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RECONSTRUCTING THE POTENTIAL FUNCTION FOR INDEFINITE STURM-LIOUVILLE PROBLEMS USING INFINITE PRODUCT FORMS

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Abstract. In this article we consider the linear second-order equation of Sturm-Liouville type

$$y'' + (\lambda\phi^2(t) - q(t))y = 0, \quad 0 \le t \le 1,$$

where λ is a real parameter, $q(t)$ is the potential function and $\phi^2(t)$ is the weight function. We use the infinite product representation of the derivative of the solution to the differential equation with Dirichlet-Neumann conditions, and for the system of dual equations which is needed for expressing inverse problem and for retrieving potential. It must be mentioned that the weight function has a zero whose order is an integer called a *turning point*.

1. Introduction

We consider the indefinite Sturm-Liouville equation

$$ly := -y'' + q(t)y = \lambda\phi^2(t)y, \quad 0 \le t \le x, \tag{1.1}$$

with Dirichlet conditions

$$y(0) = y(x) = 0, \tag{1.2}$$

and with Dirichlet-Neumann conditions

$$y(0) = y'(x) = 0, \tag{1.3}$$

where $\lambda = \rho^2$ is the spectral parameter, x is a fixed point in the interval $(0, 1)$ and also the weight function $\phi^2(t)$ and the potential function $q(t)$ satisfies

- $\phi^2(t) = (t - t_0)^{l_0}\phi_0(t)$ is real and has one zero, t_0, so called turning point of odd order $l_0 \in \mathbf{N}$ in $[0, 1]$ and also $\phi_0(t)$ is positive and twice continuously differentiable.
- $q(t)$ is bounded and integrable on $[0, 1]$.

The asymptotic solutions of (1.1) depend on a complex parameter ρ as $|\rho| \to \infty$. We assume t_0 to be a turning point of type IV; i.e., l_0 is odd. The operator l defined in (1.1) is called the indefinite Sturm-Liouville operator. The differential equation (1.1) with conditions (1.2) and (1.3) are denoted by $L_1(\phi^2(t), q(t), x)$ and $L_2(\phi^2(t), q(t), x)$, respectively.

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Inverse spectral theory can be considered as the determination of an operator (usually differential) from its spectral data. The literature on such subjects is immense for the definite cases in which the weight function $\phi^2(t)$ is positive throughout the interval such as [16, 22, 15, 20], while for the indefinite cases is not. The transformation operator method and the Gelfand-Levitan integral equation with respect to the kernel of the transformation operator [12] in this case is not suitable for the solution of the inverse problems.

The eigenvalue problem for the indefinite Sturm-Liouville problem has been discussed in [14]. The potential $q(x)$ of an indefinite Sturm-Liouville problem has been determined uniquely by three spectra in [7]. We present a new approach to reconstruct the operator l (indefinite Sturm-Liouville operator); i.e., retrieving potential function $q(t)$ in (1.1) from its spectral data. The question at the core of this paper involves the determination of the infinite product representation for the derivative of the solution of the indefinite Sturm-Liouville problem as well as the reconstruction of the potential function $q(t)$ by means of two spectra, while the weight function $\phi^2(t)$ is given.

Differential equations with an indefinite weight function appear in several mathematical physics problems. For instance, turning points correspond to the limit of motion of a wave mechanical particle bounded by a potential field. Turning points arise also in various fields such as optics, elasticity, spectroscopy, stratification and radio engineering problems to design directional couplers for non-uniform electronic lines (see [3, 17, 18, 19, 23, 25] for further references).

