Au_n \to f$ a solution in the sense of sequences of equation $Au = f$ (see [15]). If this sequence converges to a function u_0 in the norm of the space E, then u_0 is a solution of this equation. Solution in the sense of sequences is equivalent in this sense to the usual solution. However, in the case of non-Fredholm operators this convergence may not hold or it can occur in some weaker sense. In this case, solution in the sense of sequences may not imply the existence of the usual solution. In this work we will find sufficient conditions of equivalence of solutions in the sense of sequences and the usual solutions. In the other words, the conditions on sequences f_n under which the corresponding sequences u_n are strongly convergent.

In the first part of the article we consider the equation

$$-\Delta u - au = f(x), \quad x \in \mathbb{R}^d,\ d \in \mathbb{N}, \tag{1.2}$$

where $a \geq 0$ is a constant and the right side is square integrable. Note that for the operator $-\Delta - a$ on $L^2(\mathbb{R}^d)$ the essential spectrum fills the semi-axis $[-a,\ \infty)$ such that its inverse from $L^2(\mathbb{R}^d)$ to $H^2(\mathbb{R}^d)$ is not bounded. Let us write down the corresponding sequence of equations with $n \in \mathbb{N}$ as

$$-\Delta u_n - au_n = f_n(x), \quad x \in \mathbb{R}^d,\ d \in \mathbb{N}, \tag{1.3}$$

with the right sides convergent to the right side of (1.2) in $L^2(\mathbb{R}^d)$ as $n \to \infty$. The inner product of two functions

$$(f(x), g(x))_{L^2(\mathbb{R}^d)} := \int_{\mathbb{R}^d} f(x)\bar{g}(x)dx,$$

with a slight abuse of notations when these functions are not square integrable. Indeed, if $f(x) \in L^1(\mathbb{R}^d)$ and $g(x)$ is bounded, then clearly the integral considered above makes sense, like for instance in the case of functions involved in the orthogonality conditions of Theorems 1.1 and 1.2 below. In the space of three dimensions for some $A(x) = (A_1(x), A_2(x), A_3(x))$, the inner product $(f(x), A(x))_{L^2(\mathbb{R}^3)}$ is the vector with the coordinates

$$\int_{\mathbb{R}^3} f(x)\bar{A}_k(x)dx, \quad k = 1, 2, 3.$$

We start with formulating the proposition in one dimension. We will consider the space $H^2(\mathbb{R}^d)$ with the norm

$$\|u\|^2_{H^2(\mathbb{R}^d)} := \|u\|^2_{L^2(\mathbb{R}^d)} + \|\Delta u\|^2_{L^2(\mathbb{R}^d)}. \tag{1.4}$$

Theorem 1.1. *Let $n \in \mathbb{N}$ and $f_n(x) \in L^2(\mathbb{R})$, such that $f_n(x) \to f(x)$ in $L^2(\mathbb{R})$ as $n \to \infty$.*

(a) When $a > 0$ let $xf_n(x) \in L^1(\mathbb{R})$, such that $xf_n(x) \to xf(x)$ in $L^1(\mathbb{R})$ as $n \to \infty$ and the orthogonality conditions

$$\Big(f_n(x), \frac{e^{\pm i\sqrt{a}x}}{\sqrt{2\pi}}\Big)_{L^2(\mathbb{R})} = 0 \tag{1.5}$$

hold for all $n \in \mathbb{N}$. Then equations (1.2) and (1.3) admit unique solutions $u(x) \in H^2(\mathbb{R})$ and $u_n(x) \in H^2(\mathbb{R})$ respectively, such that $u_n(x) \to u(x)$ in $H^2(\mathbb{R})$ as $n \to \infty$.

(b) When $a = 0$ let $x^2 f_n(x) \in L^1(\mathbb{R})$, such that $x^2 f_n(x) \to x^2 f(x)$ in $L^1(\mathbb{R})$ as $n \to \infty$ and the orthogonality relations

$$(f_n(x), 1)_{L^2(\mathbb{R})} = 0, \quad (f_n(x), x)_{L^2(\mathbb{R})} = 0 \tag{1.6}$$

hold for all $n \in \mathbb{N}$. Then problems (1.2) and (1.3) possess unique solutions $u(x) \in H^2(\mathbb{R})$ and $u_n(x) \in H^2(\mathbb{R})$ respectively, where $u_n(x) \to u(x)$ in $H^2(\mathbb{R})$ as $n \to \infty$.

Then we turn our attention to the issue in dimensions two and higher. The sphere of radius $r > 0$ in $\mathbb{R}^d$ centered at the origin will be denoted by S_r^d, of radius $r = 1$ as S^d and its Lebesgue measure by $|S^d|$. The notation B^d will stand for the unit ball in the space of d dimensions with the center at the origin and $|B^d|$ for its Lebesgue measure.

Theorem 1.2. *Let $d \geq 2$, $n \in \mathbb{N}$ and $f_n(x) \in L^2(\mathbb{R}^d)$, such that $f_n(x) \to f(x)$ in $L^2(\mathbb{R}^d)$ as $n \to \infty$.*

(a) When $a > 0$ let $|x| f_n(x) \in L^1(\mathbb{R}^d)$, such that $|x| f_n(x) \to |x| f(x)$ in $L^1(\mathbb{R}^d)$ as $n \to \infty$ and the orthogonality conditions

$$\Big(f_n(x), \frac{e^{ipx}}{(2\pi)^{d/2}}\Big)_{L^2(\mathbb{R}^d)} = 0, \quad p \in S^d_{\sqrt{a}} \text{ a.e., } d \geq 2 \tag{1.7}$$

hold for all $n \in \mathbb{N}$. Then equations (1.2) and (1.3) admit unique solutions $u(x) \in H^2(\mathbb{R}^d)$ and $u_n(x) \in H^2(\mathbb{R}^d)$ respectively, such that $u_n(x) \to u(x)$ in $H^2(\mathbb{R}^d)$ as $n \to \infty$.

(b) When $a = 0$ and $d = 2$ let $|x|^2 f_n(x) \in L^1(\mathbb{R}^2)$, such that $|x|^2 f_n(x) \to |x|^2 f(x)$ in $L^1(\mathbb{R}^2)$ as $n \to \infty$ and the orthogonality relations

$$(f_n(x), 1)_{L^2(\mathbb{R}^2)} = 0, \quad (f_n(x), x_m)_{L^2(\mathbb{R}^2)} = 0, \quad m = 1, 2 \tag{1.8}$$

hold for all $n \in \mathbb{N}$. Then problems (1.2) and (1.3) have unique solutions $u(x) \in H^2(\mathbb{R}^2)$ and $u_n(x) \in H^2(\mathbb{R}^2)$ respectively, such that $u_n(x) \to u(x)$ in $H^2(\mathbb{R}^2)$ as $n \to \infty$.

