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ALMOST PERIODIC SOLUTIONS OF DISTRIBUTED PARAMETER BIOCHEMICAL SYSTEMS WITH TIME DELAY AND TIME VARYING INPUT

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Abstract. In this article we study the existence of almost periodic solutions for distributed parameters biochemical system, with time delay when the input S_{in} is time dependent. This study is motivated by the input begin time dependent in many applications, and by the importance of almost periodically varying environments. Using the semigroup method, we prove that if the input is almost periodic then the system has an almost periodic solution.

1. Introduction

The article is concerned with the existence of almost periodic solutions of a distributed parameters biochemical system, with time delay in the growth response when the input is time dependent ($S_{\text{in}} = S_{\text{in}}(t)$) and is an Almost periodic function of the time t. The study of the existence of periodic and almost periodic solutions is important in the theory of bioengineering systems because of the periodically and almost periodically time varying environments. In the previous decades, many authors have introduced various types of delay into the dynamical models of bioengineering systems to understand the oscillations (periodic solutions) observed experimentally on chemostat systems (see references in [6]). However, most of these studies were focused on time varying systems and few consider space varying cases. Recently, Drame et al [6] studied a distributed parameters biochemical system with time delay in the growth response. They proved the existence of periodic solutions (oscillations) for large values of the delay parameter. In [6], the authors assumed the inlet substrate concentration to be constant. However, in applications, this quantity may be time dependent and periodic or almost periodic in time. In this paper, we consider a distributed parameter biochemical system with time delay and almost periodic time varying inlet substrate concentration. The basis of the model under study is derived from the work performed on anaerobic digestion in the pilot fixed bed reactor of the LBE-INRA in Narbonne (France) and is inspired from the dynamical models built and validated on the process (see [1, 14]). Using semigroup approach, we prove the existence of almost periodic solutions for the system. We

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consider the dynamical system

$$\begin{aligned} \frac{\partial S(t)}{\partial t} &= d\frac{\partial^2 S(t)}{\partial x^2} - q\frac{\partial S(t)}{\partial x} - k\mu(S(t), X(t))X(t) \\ \frac{\partial X(t)}{\partial t} &= -k_d X(t) + \mu(S(t-r), X(t-r))X(t-r), \end{aligned} \tag{1.1}$$

with the boundary conditions: for all $t \geq 0$,

$$d\frac{\partial S}{\partial x}(t,0) - qS(t,0) + qS_{\text{in}}(t) = 0 \quad \text{and} \quad \frac{\partial S}{\partial x}(t,L) = 0, \tag{1.2}$$

and initial condition: for all $-r \leq s \leq 0$, $0 \leq x \leq L$,

$$S(s,x) = \Phi_1(s,x), \quad X(s,x) = \Phi_2(s,x) \tag{1.3}$$

where the initial data Φ_1 and Φ_2 are continuous functions on $[-\tau, 0] \times [0, L]$.

The parameters $d, q, k, k_d, S_{\text{in}}, \mu, r, L$ are positive and represent the diffusion coefficient, the superficial fluid velocity, the yield coefficient, the death rate of the biomass, the inlet limiting substrate concentration, the specific growth function or growth response, the delay parameter and the length of the reactor, respectively. In the rest of the paper, we will assume that the length L is 1.