The presence of turning points yields fundamental qualitative changes in the study of this kind of differential equation. In problem $L_1(\phi^2(t), q(t), x)$, for the special case $\phi^2(t) = t$ in the interval $[-1, 1]$, Jodayree et al. obtained the infinite product representation of the solution in the closed form [9]:

$$U(x,\lambda) = \begin{cases} \frac{p(x)}{(-x)^{1/4}} \prod_{k\geq 1} \frac{\lambda-\lambda_k(x)}{z_k^2(x)}, & -1 \leq x < 0, \\ \frac{\pi\sqrt{x}}{6} \prod_{k\geq 1} \frac{(\lambda-r_k(x))p^2(0)}{\tilde{j}_k^2} \prod_{k\geq 1} \frac{f^2(x)(u_k(x)-\lambda)}{\tilde{j}_k^2}, & 0 < x \leq 1, \end{cases}$$

where x is a fixed point in $(-1, 1)$, $U(x, \lambda)$ is the solution of the differential equation $L_1(t, q(t), x)$ which satisfies the initial condition

$$U(-1, \lambda) = 0, \quad \frac{\partial U}{\partial t}(-1, \lambda) = 1.$$

Here $p(x) = -(2/3)(-x)^{3/2} + 2/3$, $f(x) = (2/3)x^{3/2}$, $z_k(x) = k\pi/p(x)$, $\tilde{j}_k$ are the positive zeros of Bessel function $J_1'(z)$, $\{\lambda_k(x)\}$ are the eigenvalues of problem with the Dirichlet condition on $[-1, x]$ for $x < 0$ and $\{r_k(x)\}$ and $\{u_k(x)\}$ are the negative and positive eigenvalues of problem, respectively, with the Dirichlet condition on $[-1, x]$ for $x > 0$. Finding the solution in the infinite product form led to construct the dual equations which are necessary to retrieve the potential function $q(t)$ in the inverse problem [10].

Barcilon [2] introduced $\{\lambda_n(x)\}$ and $\{\mu_n(x)\}$ as the eigenvalues of classical Sturm-Liouville equation (vibrating string equation),

$$y'' + \lambda\phi^2(t)y = 0, \quad x \leq t \leq L, \tag{1.4}$$

with conditions $y(x) = y(L) = 0$ and $y'(x) = y(L) = 0$, respectively, in where x is a fixed point in the interval $(0, L)$. In contrast with problem (1.1), the function $\phi^2(t)$ is positive throughout the interval $(0, L)$. It has been shown that if $u(t, \lambda)$ is the solution of equation (1.4) with the initial conditions $u(L,\lambda)=0$ and $\frac{\partial u}{\partial t}(L,\lambda)=1$, then by using Hadamard's factorization, for fixed x belonging to $[0,L]$, it can be written

$$u(x,\lambda)=-(L-x)\prod_{k=1}^{\infty}(1-\frac{\lambda}{\lambda_n(x)}),$$
$$u'(x,\lambda)=\prod_{k=1}^{\infty}(1-\frac{\lambda}{\mu_n(x)}),$$

which are the infinite product form of the solution and its derivative for vibrating string problem [2]. He also derived the dual equations of problem (1.4) in the form

$$\begin{gathered}\frac{d\lambda_n(x)}{dx}=-\frac{\lambda_n(x)}{L-x}\frac{\prod_{k=1}^{\infty}(1-\frac{\lambda_n(x)}{\mu_k(x)})}{\prod_{k\neq n}^{\infty}(1-\frac{\lambda_n(x)}{\lambda_k(x)})},\\ \frac{d\mu_n(x)}{dx}=\mu_n^2(x)\phi^2(x)(L-x)\frac{\prod_{k=1}^{\infty}(1-\frac{\mu_n(x)}{\lambda_k(x)})}{\prod_{k\neq n}^{\infty}(1-\frac{\mu_n(x)}{\mu_k(x)})},\end{gathered}\tag{1.5}$$

with the initial condition

$$\lambda_n(0)=\lambda_n,\quad \mu_n(0)=\mu_n.$$

In fact, the pair of sequences $(\lambda_n(0),\mu_n(0))$ suffices as data to guarantee the existence and uniqueness of function $\phi^2(t)$ in (1.4) [2]. Hence, by using the solution $(\lambda_n(x),\mu_n(x))$ of (1.5), one can construct the original equation (1.4). For this reason, the equation (1.5) is referred to as dual equation of (1.4) in the classical literature.