(c) When $a = 0$ and $d = 3, 4$ let $|x| f_n(x) \in L^1(\mathbb{R}^d)$, such that $|x| f_n(x) \to |x| f(x)$ in $L^1(\mathbb{R}^d)$ as $n \to \infty$ and the orthogonality condition

$$(f_n(x), 1)_{L^2(\mathbb{R}^d)} = 0, \quad d = 3, 4 \tag{1.9}$$

holds for all $n \in \mathbb{N}$. Then problems (1.2) and (1.3) admit unique solutions $u(x) \in H^2(\mathbb{R}^d)$ and $u_n(x) \in H^2(\mathbb{R}^d)$ respectively, such that $u_n(x) \to u(x)$ in $H^2(\mathbb{R}^d)$ as $n \to \infty$.

(d) When $a = 0$ and $d \geq 5$ let $f_n(x) \in L^1(\mathbb{R}^d)$, such that $f_n(x) \to f(x)$ in $L^1(\mathbb{R}^d)$ as $n \to \infty$. Then equations (1.2) and (1.3) have unique solutions $u(x) \in H^2(\mathbb{R}^d)$ and $u_n(x) \in H^2(\mathbb{R}^d)$ respectively, such that $u_n(x) \to u(x)$ in $H^2(\mathbb{R}^d)$ as $n \to \infty$.

Note that when $a = 0$ and the dimension of the problem is at least five, orthogonality conditions in the Theorem above are not required (see e.g. [23, Lemmas 6 and 7]). We will be using the hat symbol to denote the standard Fourier transform

$$\widehat{f}(p) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x) e^{-ipx} dx, \ p \in \mathbb{R}^d, \quad d \in \mathbb{N}. \tag{1.10}$$

In the second part of the work we study the equation

$$-\Delta u + V(x) u - au = f(x), \quad x \in \mathbb{R}^3, \ a \geq 0, \tag{1.11}$$

where the right side is square integrable. The correspondent sequence of equations for $n \in \mathbb{N}$ will be

$$-\Delta u_n + V(x)u_n - au_n = f_n(x), \quad x \in \mathbb{R}^3, \ a \geq 0, \tag{1.12}$$

where the right sides converge to the right side of (1.11) in $L^2(\mathbb{R}^3)$ as $n \to \infty$. Let us make the following technical assumptions on the scalar potential involved in equations above. Note that the conditions on $V(x)$, which is shallow and short-range will be analogous to those stated in [17, Assumption 1.1] (see also [18, 19]). The essential spectrum of such a Schrödinger operator fills the nonnegative semi-axis (see e.g. [10]).

Assumption 1.3. The potential function $V(x) : \mathbb{R}^3 \to \mathbb{R}$ satisfies the estimate

$$|V(x)| \leq \frac{C}{1 + |x|^{3.5+\delta}}$$

for some $\delta > 0$ and $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ a.e. such that

$$4^{1/9}\frac{9}{8}(4\pi)^{-2/3}\|V\|_{L^\infty(\mathbb{R}^3)}^{1/9}\|V\|_{L^{4/3}(\mathbb{R}^3)}^{8/9} < 1 \quad \text{and} \quad \sqrt{c_{HLS}}\|V\|_{L^{3/2}(\mathbb{R}^3)} < 4\pi.$$

Here and further down C stands for a finite positive constant and c_{HLS} given on p.98 of [12] is the constant in the Hardy-Littlewood-Sobolev inequality

$$\Big|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{f_1(x)f_1(y)}{|x-y|^2}\,dx\,dy\Big| \leq c_{HLS}\|f_1\|_{L^{3/2}(\mathbb{R}^3)}^2, \quad f_1 \in L^{3/2}(\mathbb{R}^3).$$

According to [17, Lemma 2.3], under Assumption 1.3 above on the potential function, the operator $-\Delta + V(x) - a$ on $L^2(\mathbb{R}^3)$ is self-adjoint and unitarily equivalent to $-\Delta - a$ via the wave operators (see [11], [14])

$$\Omega^\pm := \text{s-lim}_{t\to\mp\infty}\, e^{it(-\Delta+V)}e^{it\Delta},$$

where the limit is understood in the strong L^2 sense (see e.g. [13] p.34, [6] p.90). Hence $-\Delta + V(x) - a$ on $L^2(\mathbb{R}^3)$ has only the essential spectrum $\sigma_{ess}(-\Delta + V(x) - a) = [-a, \ \infty)$. By means of the spectral theorem, its functions of the continuous spectrum satisfying

$$[-\Delta + V(x)]\varphi_k(x) = k^2\varphi_k(x), \quad k \in \mathbb{R}^3, \tag{1.13}$$

in the integral formulation the Lippmann-Schwinger equation for the perturbed plane waves (see e.g. [13] p.98)

$$\varphi_k(x) = \frac{e^{ikx}}{(2\pi)^{3/2}} - \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{e^{i|k||x-y|}}{|x-y|}(V\varphi_k)(y)dy \tag{1.14}$$

and the orthogonality relations

$$(\varphi_k(x), \varphi_q(x))_{L^2(\mathbb{R}^3)} = \delta(k-q), \quad k, q \in \mathbb{R}^3 \tag{1.15}$$

form the complete system in $L^2(\mathbb{R}^3)$. In particular, when the vector $k = 0$, we have $\varphi_0(x)$. Let us denote the generalized Fourier transform with respect to these functions using the tilde symbol as

$$\tilde{f}(k) := (f(x), \varphi_k(x))_{L^2(\mathbb{R}^3)}, \quad k \in \mathbb{R}^3.$$

The integral operator involved in (1.14) is being denoted as

$$(Q\varphi)(x) := -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{e^{i|k||x-y|}}{|x-y|}(V\varphi)(y)dy, \quad \varphi \in L^\infty(\mathbb{R}^3).$$

Let us consider $Q : L^\infty(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$. Under Assumption 1.3, according to [17, Lemma 2.1] the operator norm $\|Q\|_\infty < 1$, in fact it is bounded above by a quantity independent of k which is expressed in terms of the appropriate $L^p(\mathbb{R}^3)$ norms of the potential function $V(x)$. We have the following statement.

Theorem 1.4. *Let Assumption 1.3 hold, $n \in \mathbb{N}$ and $f_n(x) \in L^2(\mathbb{R}^3)$, such that $f_n(x) \to f(x)$ in $L^2(\mathbb{R}^3)$ as $n \to \infty$. Assume also that $|x|f_n(x) \in L^1(\mathbb{R}^3)$, such that $|x|f_n(x) \to |x|f(x)$ in $L^1(\mathbb{R}^3)$ as $n \to \infty$.*

(a) When $a > 0$ let the orthogonality conditions

$$(f_n(x), \varphi_k(x))_{L^2(\mathbb{R}^3)} = 0, \ k \in S^3_{\sqrt{a}} \ a.e. \tag{1.16}$$

hold for all $n \in \mathbb{N}$. Then equations (1.11) *and* (1.12) *admit unique solutions $u(x) \in H^2(\mathbb{R}^3)$ and $u_n(x) \in H^2(\mathbb{R}^3)$ respectively, such that $u_n(x) \to u(x)$ in $H^2(\mathbb{R}^3)$ as $n \to \infty$.*

(b) When $a = 0$ let the orthogonality relation

$$(f_n(x), \varphi_0(x))_{L^2(\mathbb{R}^3)} = 0 \tag{1.17}$$

hold for all $n \in \mathbb{N}$. Then equations (1.11) *and* (1.12) *possess unique solutions $u(x) \in H^2(\mathbb{R}^3)$ and $u_n(x) \in H^2(\mathbb{R}^3)$ respectively, such that $u_n(x) \to u(x)$ in $H^2(\mathbb{R}^3)$ as $n \to \infty$.*

Note that (1.16) and (1.17) are the orthogonality conditions to the functions of the continuous spectrum of our Schrödinger operator, as distinct from the Limiting Absorption Principle in which one needs to orthogonalize to the standard Fourier harmonics (see e.g. [9, Lemma 2.3 and Proposition 2.4]).