In the right hand side of the first equation of (1.1), the last term represents the growth response while the other terms represent the hydrodynamics (i.e. diffusion and convection terms). As the biomass is assumed to be fixed, there is no hydrodynamic term in the right hand side of the second equation of (1.1). The first term in right hand side of this equation represents the mortality of biomass and the last one represents the growth response with delay. The delay is considered in the second equation of (1.1) only as it is assumed to be caused by memory effects of micro-organisms, on one hand. On the other hand, the substrate is apparently instantaneously consumed although the biomass growth takes place with some delay. This can be explained e.g. by the absorption of the (liquid) substrate into the (solid) biomass, a phenomenon that might be fast with respect to the conversion of substrate into biomass, at least fast enough to emphasize the disappearance of the substrate from the liquid medium before its conversion into biomass exhibits its effects. The resulting dynamical model is a system of partial functional differential equations with almost periodic boundary conditions. One of the most attracting areas of qualitative theory of partial functional differential equations is the existence of periodic and almost periodic solutions due to the important roles of periodically and almost periodically varying environments play in many biological and ecological dynamical systems. Compared with periodic effects, almost periodic effects are more frequent [11]. The existence of almost periodic (or pseudo-, weighted almost periodic) solutions of partial differential equations has been increasingly studied by many authors (see e.g. [3, 4, 13] and references therein). In these studies, the semigroup or evolution family is compact and exponentially stable, and therefore cannot be applied to the system under consideration here. As mentioned above, the current work is motivated, on one hand, by a recent paper by Drame et al [6], where the authors considered existence question of periodic solutions of biochemical distributed parameters system with time delay. On the other hand, in applications the inlet concentration of substrate in biological processes are time varying and may be periodic or almost periodic function of time.

The fundamental assertion we prove in this paper is that if the input $S_{\rm in}(t)$ is an almost periodic function of t then the system (1.1)-(1.3) has an almost periodic. So, we make the following assumption:

Assumption A1. The input $S_{\rm in}(t)$ is an Almost Periodic function of t, that is: for any $\varepsilon > 0$, there exists $l(\varepsilon) > 0$ such that any interval of length $l(\varepsilon)$ contains a number τ such that

$$|S_{\rm in}(t+\tau) - S_{\rm in}(t)| < \varepsilon \quad \text{for all } t \in \mathbb{R}.$$

To guaranty the existence and some regularity of solution of the system (1.1)-(1.3), we make the assumption that:

Assumption A2. The function μ is smooth (to say of class C^2, for example) and bounded on $(0,1) \times (0,\infty)$.

In the next section, we give some preliminary properties of the system (1.1)-(1.3) and recall some definitions and properties of Almost periodic functions.

2. Preliminaries

Let us consider the following state spaces $Z_1 = Z_2 = C[0,1]$, $Z = Z_1 \times Z_2$, $\mathcal{C} = C([-r,0];Z)$ and the positive cones

$$Z^+ = \{v \in Z : v_i(z) \geq 0,\, \forall z \in [0,1],\, i = 1,2\},$$
$$\mathcal{C}^+ = \{\varphi \in \mathcal{C} : \varphi(s) \in Z^+,\, \forall -r \leq s \leq 0\}.$$

We also adopt the following notation: for a continuous function $w : [-r, b) \to Z$, $b > 0$, we define $w_t \in \mathcal{C}$, $t \in [0, b)$, by $w_t(s) = w(t+s)$ for all $-r \leq s \leq 0$. Therefore, it is not difficult to see that the map $t \to w_t$ is continuous from $[0,b)$ into $\mathcal{C}$.

Definition 2.1 ([2, p. 55])**.** *A function $F(t,x)$ is called almost periodic in t, uniformly with respect to $x \in Z$, if for any $\varepsilon > 0$ there exists a number $l(\varepsilon)$ such that any interval of the real line of length $l(\varepsilon)$ contains at least one number τ such that*

$$|F(t+\tau,x) - F(t,x)| < \varepsilon \quad \text{for all } t \in \mathbb{R},\ x \in Z.$$

Theorem 2.2 ([2, Theorem 2.6])**.** *A necessary and sufficient condition for a function $F(t,x)$ to be almost periodic in t, uniformly with respect to $x \in \Omega$, where Ω is a bounded and closed set, is that for any sequence $\{F(t+\tau_n,x)\}$, we can extract a subsequence $\{F(t+r_n,x)\}$ that satisfies the Cauchy uniform convergence.*

Let us consider the following auxiliary problem from (1.1)-(1.2) (recall $L = 1$):

$$\begin{gathered} \frac{\partial w(t)}{\partial t} = d\frac{\partial^2 w(t)}{\partial x^2} - q\frac{\partial w(t)}{\partial x}, \\ d\frac{\partial w}{\partial x}(t,0) = qw(t,0) - qS_{\rm in}(t), \quad \frac{\partial w}{\partial x}(t,1) = 0. \end{gathered} \tag{2.1}$$