Pranger [21] studied the recovery of the function $\phi^2(t)$ from the eigenvalues in equation (1.4) with the Dirichlet boundary condition on the interval $[0,1]$, replacing $\{\mu_n\}$ by $\{\lambda_n'\}$ and introducing the infinite product form of the solution to construct the dual equation

$$\lambda_n''+\frac{2}{x}\lambda_n+2\lambda_n\lambda_n'\sum_{j\neq n}(\frac{\lambda_j'}{\lambda_j^2})(1-\frac{\lambda_n}{\lambda_j})^{-1}-2\frac{(\lambda_n')^2}{\lambda_n}=0,$$

where $\{\lambda_n\}$ are eigenvalues of equation (1.4) on the interval $[0,x]$, $0<x\le 1$. It is well known that if there is a $c>0$ so that $\phi^2(t)\ge c$ for all t and $\phi^2(t)\in C^2(0,L)$, then Equation (1.4) can be transformed into the canonical Sturm-Liouville equation [8]

$$y''+(\lambda-q)y=0.$$

In section 2 we introduce some notation which we use throughout this article. In section 3 we find the infinite product form for the derivative of the solution of the indefinite Sturm-Liouville equation (1.1) before and after the turning point at the interval $(0,1)$. The main results of the paper are expressed by theorems 4.2 and 4.3. The infinite product representation for the solution of problem (1.1) in [13] and its derivative given here, enable us to construct the dual equations of this problem, in section 4, which this system of equations identifies the two spectra of eigenvalues for an arbitrary fixed point in the whole interval. Using these two spectra, one can retrieve the potential function $q(t)$ by the algorithm stated in the end of section 4.

2. Preliminaries

Let $\epsilon > 0$ be fixed and sufficiently small, and let $D_\epsilon = [0, t_0 - \epsilon] \cup [t_0 + \epsilon, 1]$. Further, we set $\mu = \frac{1}{2+l_0}$ (l_0 is order of turning point), $\lambda = \rho^2$ (ρ is a complex parameter) and $\theta = 4\mu$. We also denote

$$I_+ = \{t : \phi^2(t) > 0\}, \quad I_- = \{t : \phi^2(t) < 0\},$$

$$\xi(t) = \begin{cases} 0 & \text{for } t \in I_+(t), \\ 1 & \text{for } t \in I_-(t), \end{cases}$$

$$\phi_+^2(t) = \max(0, \phi^2(t)), \quad \phi_-^2(t) = \max(0, -\phi^2(t)),$$

$$K_\pm(t) = \begin{cases} 1 & \text{for } t \in I_-(t), \\ \frac{1}{2}\csc(\frac{\pi\mu}{2})\exp(\mp i\frac{\pi}{4}) & \text{for } t \in I_+(t), \end{cases}$$

$$K_\pm^*(t) = \begin{cases} \pm i & \text{for } t \in I_-(t), \\ 2\sin(\frac{\pi\mu}{2})\exp(\pm i\frac{\pi}{4}) & \text{for } t \in I_+(t), \end{cases}$$

Let

$$S_k = \{\rho : \arg\rho \in [\frac{k\pi}{4}, \frac{(k+1)\pi}{4}]\}, \quad k = 0, 1.$$

Here the choice of the root ϕ of ϕ^2 depends on the interval and the sector under consideration and has to be determined carefully. Due to the type of turning point t_0, we have

$$\phi(t) = \begin{cases} |\phi(t)| & \text{for } t > t_0, \\ |\phi(t)| e^{i\frac{\pi}{2}l_0} & \text{for } t < t_0. \end{cases}$$