2. Proof of the generalization of the solvability in the sense of sequences

Application of the standard Fourier transform (1.10) to both sides of equations (1.2) and (1.3) for $p \in \mathbb{R}^d, \ d \in \mathbb{N}$ yields

$$\widehat{u}(p) = \frac{\widehat{f}(p)}{p^2 - a}, \quad \widehat{u}_n(p) = \frac{\widehat{f}_n(p)}{p^2 - a}, \quad a \geq 0, \quad n \in \mathbb{N}.$$

When $a = 0$ we write their difference as

$$\widehat{u}_n(p) - \widehat{u}(p) = \frac{\widehat{f}_n(p) - \widehat{f}(p)}{p^2} \chi_{\{p \in \mathbb{R}^d : |p| \leq 1\}} + \frac{\widehat{f}_n(p) - \widehat{f}(p)}{p^2} \chi_{\{p \in \mathbb{R}^d : |p| > 1\}}. \tag{2.1}$$

Here and further down χ_A will stand for the characteristic function of a set $A \subseteq \mathbb{R}^d$. The complement of a set will be designated as A^c. Denote the second term in the right side of (2.1) as $\xi_n^{d,0}(p)$.

When $a > 0$ and the dimension $d = 1$ we introduce the following set as the union of intervals on the real line

$$I_\delta = I_\delta^- \cup I_\delta^+ := [-\sqrt{a} - \delta, -\sqrt{a} + \delta] \cup [\sqrt{a} - \delta, \sqrt{a} + \delta], \quad 0 < \delta < \sqrt{a},$$

which enables us to express in this case

$$\widehat{u}_n(p) - \widehat{u}(p) = \frac{\widehat{f}_n(p) - \widehat{f}(p)}{p^2 - a} \chi_{I_\delta^-}(p) + \frac{\widehat{f}_n(p) - \widehat{f}(p)}{p^2 - a} \chi_{I_\delta^+}(p) + \frac{\widehat{f}_n(p) - \widehat{f}(p)}{p^2 - a} \chi_{I_\delta^c}(p). \tag{2.2}$$

Denote the last term in the right side of (2.2) as $\xi_n^{1, \ a}(p)$.

For $a>0$ and dimensions $d\ge 2$ we introduce the following set as the layer in $\mathbb{R}^d$:

$$A_\sigma := \{p\in\mathbb{R}^d \mid \sqrt{a}-\sigma\le |p|\le \sqrt{a}+\sigma\},\quad 0<\sigma<\sqrt{a}$$

and express

$$\widehat{u}_n(p)-\widehat{u}(p)=\frac{\widehat{f}_n(p)-\widehat{f}(p)}{p^2-a}\chi_{A_\sigma}+\frac{\widehat{f}_n(p)-\widehat{f}(p)}{p^2-a}\chi_{A_\sigma{}^c}. \tag{2.3}$$

Denote the second term in the right side of (2.3) as $\xi_n^{d,a}(p)$.

Proof of Theorem 1.1. (a) We express the first term in the right side of (2.2) as

$$\frac{\widehat{f}_n(-\sqrt{a})-\widehat{f}(-\sqrt{a})+\int_{-\sqrt{a}}^{p}\frac{d}{dq}[\widehat{f}_n(q)-\widehat{f}(q)]dq}{p^2-a}\chi_{I_\delta^-}(p). \tag{2.4}$$

Note that by means of orthogonality conditions (1.5) and part (a) of Lemma 3.3 with $w(x)=\frac{e^{\pm i\sqrt{a}x}}{\sqrt{2\pi}}$, we have

$$\Big(f_n(x),\frac{e^{\pm i\sqrt{a}x}}{\sqrt{2\pi}}\Big)_{L^2(\mathbb{R})}=0,\quad n\in\mathbb{N},\quad \Big(f(x),\frac{e^{\pm i\sqrt{a}x}}{\sqrt{2\pi}}\Big)_{L^2(\mathbb{R})}=0, \tag{2.5}$$

such that $\widehat{f}_n(\pm\sqrt{a})$ and $\widehat{f}(\pm\sqrt{a})$ vanish and via [23, Lemma 5] equations (1.2) and (1.3) considered in one dimension with $a>0$ admit unique solutions $u(x)\in H^2(\mathbb{R})$ and $u_n(x)\in H^2(\mathbb{R})$ respectively. By using the trivial estimate

$$\Big|\frac{d}{dq}[\widehat{f}_n(q)-\widehat{f}(q)]\Big|\le\frac{1}{\sqrt{2\pi}}\|xf_n-xf\|_{L^1(\mathbb{R})},\ q\in\mathbb{R}, \tag{2.6}$$

we easily derive the upper bound on the absolute value of (2.4) as

$$\frac{1}{\sqrt{2\pi}}\frac{\|xf_n-xf\|_{L^1(\mathbb{R})}}{2\sqrt{a}-\delta}\chi_{I_\delta^-}(p).$$

Therefore, the $L^2(\mathbb{R})$ norm of the first term in the right side of (2.2) can be estimated from above by

$$\sqrt{\frac{\delta}{\pi}}\frac{\|xf_n-xf\|_{L^1(\mathbb{R})}}{2\sqrt{a}-\delta}\to 0,\ n\to\infty \tag{2.7}$$

according to one of the assumptions of the theorem. Similarly to (2.4) and using relations (2.5), we write the second term in the right side of (2.2) as

$$\frac{\int_{\sqrt{a}}^{p}\frac{d}{dq}[\widehat{f}_n(q)-\widehat{f}(q)]dq}{p^2-a}\chi_{I_\delta^+}(p), \tag{2.8}$$

which can be easily estimated from above in the absolute value by means of (2.6) by

$$\frac{1}{\sqrt{2\pi}}\frac{\|xf_n-xf\|_{L^1(\mathbb{R})}}{2\sqrt{a}-\delta}\chi_{I_\delta^+}(p).$$

Hence, the $L^2(\mathbb{R})$ norm of the second term in the right side of (2.2) admits the upper bound (2.7) as well. Thus, via Lemma 3.2, which guarantees that

$$\lim_{n\to\infty}\|\xi_n^{1,\ a}(p)\|_{L^2(\mathbb{R})}=0,$$

we have $u_n(x)\to u(x)$ in $L^2(\mathbb{R})$ as $n\to\infty$ and complete the proof of part a) of the theorem by means of part (a) of Lemma 3.1.