In section 3, we will show that under assumption A1, the auxiliary problem (2.1) has an almost periodic solution. Observe that if w is an almost periodic solution of (2.1) then the function f, defined by: for any $u \in \mathcal{C}$,

$$(f_1(t,u))(x) = -k\mu(u_1(0,x) + w(t,x), u_2(0,x))u_2(0,x)$$
$$(f_2(t,u))(x) = \mu(u_1(-r,x) + w(t,x), u_2(-r,x))u_2(-r,x)$$

and $f = (f_1, f_2)$, is almost periodic in t uniformly with respect to u. Now, let $w(t,x)$ be a solution of (2.1) and let us introduce

$$u_1(t,x) = S(t,x) - w(t,x) \text{ and } u_2(t,x) = X(t,x) \quad \text{for all } t \geq 0 \text{ and } 0 \leq x \leq 1,$$

where S and X are as in (1.1)-(1.3). Then $u = (u_1, u_2)$ satisfies the equations

$$\begin{gathered}
\frac{\partial u_1(t)}{\partial t} = d\frac{\partial^2 u_1(t)}{\partial x^2} - q\frac{\partial u_1(t)}{\partial x} + f_1(t, u_{1t}, u_{2t}) \\
\frac{\partial u_2(t)}{\partial t} = -k_d u_2(t) + f_2(t, u_{1t}, u_{2t}), \\
d\frac{\partial u_1}{\partial x}(t,0) = qu_1(t,0), \quad \frac{\partial u_1}{\partial x}(t,1) = 0.
\end{gathered} \tag{2.2}$$

Let us define the operators

$$\begin{gathered}
D(A_1) = \{u_1 \in C^2[0,1] : d\frac{\partial u_1}{\partial x}(0) = qu_1(0),\ \frac{\partial u_1}{\partial x}(1) = 0\}; \\
A_1 u_1 = d\frac{\partial^2 u_1}{\partial x^2} - q\frac{\partial u_1}{\partial x}; \\
D(A_2) = C[0,1], \quad A_2 = -k_d I; \\
D(A) = D(A_1) \otimes D(A_2), \quad A = \operatorname{diag}(A_1, A_2).
\end{gathered}$$

By the same arguments as in [5], we can show that the operator A is the infinitesimal generator of a C_0-semigroup of bounded linear operators $T(t)$ on Z, given by $T(t) = \operatorname{diag}(T_1(t), T_2(t))$, where $T_1(t)$ and $T_2(t)$ are the C_0-semigroups generated by A_1 and A_2, respectively. Moreover, the semigroup $T_1(t)$ is compact in $C^1[0,1]$ and $T(t)$ is analytic. The system (2.2) can be written as the following abstract Cauchy problem

$$\begin{gathered}
\frac{du(t)}{dt} = Au(t) + f(t, u_t), \\
u(s) = u_0(s), \quad \text{for } -r \leq s \leq 0,
\end{gathered} \tag{2.3}$$

where, as in (1.3), the initial data u_0 is in $C([-\tau, 0], Z)$.

Definition 2.3.

- A mild solution of (2.3) (equivalently of (1.1)-(1.3)) is a continuous function $u : [-\tau, t_u) \to Z$, with $t_u > 0$, satisfying
$$\begin{gathered}
u(t) = T(t)\theta(0) + \int_0^t T(t-s) f(s, u_s) ds, \quad 0 \leq t \leq t_u \\
u_0 = \theta \in \mathcal{C},
\end{gathered}$$
where $u_s \in \mathcal{C}$ is given by $u_s(\sigma) = u(s+\sigma)$, $-\tau \leq \sigma \leq 0$.
- A function $u \in C([-\tau, t_u), Z) \cap C^1((0, t_u), Z)$ satisfying $u(t) \in D(A)$, for $0 < t < t_u$, and $u(t)$ satisfies (2.3) is called a classical solution.