In [5] it is shown that for each fixed sector S_k ($k = 0, 1$), there exist Fundamental Solutions (FS) of (1.1) $\{z_1(t,\rho), z_2(t,\rho)\}$, $t \in (0,1)$, $\rho \in S_k$ such that the functions $(t,\rho) \to z_s^{(j)}(t,\rho)(s = 1,2; j = 0,1)$ are continuous and holomorphic for $t \in (0,1)$, $\rho \in S_k$. Moreover, for $|\rho| \to \infty$, $\rho \in S_k$, $t \in D_\epsilon$, $j = 0, 1$

$$\begin{aligned} z_1^{(j)}(t,\rho) &= (\pm i\rho)^j |\phi(t)|^{j-\frac{1}{2}} (e^{\mp i\frac{\pi}{2}\xi(t)})^j e^{\rho\int_0^t |\phi_-(\tau)|d\tau} \\ &\quad \times e^{\pm i\rho\int_0^t |\phi_+(\tau)|d\tau} K_\pm(t)\kappa(t,\rho), \end{aligned} \tag{2.1}$$

$$\begin{aligned} z_2^{(j)}(t,\rho) &= (\mp i\rho)^j |\phi(t)|^{j-\frac{1}{2}} (e^{\mp i\frac{\pi}{2}\xi(t)})^j e^{-\rho\int_0^t |\phi_-(\tau)|d\tau} \\ &\quad \times e^{\mp i\rho\int_0^t |\phi_+(\tau)|d\tau} K_\pm^*(t)\kappa(t,\rho), \end{aligned} \tag{2.2}$$

$$\omega(\lambda) = \begin{vmatrix} z_1(t,\rho) & z_2(t,\rho) \\ z_1'(t,\rho) & z_2'(t,\rho) \end{vmatrix} = \mp(2i\rho)[1].$$

Here and in the following:

- (i) The upper or lower signs in formulae correspond to the sectors S_0, S_1 respectively.
- (ii) $[1] = 1 + O(\frac{1}{\rho^\theta})$ uniformly in $t \in D_\epsilon$.
- (iii) $\kappa(t,\rho) = O(1)$ as $|\rho| \to \infty$, $\rho \in S_k$.

3. Infinite product representation

Let $S(t,\lambda)$ be the solution of equation (1.1) with initial conditions

$$S(0,\lambda) = 0, \quad S'(0,\lambda) = 1. \tag{3.1}$$

Using $\{z_1(t,\rho), z_2(t,\rho)\}$, we can write

$$S(t,\lambda) = c_1 z_1(t,\rho) + c_2 z_2(t,\rho).$$

By imposing the initial conditions (3.1) we have

$$\begin{gathered} c_1 z_1(0,\rho) + c_2 z_2(0,\rho) = 0, \\ c_1 z_1'(0,\rho) + c_2 z_2'(0,\rho) = 1. \end{gathered}$$

After getting c_1 and c_2 by using Cramer's rule we obtain

$$S(t,\lambda) = \frac{z_1(0,\rho)z_2(t,\rho) - z_2(0,\rho)z_1(t,\rho)}{\omega(\lambda)}, \tag{3.2}$$

$$S'(t,\lambda) = \frac{z_1(0,\rho)z_2'(t,\rho) - z_2(0,\rho)z_1'(t,\rho)}{\omega(\lambda)}. \tag{3.3}$$

According to [5], we can also write the fundamental solutions $\{z_1(t,\rho), z_2(t,\rho)\}$, of (1.1), in the asymptotic form

$$z_1(t,\rho) = \begin{cases} |\phi(t)|^{-1/2} e^{\rho\int_0^t |\phi(\tau)|d\tau}[1] & 0 \le t < t_0, \\ \frac{1}{2}\csc(\frac{\pi\mu}{2})|\phi(t)|^{-1/2} e^{\rho\int_0^{t_0} |\phi(\tau)|d\tau} & \\ \times\big\{e^{i\rho\int_{t_0}^t |\phi(\tau)|d\tau - i\frac{\pi}{4}}[1] + e^{-i\rho\int_{t_0}^t |\phi(\tau)|d\tau + i\frac{\pi}{4}}[1]\big\} & t_0 < t \le 1, \end{cases} \tag{3.4}$$