(b) By means of orthogonality relations (1.6) for all $n \in \mathbb{N}$ we have

$$\widehat{f}_n(0) = 0, \quad \frac{d\widehat{f}_n}{dp}(0) = 0. \tag{2.9}$$

Then part (b) of Lemma 3.3 yields

$$(f(x), 1)_{L^2(\mathbb{R})} = 0, \quad (f(x), x)_{L^2(\mathbb{R})} = 0,$$

such that

$$\widehat{f}(0) = 0, \quad \frac{d\widehat{f}}{dp}(0) = 0. \tag{2.10}$$

Via part (b) of [23, Lemma 5] equations (1.2) and (1.3) studied in one dimension with $a = 0$ admit unique solutions $u(x) \in H^2(\mathbb{R})$ and $u_n(x) \in H^2(\mathbb{R})$ respectively. Identities (2.9) and (2.10) yield the representation formula

$$\widehat{f}_n(p) - \widehat{f}(p) = \int_0^p \Big(\int_0^s \frac{d^2}{dq^2}[\widehat{f}_n(q) - \widehat{f}(q)]dq \Big) ds, \quad p \in \mathbb{R},$$

which we are going to use along with the inequality

$$\Big|\frac{d^2}{dq^2}[\widehat{f}_n(q) - \widehat{f}(q)]\Big| \le \frac{1}{\sqrt{2\pi}} \|x^2 f_n - x^2 f\|_{L^1(\mathbb{R})}, \ q \in \mathbb{R}.$$

Thus, for the first term in the right side of (2.1) in one dimension we have the upper bound in the absolute value as

$$\frac{1}{2\sqrt{2\pi}} \|x^2 f_n - x^2 f\|_{L^1(\mathbb{R})} \chi_{\{p \in \mathbb{R}:\ |p| \le 1\}}$$

and in the $L^2(\mathbb{R})$ norm as

$$\frac{1}{2\sqrt{\pi}} \|x^2 f_n - x^2 f\|_{L^1(\mathbb{R})} \to 0, \ n \to \infty$$

according to one of the assumptions of the theorem. By means of Lemma 3.2

$$\lim_{n\to\infty} \|\xi_n^{1,\ 0}(p)\|_{L^2(\mathbb{R})} = 0$$

and we arrive at $u_n(x) \to u(x)$ in $L^2(\mathbb{R})$ as $n \to \infty$. We complete the proof of the theorem via part (a) of Lemma 3.1. □

Proof of Theorem 1.2. (a) Orthogonality conditions (1.7) along with part (a) of Lemma 3.3 with $w(x) = \frac{e^{ipx}}{(2\pi)^{d/2}}$, $p \in S^d_{\sqrt{a}}$ a.e. imply

$$\Big(f(x), \frac{e^{ipx}}{(2\pi)^{d/2}} \Big)_{L^2(\mathbb{R}^d)} = 0, \quad p \in S^d_{\sqrt{a}} \ a.e., \tag{2.11}$$

such that by means of part a) of [23, Lemma 6] equations (1.2) and (1.3) with $a > 0$ admit unique solutions $u(x) \in H^2(\mathbb{R}^d)$ and $u_n(x) \in H^2(\mathbb{R}^d)$ respectively for $d \ge 2$. Due to (1.7) and (2.11), we have

$$\widehat{f}_n(\sqrt{a}, \omega) = 0, \quad \widehat{f}(\sqrt{a}, \omega) = 0 \quad \text{a.e.} \tag{2.12}$$

Here and below ω stands for the angle variables on the sphere centered at the origin of a given radius. Via identities (2.12) the first term in the right side of (2.3) can be written as

$$\frac{\int_{\sqrt{a}}^{|p|} \frac{\partial}{\partial s}[\widehat{f}_n(s, \omega) - \widehat{f}(s, \omega)]ds}{p^2 - a} \chi_{A_\sigma}. \tag{2.13}$$

Clearly, for $q \in \mathbb{R}^d$, $d \geq 2$ we have the inequality

$$\Big|\frac{\partial}{\partial |q|}[\widehat{f}_n(|q|,\omega) - \widehat{f}(|q|,\omega)]\Big| \leq \frac{1}{(2\pi)^{d/2}}\||x|f_n - |x|f\|_{L^1(\mathbb{R}^d)}, \tag{2.14}$$

such that expression (2.13) can be estimated from above in the absolute value by

$$\frac{1}{(2\pi)^{d/2}\sqrt{a}}\||x|f_n - |x|f\|_{L^1(\mathbb{R}^d)}\chi_{A_\sigma}$$

and therefore in the $L^2(\mathbb{R}^d)$ norm by

$$\frac{1}{(2\pi)^{d/2}\sqrt{a}}\||x|f_n - |x|f\|_{L^1(\mathbb{R}^d)}\sqrt{|B^d|[(\sqrt{a}+\sigma)^d - (\sqrt{a}-\sigma)^d]} \to 0,$$

as $n \to \infty$ due to one of the assumptions of the theorem. Hence, according to Lemma 3.2

$$\lim_{n\to\infty}\|\xi_n^{d,a}(p)\|_{L^2(\mathbb{R}^d)} = 0$$

and we arrive at $u_n(x) \to u(x)$ in $L^2(\mathbb{R}^d)$, $d \geq 2$ as $n \to \infty$. We complete the proof of part a) of the theorem via part (a) of Lemma 3.1.

(b) By means of orthogonality conditions (1.8) along with part (b) of Lemma 3.3 we have

$$(f(x),1)_{L^2(\mathbb{R}^2)} = 0, \quad (f(x),x_m)_{L^2(\mathbb{R}^2)} = 0, \quad m = 1,2. \tag{2.15}$$

Thus via part b) of [23, Lemma 6] equations (1.2) and (1.3) with $a = 0$ considered in two dimensions admit unique solutions $u(x) \in H^2(\mathbb{R}^2)$ and $u_n(x) \in H^2(\mathbb{R}^2)$ respectively. Identities (1.8) and (2.15) imply $\widehat{f}_n(0) = 0$, $n \in \mathbb{N}$ and $\widehat{f}(0) = 0$. Let θ denote the angle between two vectors $p = (|p|,\theta_p)$ and $x = (|x|,\theta_x)$ in $\mathbb{R}^2$. Then

$$\frac{\partial \widehat{f}_n}{\partial |p|}(0,\theta_p) = -\frac{i}{2\pi}\int_{\mathbb{R}^2} f_n(x)|x|cos\theta dx$$

can be easily expressed as

$$-\frac{i}{2\pi}\{\cos\theta_p\int_{\mathbb{R}^2} f_n(x)x_1dx + \sin\theta_p\int_{\mathbb{R}^2} f_n(x)x_2dx\} = 0$$

due to orthogonality relations (1.8). Analogously, we can write $\frac{\partial \widehat{f}}{\partial |p|}(0,\theta_p)$ as

$$-\frac{i}{2\pi}\{\cos\theta_p\int_{\mathbb{R}^2} f(x)x_1dx + \sin\theta_p\int_{\mathbb{R}^2} f(x)x_2dx\} = 0$$

via orthogonality conditions (2.15). The argument above implies

$$\widehat{f}_n(p) - \widehat{f}(p) = \int_0^{|p|}\Big(\int_0^s \frac{\partial^2}{\partial \xi^2}[\widehat{f}_n(\xi,\theta_p) - \widehat{f}(\xi,\theta_p)]d\xi\Big)ds.$$