3. The auxiliary problem and existence of solutions for (2.3)

As we mentioned earlier, in this section we prove that under assumption A1 the auxiliary problem (2.1) has an almost periodic solution, by using the method of sub-super solutions. Later, we will also discuss the existence of solutions of (2.3).

Since $S_{\rm in}(t)$ is almost periodic, by [2, Theorem 1.2], $S_{\rm in}(t)$ is continuous and bounded on the real line. Let $\underline{S} \geq 0$ and $\bar{S} \geq 0$ be such that

$$\underline{S} \leq S_{\rm in}(t) \leq \frac{1}{2}\bar{S}, \quad \text{for all } t \in \mathbb{R},$$

and let $S_* = \frac{q}{2d}\underline{S}$, and $S^* = \frac{q}{2d}\bar{S}$. To prove the following remark, we assume that:
Assumption A3: $1 + q \leq \frac{d}{2}$.

Remark 3.1. Assume that A3 holds. Then there exists a function g such that for all $t \geq 0$

$$0 \leq g(t) \leq A = \min(S_*, \frac{2q^2\underline{S}}{4d + qd}), \quad 0 \leq g'(t) \leq 8dg(t) - 4qS_*, \quad t \geq 0.$$

Proof. Let $y(t) = \frac{At}{1+t}$, for $t \geq 0$. We have $y(t) \leq A$, for $t \geq 0$. In addition, for $t \geq 1$, $y(t) \geq \frac{A}{2}$. On the other hand, under assumption **A3**, we have $A \geq \frac{2q}{d}S_*$. Therefore, for $t \geq 1$, we obtain

$$8dy(t) - 4qS_* \geq 8dy(t) - 2dA \geq 8dy(t) - 4dy(t) = 4dy(t) \geq 2dA.$$

Also, $y'(t) = \frac{A}{(1+t)^2} \leq 2dA$ for all $t \geq -1 + \frac{1}{\sqrt{2d}}$. Taking $t_0 = \max(1, -1 + \frac{1}{\sqrt{2d}})$, we obtain $0 \leq y'(t) \leq 8dy(t) - 4qS_*$ for all $t \geq t_0$. Finally, define the function g to be $g(t) = y(t + t_0)$ for all $t \geq 0$. □

Lemma 3.2. *Assume* A1 *and* A3 *hold. Then* (2.1) *has an almost periodic solution, $w(t,x)$, in the sense of definition 2.1.*

Proof. By [7, Theorem 22.3], equation (2.1) will have a stable periodic solution, which in turn will be almost periodic, if it has a properly ordered pair of strict sub- and super solutions. Let

$$\varphi(t,x) = g(t)(\frac{1}{2} - x)^2 + S_*(-1 - x), \quad \psi(t,x) = -g(t)(\frac{1}{2} - x)^2 + S_*(1 + x),$$

for $t \geq 0$, $0 \leq x \leq 1$, where g is as given in Remark 3.1. We will show that φ and ψ are properly ordered pair of strict sub- and super solutions of the equation (2.1).

• For φ: The scripts φ_t, φ_x, and φ_{xx} represent the first and second partial derivatives of φ with respect to t and x, respectively. We have

$$\begin{gathered} \varphi_t(t,x) = g'(t)(\frac{1}{2} - x)^2 \leq \frac{1}{4}g'(t), \\ \varphi_x(t,x) = -2g(t)(\frac{1}{2} - x) - S_*, \\ \varphi_{xx}(t,x) = 2g(t). \end{gathered}$$

(i) If $g'(t) \leq 8dg(t) - 4qS_*$ as in Remark 3.1, then $\frac{1}{4}g'(t) \leq 2dg(t) + 2qg(t)(\frac{1}{2} - x) + qS_*$ and therefore,

$$\varphi_t(t,x) \leq d\varphi_{xx} - q\varphi_x.$$

(ii) $d\varphi_x(t,0) - q\varphi(t,0) = -dg(t) - dS_* - \frac{q}{4}g(t) + qS_* = -(d + \frac{q}{4})g(t) - \frac{q}{2}\underline{S} + \frac{q^2}{2d}\underline{S}$. Therefore, if $0 \leq g(t) \leq A$ as in Remark 3.1, then

$$d\varphi_x(t,0) - q\varphi(t,0) \geq -\frac{q}{2}\underline{S} \geq -qS_{\rm in}(t).$$

(iii) $\varphi_x(t,1) = g(t) - S_* < 0$ since g satisfies the conditions in Remark 3.1.

Combining (i), (ii), and (iii), it follows that φ is a strict sub-solution of (2.1).

• For ψ. Similarly, the scripts ψ_t, ψ_x, and ψ_{xx} represent the first and second partial derivatives of ψ with respect to t and x, respectively. We have

$$\psi_t(t,x) = -g'(t)(\frac{1}{2} - x)^2 \geq -\frac{1}{4}g'(t),$$
$$\psi_x(t,x) = 2g(t)(\frac{1}{2} - x) + S^*,$$
$$\psi_{xx}(t,x) = -2g(t).$$

(i) If $g'(t) \leq 8dg(t) - 4qS_*$ as in Remark 3.1, then $-\frac{1}{4}g'(t) \geq -2dg(t) - 2qg(t)(\frac{1}{2} - x) + qS^*$ and therefore,

$$\psi_t(t,x) \geq d\psi_{xx} - q\psi_x.$$

(ii) $d\psi_x(t,0) - q\psi(t,0) = dg(t) - dS^* + \frac{q}{4}g(t) - qS^* = (d + \frac{q}{4})g(t) - \frac{q}{2}\bar{S} - \frac{q^2}{2d}\bar{S}$. Therefore, if $0 \leq g(t) \leq A$ as in Remark 3.1, then

$$d\psi_x(t,0) - q\psi(t,0) \leq -\frac{q}{2}\bar{S} \leq -qS_{\rm in}(t).$$

(iii) $\psi_x(t,1) = -g(t) + S^* > 0$ since g satisfies the conditions in Remark 3.1.

Combining (i), (ii), and (iii), it follows that ψ is a strict super-solution of (2.1).

Now, let us show that $\varphi(t,x) \leq \psi(t,x)$ for all $t \geq 0$ and $0 \leq x \leq 1$. Using Remark 3.1,

$$\psi(t,x) - \varphi(t,x) = -2g(t)(\frac{1}{2} - x)^2 + (S^* + S_*)(1 + x) \geq (S^* + S_*)(1 + x - (\frac{1}{2} - x)^2) \geq 0,$$

for all $t \geq 0$ and $0 \leq x \leq 1$.

Then, φ and ψ are properly ordered pair of strict sub- and super solutions of (2.1). Therefore, by [7, Theorem 22.3], the equation (2.1) has a stable periodic solution, $w(t,x)$, which in turn is almost periodic. □

From assumption A2, the function $f : \mathcal{C} \to Z$ is continuously differentiable. Also, the semigroup $T(t)$ is a C_0-semigroup on Z. Then, by the usual existence and regularity theorem (see [6, Theorem 3,1], [8, Theorem 1], [10, Theorem 1.5, p. 187]), we have the following theorem.

Theorem 3.3. *Assume* A2 *holds. For any $\theta \in \mathcal{C}$, system* (2.3) *(and equivalently* (1.1)-(1.3)*) has a unique mild solution $u(t)$ with initial condition θ. Moreover, $u(t)$ is a classical solution of* (2.3) *for all $t > 0$. Also, if we denote by $\mathbb{T}(t)\theta = u(t,\theta)$ the solution of* (2.3)*, then $\mathbb{T}(t)$ is a nonlinear C_0-semigroup on Z.*

4. Main Result

In this section, we prove our main result on the existence of almost periodic solutions of the system (2.2) which is equivalent to the distributed parameters biochemical system (1.1)-(1.3). Let us first prove the following lemma.