Clearly, for $p \in \mathbb{R}^2$ we have the inequality

$$\Big|\frac{\partial^2}{\partial |q|^2}[\widehat{f}_n(q) - \widehat{f}(q)]\Big| \leq \frac{1}{2\pi}\||x|^2f_n - |x|^2f\|_{L^1(\mathbb{R}^2)},$$

which yields the upper bound

$$|\widehat{f}_n(p) - \widehat{f}(p)| \leq \frac{1}{4\pi}\||x|^2f_n - |x|^2f\|_{L^1(\mathbb{R}^2)}|p|^2, \quad p \in \mathbb{R}^2.$$

Thus the first term in the right side of (2.1) admits the estimate from above in the absolute value as

$$\frac{1}{4\pi}\||x|^2 f_n - |x|^2 f\|_{L^1(\mathbb{R}^2)}\chi_{\{p\in\mathbb{R}^2 : |p|\le 1\}}$$

and in the $L^2(\mathbb{R}^2)$ norm as

$$\frac{1}{2\sqrt{\pi}}\||x|^2 f_n - |x|^2 f\|_{L^1(\mathbb{R}^2)} \to 0, \quad n\to\infty$$

according to one of the assumptions of the theorem. By means of Lemma 3.2 we have

$$\lim_{n\to\infty}\|\xi_n^{2,\ 0}(p)\|_{L^2(\mathbb{R}^2)} = 0$$

and then via part a) of Lemma 3.1 we obtain $u_n(x)\to u(x)$ in $H^2(\mathbb{R}^2)$ as $n\to\infty$.

(c) Orthogonality condition (1.9) and part a) of Lemma 3.3 with $w(x)=1,\ x\in\mathbb{R}^d$ yield

$$(f(x),1)_{L^2(\mathbb{R}^d)} = 0, \quad d=3,4. \tag{2.16}$$

Part (c) of [23, Lemma 6] implies that equations (1.2) and (1.3) with $a=0$ in dimensions $d=3,4$ admit unique solutions $u(x)\in H^2(\mathbb{R}^d)$ and $u_n(x)\in H^2(\mathbb{R}^d)$ respectively. Due to (1.9) and (2.16), we have $\widehat{f}_n(0)=0$ and $\widehat{f}(0)=0$. Hence we can write the first term in the right side of (2.1) as

$$\frac{\int_0^{|p|}\frac{\partial}{\partial|q|}[\widehat{f}_n(|q|,\omega)-\widehat{f}(|q|,\omega)]d|q|}{p^2}\chi_{\{p\in\mathbb{R}^d : |p|\le 1\}}.$$

By applying inequality (2.14) to the expression above we easily obtain the upper bound for it in the absolute value as

$$\frac{1}{(2\pi)^{d/2}}\||x|f_n - |x|f\|_{L^1(\mathbb{R}^d)}\frac{\chi_{\{p\in\mathbb{R}^d : |p|\le 1\}}}{|p|}$$

and in the $L^2(\mathbb{R}^d)$ norm as

$$\frac{1}{(2\pi)^{d/2}}\||x|f_n - |x|f\|_{L^1(\mathbb{R}^d)}\sqrt{\int_0^1 |S^d||p|^{d-3}d|p|} \to 0,\ n\to\infty,\ d=3,4$$

due to one of the assumptions of the theorem. By means of Lemma 3.2 we have

$$\lim_{n\to\infty}\|\xi_n^{d,\ 0}(p)\|_{L^2(\mathbb{R}^d)} = 0, \quad d=3,4.$$

Part (a) of Lemma 3.1 implies $u_n(x)\to u(x)$ in $H^2(\mathbb{R}^d)$, $d=3,4$ as $n\to\infty$.

(d) In dimensions $d\ge 5$ equations (1.2) and (1.3) with $a=0$ admit unique solutions $u(x)\in H^2(\mathbb{R}^d)$ and $u_n(x)\in H^2(\mathbb{R}^d)$ respectively by means of [23, Lemma 7]. No orthogonality conditions are required in this case. We have the following trivial inequality

$$|\widehat{f}_n(p)-\widehat{f}(p)| \le \frac{1}{(2\pi)^{d/2}}\|f_n - f\|_{L^1(\mathbb{R}^d)},\ p\in\mathbb{R}^d,$$

which yields the upper bound in the absolute value on the first term in the right side of (2.1) as

$$\frac{1}{(2\pi)^{d/2}}\|f_n - f\|_{L^1(\mathbb{R}^d)}\frac{\chi_{\{p\in\mathbb{R}^d : |p|\le 1\}}}{p^2},$$

such that we obtain the upper bound in the $L^2(\mathbb{R}^d)$ norm for it as

$$\frac{1}{(2\pi)^{d/2}}\|f_n - f\|_{L^1(\mathbb{R}^d)}\sqrt{\int_0^1 |S^d||p|^{d-5}d|p|} \to 0,\ n\to\infty,\ d\geq 5$$

due to one of the assumptions of the theorem. By means of Lemma 3.2 we have

$$\lim_{n\to\infty}\|\xi_n^{d,\ 0}(p)\|_{L^2(\mathbb{R}^d)} = 0,\ d\geq 5.$$

Part (a) of Lemma 3.1 yields $u_n(x)\to u(x)$ in $H^2(\mathbb{R}^d)$, $d\geq 5$ as $n\to\infty$. $\square$

Let us apply the generalized Fourier transform with respect to the functions of the continuous spectrum of the Schrödinger operator to both sides of equations (1.11) and (1.12), which yields

$$\tilde{u}(k) = \frac{\tilde{f}(k)}{k^2-a},\quad \tilde{u}_n(k) = \frac{\tilde{f}_n(k)}{k^2-a},\quad k\in\mathbb{R}^3,\ a\geq 0.$$

For $a=0$ we express the difference of the transforms above as

$$\tilde{u}_n(k)-\tilde{u}(k) = \frac{\tilde{f}_n(k)-\tilde{f}(k)}{k^2}\chi_{\{k\in\mathbb{R}^3:|k|\leq 1\}} + \frac{\tilde{f}_n(k)-\tilde{f}(k)}{k^2}\chi_{\{k\in\mathbb{R}^3:|k|>1\}}. \tag{2.17}$$

Let $\eta_n^0(k)$ stand for the second term in the right side of (2.17).