Lemma 4.1. *Under assumption* A2*, the solutions of* (2.2) *are bounded.*

Proof. Let $u(t) = (u_1(t), u_2(t))$ be a solution of (2.2). Let us start with the second component $u_2(t)$. In integral form, we have

$$u_{2t}(.) = T_2(t)\theta_2(.) + \int_0^t T_2(t-s)f_2(s,u_s)ds.$$

Therefore,

$$|u_{2t}(.)| \leq |\theta_2(.)| + N \int_0^t e^{-\gamma(t-s)} |u_{2s}| ds,$$

where N is an upper bound of the function μ. Recall that

$$(f_2(t,u))(x) = \mu(u_1(-r,x) + w(t,x), u_2(-r,x))u_2(-r,x).$$

Applying the Gronwall inequality, we obtain

$$|u_{2t}(.)| \leq |\theta_2(.)| \exp\Big(N \int_0^t e^{-\gamma(t-s)} ds\Big) \leq |\theta_2(.)| \exp\big(\frac{N}{\gamma}\big).$$

Therefore, u_2 is bounded in Z_2. Now, observe that $(f_1(t,u))(x) = -k\mu(u_1(0,x) + w(t,x), u_2(0,x))u_2(0,x)$. Since the function μ is bounded and we just proved that u_2 is also bounded, then $f_1(s, u_s)$ is also bounded. Using the integral form

$$u_1(t) = T_1(t)\theta_1(0) + \int_0^t T_1(t-s) f_1(s, u_s) ds$$

of u_1, we deduce from [10, p.236, Lemma 2.4] that u_1 has a compact closure in Z_1. Therefore, u_1 is bounded in Z_1. □

Now, we can state and prove our main result.

Theorem 4.2. *Assume* A1–A3 *hold. Then, system* (2.2) *has an almost periodic solution,* $u = (u_1, u_2)$*, in the sense of definition 2.1.*

Proof. Observe that the function f in the system (2.2) is almost periodic in t, uniformly with respect to u, since the function $w(t,x)$ in the expression of f is an almost periodic solution of (2.1).

Let $u(t) = (u_1(t), u_2(t))$ be a bounded solution of (2.2) corresponding to the almost periodic solution $w(t,x)$ of (2.1), with initial condition $u(s) = (\theta_1(s), \theta_2(s))$, $r \leq s \leq 0$. In integral form, we have

$$u_1(t) = T_1(t)\theta_1(0) + \int_0^t T_1(t-s) f_1(s, u_s) ds,$$
$$u_2(t) = T_2(t)\theta_2(0) + \int_0^t T_2(t-s) f_2(s, u_s) ds.$$

Since the semigroup $T_1(t)$ is compact and f_1 is bounded, then we can deduce from [10, p. 236, Lemma 2.4] that $u_1(t)$ has a compact closure in Z_1. Therefore, for any sequence $u_1(t + \tau_n)$, we can extract a subsequence $u_1(t + r_n)$ that satisfies the Cauchy uniform convergence in Z_1.

Now, let us consider the component $u_2(t)$ of the solution. Let $u_2(t + \tau_n)$ be a sequence. Since $f_2(t,u)$ is almost periodic in t, uniformly in u, by Theorem 2.2, we can extract a subsequence $f(t + r_n, u)$ which satisfies the Cauchy uniform convergence. In integral form, we have

$$u_{2t}(.) = T_2(t)\theta_2(.) + \int_0^t T_2(t-s) f_2(s, u_s) ds.$$

Let i, j be positive integers. We have

$$|u_{2(t+i)} - u_{2(t+j)}| \leq |T_2(t+i) - T_2(t+j)||\theta_2(.)| + |\int_0^{t+i} T_2(t+i-s) f_2(s, u_s) ds$$

$$- \int_0^{t+j} T_2(t+j-s) f_2(s,u_s)ds|$$

Introducing the change of variables in the integrals, we obtain:

$$\begin{aligned}
&|u_{2(t+i)} - u_{2(t+j)}| \\
&\le |e^{-\gamma(t+i)} - e^{-\gamma(t+j)}||\theta_2(.)| + \int_0^t T_2(t-s)|f_2(s+i,u_{s+i}) - f_2(s+j,u_{s+j})|ds \\
&\quad + |-\int_{-i}^0 T_2(t-s) f_2(s+i,u_{s+i})ds + \int_{-j}^0 T_2(t-s) f_2(s+j,u_{s+j})ds| \\
&\le |e^{-\gamma(t+i)} - e^{-\gamma(t+j)}||\theta_2(.)| + \int_0^t T_2(t-s)|f_2(s+i,u_{s+i}) - f_2(s+i,u_{s+j})|ds \\
&\quad + \int_0^t T_2(t-s)|f_2(s+i,u_{s+j}) - f_2(s+j,u_{s+j})|ds \\
&\quad + |-\int_{-i}^0 T_2(t-s) f_2(s+i,u_{s+i})ds + \int_{-j}^0 T_2(t-s) f_2(s+j,u_{s+j})ds| \\
&\le |e^{-\gamma(t+i)} - e^{-\gamma(t+j)}||\theta_2(.)| + L\int_0^t e^{-\gamma(t-s)}|u_{1(s+i)}) - u_{1(s+j)})|ds \\
&\quad + L\int_0^t e^{-\gamma(t-s)}|u_{2(s+i)}) - u_{2(s+j)})|ds \\
&\quad + \int_0^t e^{-\gamma(t-s)}|f_2(s+i,u_{s+j}) - f_2(s+j,u_{s+j})|ds \\
&\quad + |-\int_{-i}^0 T_2(t-s) f_2(s+i,u_{s+i})ds + \int_{-j}^0 T_2(t-s) f_2(s+j,u_{s+j})ds|,
\end{aligned}$$

where L is the Lipschitz constant of of the function f_2. Observe that

(i) For any $\varepsilon > 0$, there exists $l_1(\varepsilon) > 0$, such that if $i,\ j \ge l_1(\varepsilon)$ then $|e^{-\gamma(t+i)} - e^{-\gamma(t+j)}| < \varepsilon$.

(ii) Since $u_1(t)$ has a compact closure, then: for any $\varepsilon > 0$, there exists $l_2(\varepsilon) > 0$, such that if $i,j \ge l_2(\varepsilon)$, then $|u_{1(t+i)} - u_{1(t+j)}| < \varepsilon$. Therefore,

$$L\int_0^t e^{-\gamma(t-s)}|u_{1(s+i)} - u_{1(s+j)}|ds \le L\varepsilon \int_0^t e^{-\gamma(t-s)}ds = \frac{L\varepsilon}{\gamma}(1-e^{-\gamma t}) < \frac{L\varepsilon}{\gamma}.$$

(iii) By Theorem 2.2: for any $\varepsilon > 0$, there exists $l_3(\varepsilon) > 0$, such that if $i,j \ge l_3(\varepsilon)$ then

$$|f_2(s+i,u_{s+j}) - f_2(s+j,u_{s+j})| < \varepsilon.$$

Therefore,

$$\int_0^t e^{-\gamma(t-s)}|f_2(s+i,u_{s+j}) - f_2(s+j,u_{s+j})|ds \le \varepsilon \int_0^t e^{-\gamma(t-s)}ds \le \frac{\varepsilon}{\gamma}.$$

(iv) Finally, for the last integral we have:

$$\begin{aligned}
I &= |-\int_{-i}^0 T_2(t-s) f_2(s+i,u_{s+i})ds + \int_{-j}^0 T_2(t-s) f_2(s+j,u_{s+j})ds| \\
&\le \int_{-i}^0 T_2(t-s)|f_2(s+j,u_{s+j}) - f_2(s+i,u_{s+i})|ds
\end{aligned}$$

$$
\begin{aligned}
&+ \int_{-j}^{-i} T_2(t-s)|f_2(s+j,u_{s+j})|ds \\
&\le \int_{-i}^{0} T_2(t-s)|f_2(s+j,u_{s+j}) - f_2(s+j,u_{s+i})|ds \\
&\quad + \int_{-i}^{0} T_2(t-s)|f_2(s+j,u_{s+i}) - f_2(s+i,u_{s+i})|ds + N_2 \int_{-j}^{-i} e^{-\gamma(t-s)}ds,
\end{aligned}
$$