When $a>0$ we introduce the spherical layer in the space of three dimensions as

$$B_\sigma := \{k\in\mathbb{R}^3 : \sqrt{a}-\sigma\leq |k|\leq\sqrt{a}+\sigma\},\quad 0<\sigma<\sqrt{a},$$

which enables us to write

$$\tilde{u}_n(k)-\tilde{u}(k) = \frac{\tilde{f}_n(k)-\tilde{f}(k)}{k^2-a}\chi_{B_\sigma} + \frac{\tilde{f}_n(k)-\tilde{f}(k)}{k^2-a}\chi_{B_\sigma^c}. \tag{2.18}$$

The second term in the right side of (2.18) is being designated as $\eta_n^a(k)$.

Proof of Theorem 1.4. (a) Orthogonality conditions (1.16) along with [17, Corollary 2.2] and part (a) of Lemma 3.3 with $w(x)=\varphi_k(x)$, $k\in S^3_{\sqrt{a}}$ a.e. give us

$$(f(x),\varphi_k(x))_{L^2(\mathbb{R}^3)} = 0,\quad k\in S^3_{\sqrt{a}}\ \text{a.e.} \tag{2.19}$$

Then by means of [17, Theorem 1.2] equations (1.11) and (1.12) with a bounded potential function $V(x)$ and $a>0$ admit unique solutions $u(x)\in H^2(\mathbb{R}^3)$ and $u_n(x)\in H^2(\mathbb{R}^3)$ respectively. Via orthogonality relations (1.16) and (2.19) discussed above we have on $S^3_{\sqrt{a}}$ a.e.

$$\tilde{f}_n(\sqrt{a},\omega)=0,\quad \tilde{f}(\sqrt{a},\omega)=0,$$

which enables us to express the first term in the right side of (2.18) as

$$\frac{\int_{\sqrt{a}}^{|k|}\frac{\partial}{\partial|q|}[\tilde{f}_n(|q|,\omega)-\tilde{f}(|q|,\omega)]d|q|}{k^2-a}\chi_{B_\sigma}.$$

For the expression above we easily obtain the upper bound in the absolute value as

$$\|\nabla_q(\tilde{f}_n(q)-\tilde{f}(q))\|_{L^\infty(\mathbb{R}^3)}\frac{\chi_{B_\sigma}}{\sqrt{a}}$$

and in the $L^2(\mathbb{R}^3)$ norm as

$$\frac{\|\nabla_q(\tilde{f}_n(q)-\tilde{f}(q))\|_{L^\infty(\mathbb{R}^3)}}{\sqrt{a}}\sqrt{\frac{4\pi}{3}((\sqrt{a}+\sigma)^3-(\sqrt{a}-\sigma)^3)}\to 0,\quad n\to\infty$$

via Lemma 3.4. By means of Lemma 3.2 we have

$$\lim_{n\to\infty} \|\eta_n^a(k)\|_{L^2(\mathbb{R}^3)} = 0.$$

Part (b) of Lemma 3.1 yields $u_n(x) \to u(x)$ in $H^2(\mathbb{R}^3)$ as $n \to \infty$.

(b) Orthogonality relations (1.17), Corollary 2.2 of [17] and part (a) of Lemma 3.3 with $w(x) = \varphi_0(x)$ imply

$$(f(x), \varphi_0(x))_{L^2(\mathbb{R}^3)} = 0. \tag{2.20}$$

We deduce from part (b) of [17, Theorem 1.2] that equations (1.11) and (1.12) with $V(x)$ satisfying Assumption 1.3 and $a = 0$ possess unique solutions $u(x) \in H^2(\mathbb{R}^3)$ and $u_n(x) \in H^2(\mathbb{R}^3)$ respectively. Since orthogonality conditions (1.17) and (2.20) yield

$$\tilde{f}_n(0) = 0, \quad \tilde{f}(0) = 0,$$

we can express the first term in the right side of (2.17) as

$$\frac{\int_0^{|k|} \frac{\partial}{\partial |q|}[\tilde{f}_n(|q|,\omega) - \tilde{f}(|q|,\omega)]d|q|}{k^2} \chi_{\{k\in\mathbb{R}^3 : |k|\le 1\}}.$$

Obviously, for the quantity above there is an upper bound in the absolute value as

$$\|\nabla_q(\tilde{f}_n(q) - \tilde{f}(q))\|_{L^\infty(\mathbb{R}^3)} \frac{\chi_{\{k\in\mathbb{R}^3 : |k|\le 1\}}}{|k|}$$

and therefore, in the $L^2(\mathbb{R}^3)$ norm simply as

$$\sqrt{4\pi}\|\nabla_q(\tilde{f}_n(q) - \tilde{f}(q))\|_{L^\infty(\mathbb{R}^3)} \to 0, \quad n \to \infty$$

due to Lemma 3.4, Lemmas 3.2 yields

$$\lim_{n\to\infty} \|\eta_n^0(k)\|_{L^2(\mathbb{R}^3)} = 0.$$

Then by means of part (b) of Lemma 3.1 we arrive at $u_n(x) \to u(x)$ in $H^2(\mathbb{R}^3)$ as $n \to \infty$. $\square$

2.1. **Remarks.** Denote by F a space of functions which belong to $L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ and for which the norm

$$\|f\|_F = \|f\|_{L^2(\mathbb{R}^d)} + \||x|f\|_{L^1(\mathbb{R}^d)}$$

is bounded. A sequence $f_n \in F$ such that $f_n \to f$ in the norm of the space satisfies conditions of Theorems 1.1, 1.2, 1.4. Hence if we introduce a space E in such a way that the operator A acts from E into F, then its image is closed. The functionals in solvability conditions are linear bounded functionals over F.

The space E can be defined as a closure of infinitely differentiable functions with compact supports in the norm

$$\|u\|_E = \|u\|_{H^2(\mathbb{R}^d)} + \|Au\|_F.$$

The operator $A : E \to F$ is semi-Fredholm.

Similar construction can be considered in the case where $|x|^2 f \in L^1(\mathbb{R}^d)$ (Theorems 1.1, b and 1.2, b).

3. Auxiliary results

The following elementary lemma shows that to conclude the proofs of Theorems 1.1, 1.2 and 1.4 it is sufficient to show the convergence in L^2 of the solutions of the studied equations as $n \to \infty$.

Lemma 3.1. *(a) Let the conditions of Theorem 1.1 hold when $d = 1$, of Theorem 1.2 when $d \geq 2$, such that $u(x), u_n(x) \in H^2(\mathbb{R}^d)$ are the unique solutions of equations* (1.2) *and* (1.3) *respectively and $u_n(x) \to u(x)$ in $L^2(\mathbb{R}^d)$ as $n \to \infty$. Then $u_n(x) \to u(x)$ in $H^2(\mathbb{R}^d)$ as $n \to \infty$.*

(b) Let the conditions of Theorem 1.4 hold, such that $u(x), u_n(x) \in H^2(\mathbb{R}^3)$ are the unique solutions of equations (1.11) *and* (1.12) *respectively and $u_n(x) \to u(x)$ in $L^2(\mathbb{R}^3)$ as $n \to \infty$. Then $u_n(x) \to u(x)$ in $H^2(\mathbb{R}^3)$ as $n \to \infty$.*

Proof. (a) From equations (1.2) and (1.3) with $a \geq 0$ we easily deduce

$$\|\Delta u_n - \Delta u\|_{L^2(\mathbb{R}^d)} \leq a\|u_n - u\|_{L^2(\mathbb{R}^d)} + \|f_n - f\|_{L^2(\mathbb{R}^d)} \to 0, \quad n \to \infty.$$

By means of definition (1.4) we have $u_n(x) \to u(x)$ in $H^2(\mathbb{R}^d)$ as $n \to \infty$.

(b) From equations (1.11) and (1.12), for $a \geq 0$, we easily obtain

$$\|\Delta u_n - \Delta u\|_{L^2(\mathbb{R}^3)} \leq (a + \|V\|_{L^\infty(\mathbb{R}^3)})\|u_n - u\|_{L^2(\mathbb{R}^3)} + \|f_n - f\|_{L^2(\mathbb{R}^3)} \to 0, \ n \to \infty.$$

Therefore, definition (1.4) yields $u_n(x) \to u(x)$ in $H^2(\mathbb{R}^3)$ as $n \to \infty$. □

The auxiliary statement below will be helpful in establishing the convergence in L^2 of the solutions of the equations discussed above as $n \to \infty$.

Lemma 3.2. *Let $n \in \mathbb{N}$ and $f_n(x) \in L^2(\mathbb{R}^d)$, such that $f_n(x) \to f(x)$ in $L^2(\mathbb{R}^d)$ as $n \to \infty$. Then the expressions $\xi_n^{d,0}(p)$, $\xi_n^{1,a}(p)$, $\xi_n^{d,a}(p)$, $\eta_n^0(k)$, $\eta_n^a(k)$ defined in formulas* (2.1), (2.2), (2.3), (2.17) *and* (2.18) *respectively tend to zero in the corresponding $L^2(\mathbb{R}^d)$ norms as $n \to \infty$.*

Proof. Clearly, $|\xi_n^{d,\ 0}(p)| \leq |\widehat{f}_n(p) - \widehat{f}(p)|$, $p \in \mathbb{R}^d$, such that

$$\|\xi_n^{d,\ 0}(p)\|_{L^2(\mathbb{R}^d)} \leq \|f_n - f\|_{L^2(\mathbb{R}^d)} \to 0, \quad n \to \infty.$$

The definition of this expression yields $|\xi_n^{1,\ a}(p)| \leq \frac{|\widehat{f}_n(p) - \widehat{f}(p)|}{\delta^2}$, $p \in \mathbb{R}$. Hence

$$\|\xi_n^{1,\ a}(p)\|_{L^2(\mathbb{R})} \leq \frac{\|f_n - f\|_{L^2(\mathbb{R})}}{\delta^2} \to 0, \quad n \to \infty.$$

Finally, in the no potential case $|\xi_n^{d,a}(p)| \leq \frac{|\widehat{f}_n(p) - \widehat{f}(p)|}{\sqrt{a}\sigma}$, $p \in \mathbb{R}^d$, $d \geq 2$. Thus

$$\|\xi_n^{d,a}(p)\|_{L^2(\mathbb{R}^d)} \leq \frac{\|f_n - f\|_{L^2(\mathbb{R}^d)}}{\sqrt{a}\sigma} \to 0, \quad n \to \infty.$$

We easily estimate

$$|\eta_n^0(k)| \leq |\tilde{f}_n(k) - \tilde{f}(k)|, \quad k \in \mathbb{R}^3,$$

which implies

$$\|\eta_n^0(k)\|_{L^2(\mathbb{R}^3)} \leq \|f_n - f\|_{L^2(\mathbb{R}^3)} \to 0, \quad n \to \infty.$$

The trivial inequality

$$|\eta_n^a(k)| \leq \frac{|\tilde{f}_n(k) - \tilde{f}(k)|}{\sqrt{a}\sigma}, \ k \in \mathbb{R}^3$$

yields

$$\|\eta_n^a(k)\|_{L^2(\mathbb{R}^3)} \leq \frac{\|f_n - f\|_{L^2(\mathbb{R}^3)}}{\sqrt{a}\sigma} \to 0, \ n \to \infty,$$

which completes the proof of the lemma. □

The following lemma provides better information on the convergence as $n \to \infty$ of the right sides of the nonhomogeneous elliptic problems studied in the article.

Lemma 3.3. *Let $n \in \mathbb{N}$ and $f_n(x) \in L^2(\mathbb{R}^d)$, $d \in \mathbb{N}$, such that $f_n(x) \to f(x)$ in $L^2(\mathbb{R}^d)$ as $n \to \infty$.*

(a) If $|x|f_n(x) \in L^1(\mathbb{R}^d)$, such that $|x|f_n(x) \to |x|f(x)$ in $L^1(\mathbb{R}^d)$ as $n \to \infty$ then $f_n(x) \to f(x)$ in $L^1(\mathbb{R}^d)$ as $n \to \infty$. Moreover, if $(f_n(x), w(x))_{L^2(\mathbb{R}^d)} = 0$, $n \in \mathbb{N}$, with some $w(x) \in L^\infty(\mathbb{R}^d)$ then $(f(x), w(x))_{L^2(\mathbb{R}^d)} = 0$ as well.

(b) If $|x|^2 f_n(x) \in L^1(\mathbb{R}^d)$, such that $|x|^2 f_n(x) \to |x|^2 f(x)$ in $L^1(\mathbb{R}^d)$ as $n \to \infty$ then $|x|f_n(x) \to |x|f(x)$ in $L^1(\mathbb{R}^d)$ and $f_n(x) \to f(x)$ in $L^1(\mathbb{R}^d)$ as $n \to \infty$. Moreover, if $(f_n(x), 1)_{L^2(\mathbb{R}^d)} = 0$ and $(f_n(x), x_k)_{L^2(\mathbb{R}^d)} = 0$ for $n \in \mathbb{N}$ and $k = 1, ..., d$ then $(f(x), 1)_{L^2(\mathbb{R}^d)} = 0$ and $(f(x), x_k)_{L^2(\mathbb{R}^d)} = 0$ for $k = 1, \dots, d$ as well.