where N_2 is an upper bound of the function f_2. If we proceed as in (i), (ii) and (iii), we obtain: for any $\varepsilon > 0$, there exists $l_4(\varepsilon) > 0$, such that if $i,j \ge l_4(\varepsilon)$ then

$$
\begin{aligned}
I &\le \varepsilon L \int_{-i}^{0} e^{-\gamma(t-s)}ds + \varepsilon \int_{-i}^{0} e^{-\gamma(t-s)}ds + \frac{N_2}{\gamma}(e^{-\gamma(t+i)} - e^{-\gamma(t+j)}) \\
&\le \frac{\varepsilon L}{\gamma} + \frac{\varepsilon}{\gamma} + \frac{\varepsilon N_2}{\gamma} = \varepsilon\Big(\frac{L+N_2+1}{\gamma}\Big).
\end{aligned}
$$

Combining, (i), (ii), (iii) and (iv), for $i,j \ge l(\varepsilon) = \min\big(l_1(\varepsilon), l_2(\varepsilon), l_3(\varepsilon), l_4(\varepsilon)\big)$, we obtain

$$
\begin{aligned}
&|u_{2(t+i)} - u_{2(t+j)}| \\
&\le \varepsilon + \frac{L\varepsilon}{\gamma} + \frac{\varepsilon}{\gamma} + \varepsilon\big(\frac{L+N_2+1}{\gamma}\big) + L\int_0^t e^{-\gamma(t-s)}|u_{2(s+i)} - u_{2(s+j)}|ds \\
&\le \varepsilon\big(1 + \frac{L}{\gamma} + \frac{1}{\gamma} + \frac{L+N_2+1}{\gamma}\big) + L\int_0^t e^{-\gamma(t-s)}|u_{2(s+i)} - u_{2(s+j)}|ds.
\end{aligned}
$$

Applying the Gronwall inequality, we obtain

$$
\begin{aligned}
|u_{2(t+i)} - u_{2(t+j)}| &\le \varepsilon\big(\frac{2+\gamma+2L+N_2}{\gamma}\big)\exp\Big(L\int_0^t e^{-\gamma(t-s)}ds\Big) \\
&\le \varepsilon\big(\frac{2+\gamma+2L+N_2}{\gamma}\big)\exp\Big(\frac{L}{\gamma}(1-e^{-\gamma t})\Big) \\
&\le \varepsilon\big(\frac{2+\gamma+2L+N_2}{\gamma}\big)\exp\big(\frac{L}{\gamma}\big).
\end{aligned}
$$

Therefore, taking $\varepsilon' = \varepsilon\big(\frac{2+\gamma+2L+N_2}{\gamma}\big)e^{L/\gamma}$ and interchanging the roles of ε and ε', we obtain: for any $\varepsilon > 0$, there exists $l(\varepsilon) > 0$, such that if $i,j \ge l(\varepsilon)$ then $|u_{2(t+i)} - u_{2(t+j)}| < \varepsilon$. That is the component $u_2(t)$ of the solution is also almost periodic. Hence, the system (2.2) has an almost periodic solution. □

Final remarks. This paper was devoted to the qualitative analysis of a distributed parameter biochemical systems with time delay and time varying input. The basis for the system studied here is derived from the work performed on anaerobic digestion in the pilot fixed bed reactor of LBE-INRA in Narbonne (France) and is mainly inspired from the dynamical models built and validated on the process (see [1, 14]). The growth function in [1], [14] and subsequently in [5], is expressed via the law:

$$
\mu(S,X) = \mu_0 \frac{S}{K_S X + S + \frac{1}{K_i}S^2} \tag{4.1}
$$

which clearly satisfies the assumption A2 of our present paper. In (4.1), the reaction is considered autocatalytic; i.e., the biomass(microorganism) is not only a product of the reaction, but also a catalyst of that reaction. Therefore, although our present work is inspired from dynamical models built and validated on the process, our main result can be applied to many different situations with different models of the growth function (or reaction term).

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