Proof. (a) Note that $f_n(x) \in L^1(\mathbb{R}^d)$, $n \in \mathbb{N}$ via the trivial argument analogous to the one of Fact 1 of [17]. We easily estimate the norm using the Schwarz inequality as

$$\begin{aligned} \|f_n - f\|_{L^1(\mathbb{R}^d)} &\leq \sqrt{\int_{|x|\leq 1} |f_n - f|^2 dx} \sqrt{\int_{|x|\leq 1} dx} + \int_{|x|>1} |x||f_n - f| dx \\ &\leq \|f_n - f\|_{L^2(\mathbb{R}^d)} \sqrt{|B^d|} + \||x|f_n - |x|f\|_{L^1(\mathbb{R}^d)} \to 0, \quad n \to \infty. \end{aligned}$$

Then for $w(x)$, which is bounded by one of the assumptions of the lemma, we obtain

$$\begin{aligned} |(f(x), w(x))_{L^2(\mathbb{R}^d)}| &= |(f(x) - f_n(x), w(x))_{L^2(\mathbb{R}^d)}| \\ &\leq \|f_n - f\|_{L^1(\mathbb{R}^d)} \|w\|_{L^\infty(\mathbb{R}^d)} \to 0, \quad n \to \infty, \end{aligned}$$

which completes the proof of part (a) of the lemma.

(b) By means of the argument, which relies on the Schwarz inequality and the assumptions that $f_n(x) \in L^2(\mathbb{R}^d)$ and $|x|^2 f_n(x) \in L^1(\mathbb{R}^d)$, we easily obtain

$$|x|f_n(x) \in L^1(\mathbb{R}^d), \quad n \in \mathbb{N}.$$

Let us apply the Schwarz inequality again to arrive at the bound

$$\begin{aligned} \||x|f_n - |x|f\|_{L^1(\mathbb{R}^d)} &\leq \int_{|x|\leq 1} |f_n - f| dx + \int_{|x|>1} |x|^2 |f_n - f| dx \\ &\leq \|f_n - f\|_{L^2(\mathbb{R}^d)} \sqrt{|B^d|} + \||x|^2 f_n - |x|^2 f\|_{L^1(\mathbb{R}^d)} \to 0, \end{aligned}$$

as $n \to \infty$. Hence $f_n(x) \to f(x)$ in $L^1(\mathbb{R}^d)$ as $n \to \infty$ and $(f(x), 1)_{L^2(\mathbb{R}^d)} = 0$ according to the argument above of part (a) of the lemma. Here $w(x) = 1$, $x \in \mathbb{R}^d$. Finally, for $k = 1, \dots, d$ we arrive at

$$|(f(x), x_k)_{L^2(\mathbb{R}^d)}| = |(f(x) - f_n(x), x_k)_{L^2(\mathbb{R}^d)}| \leq \||x|f_n - |x|f\|_{L^1(\mathbb{R}^d)} \to 0$$

as $n \to \infty$, which completes the proof of part (b) of the lemma. □

The $L^\infty(\mathbb{R}^3)$ norm studied in the lemma below is finite due to [17, Lemma 2.4]. We go further by proving that it tends to zero.

Lemma 3.4. *Let the conditions of Theorem 1.4 hold. Then we have*

$$\|\nabla_k(\tilde{f}_n(k) - \tilde{f}(k))\|_{L^\infty(\mathbb{R}^3)} \to 0, \quad n \to \infty.$$

Proof. Clearly, we need to estimate the quantity

$$\nabla_k(\tilde{f}_n(k) - \tilde{f}(k)) = (f_n(x) - f(x), \nabla_k \varphi_k(x))_{L^2(\mathbb{R}^3)}. \tag{3.1}$$

It easily follows from the Lippmann-Schwinger equation (1.14) that

$$\nabla_k \varphi_k(x) = \frac{e^{ikx}}{(2\pi)^{3/2}} ix + (I-Q)^{-1} Q \frac{e^{ikx}}{(2\pi)^{3/2}} ix + (I-Q)^{-1}(\nabla_k Q)(I-Q)^{-1} \frac{e^{ikx}}{(2\pi)^{3/2}}.$$

Here the operator $\nabla_k Q : L^\infty(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3; \mathbb{C}^3)$ possesses the integral kernel

$$\nabla_k Q(x, y, k) = -\frac{i}{4\pi} e^{i|k||x-y|} \frac{k}{|k|} V(y).$$

Evidently, for the operator norm

$$\|\nabla_k Q\|_\infty \le \frac{1}{4\pi} \|V\|_{L^1(\mathbb{R}^3)} < \infty \tag{3.2}$$

due to the rate of decay of the potential function $V(x)$ stated in Assumption 1.3. Therefore, in order to prove the convergence to zero as $n \to \infty$ of the $L^\infty(\mathbb{R}^3)$ norm of expression (3.1), we need to estimate the three terms defined below. The first one is given by

$$R_1^n(k) := \Big(f_n(x) - f(x), \frac{e^{ikx}}{(2\pi)^{3/2}} ix\Big)_{L^2(\mathbb{R}^3)}, \quad k \in \mathbb{R}^3.$$

We easily arrive at

$$\|R_1^n(k)\|_{L^\infty(\mathbb{R}^3)} \le \frac{1}{(2\pi)^{3/2}} \||x|f_n - |x|f\|_{L^1(\mathbb{R}^3)} \to 0, \quad n \to \infty$$

according to one of our assumptions. The second term which we need to estimate is

$$R_2^n(k) := \Big(f_n(x) - f(x), (I-Q)^{-1} Q \frac{e^{ikx}}{(2\pi)^{3/2}} ix\Big)_{L^2(\mathbb{R}^3)}, \quad k \in \mathbb{R}^3.$$

Let us use the upper bound

$$\|R_2^n(k)\|_{L^\infty(\mathbb{R}^3)} \le \frac{1}{(2\pi)^{3/2}} \frac{1}{1 - \|Q\|_\infty} \|Q e^{ikx} x\|_{L^\infty(\mathbb{R}^3)} \|f_n - f\|_{L^1(\mathbb{R}^3)}.$$

In the proof of [17, Lemma 2.4] it was established that the norm $\|Qe^{ikx}x\|_{L^\infty(\mathbb{R}^3)}$ is bounded above by a finite quantity independent of k. According to the part (a) of Lemma 3.3 when $n \to \infty$, $f_n \to f$ in $L^1(\mathbb{R}^3)$. Therefore,

$$\|R_2^n(k)\|_{L^\infty(\mathbb{R}^3)} \to 0, \quad n \to \infty.$$

Finally, it remains to estimate the expression

$$R_3^n(k) := \Big(f_n(x) - f(x), (I-Q)^{-1}(\nabla_k Q)(I-Q)^{-1} \frac{e^{ikx}}{(2\pi)^{3/2}}\Big)_{L^2(\mathbb{R}^3)}, \quad k \in \mathbb{R}^3.$$

Using (3.2), we easily deduce the inequality

$$\|R_3^n(k)\|_{L^\infty(\mathbb{R}^3)} \le \frac{1}{4\pi(2\pi)^{3/2}} \frac{\|V\|_{L^1(\mathbb{R}^3)}}{(1 - \|Q\|_\infty)^2} \|f_n - f\|_{L^1(\mathbb{R}^3)} \to 0, \quad n \to \infty$$

via the statement of the part (a) of Lemma 3.3. $\square$

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