$$z_2(t,\rho) = \begin{cases} i|\phi(t)|^{-1/2} e^{-\rho\int_0^t |\phi(\tau)|d\tau}[1] & 0 \le t < t_0, \\ 2\sin(\frac{\pi\mu}{2})|\phi(t)|^{-1/2} e^{-\rho\int_0^{t_0} |\phi(\tau)|d\tau} & \\ \times e^{i\rho\int_{t_0}^t |\phi(\tau)|d\tau - i\frac{\pi}{4}}[1] & t_0 < t \le 1, \end{cases} \tag{3.5}$$

$$z_1'(t,\rho) = \begin{cases} \rho|\phi(t)|^{1/2} e^{\rho\int_0^t |\phi(\tau)|d\tau}[1] & 0 \le t < t_0, \\ \frac{i\rho}{2}\csc(\frac{\pi\mu}{2})|\phi(t)|^{1/2} e^{\rho\int_0^{t_0} |\phi(\tau)|d\tau} & \\ \times\big\{e^{i\rho\int_{t_0}^t |\phi(\tau)|d\tau - i\frac{\pi}{4}}[1] - e^{-i\rho\int_{t_0}^t |\phi(\tau)|d\tau + i\frac{\pi}{4}}[1]\big\} & t_0 < t \le 1, \end{cases} \tag{3.6}$$

$$z_2'(t,\rho) = \begin{cases} -i\rho|\phi(t)|^{1/2} e^{-\rho\int_0^t |\phi(\tau)|d\tau}[1] & 0 \le t < t_0, \\ 2i\rho\sin(\frac{\pi\mu}{2})|\phi(t)|^{1/2} & \\ \times e^{-\rho\int_0^{t_0} |\phi(\tau)|d\tau} e^{i\rho\int_{t_0}^t |\phi(\tau)|d\tau - i\frac{\pi}{4}}[1] & t_0 < t \le 1. \end{cases} \tag{3.7}$$

Then, by (3.2) and (3.3) and asymptotic forms of FS in (3.4)-(3.7), we can write

$$S(t,\lambda) = \begin{cases} \frac{|\phi(0)\phi(t)|^{-1/2}}{\rho}\sinh(\rho\int_0^t |\phi(\tau)|d\tau)[1] & 0 \le t < t_0, \\ \frac{|\phi(0)|^{-1/2}|\phi(t)|^{1/2}}{-2i\rho}\big\{D_1(\rho)e^{i\rho\int_{t_0}^t |\phi(\tau)|d\tau}[1] & \\ +D_2(\rho)e^{-i\rho\int_{t_0}^t |\phi(\tau)|d\tau}[1]\big\} & t_0 < t \le 1, \end{cases} \tag{3.8}$$

$$S'(t,\lambda) = \begin{cases} |\phi(0)|^{-1/2}|\phi(t)|^{1/2}\cosh\left(\rho\int_0^t |\phi(\tau)|d\tau\right)[1] & 0 \le t < t_0, \\ \frac{|\phi(0)|^{-1/2}|\phi(t)|^{1/2}}{-2}\big\{D_1(\rho)e^{i\rho\int_{t_0}^t |\phi(\tau)|d\tau}[1] & \\ -D_2(\rho)e^{-i\rho\int_{t_0}^t |\phi(\tau)|d\tau}[1]\big\} & t_0 < t \le 1, \end{cases} \tag{3.9}$$

where

$$\begin{gathered} D_1(\rho) = 2\sin(\frac{\pi\mu}{2})e^{-\rho\int_0^{t_0} |\phi(\tau)|d\tau - i\frac{\pi}{4}} - \frac{1}{2}\csc(\frac{\pi\mu}{2})e^{\rho\int_0^{t_0} |\phi(\tau)|d\tau - i\frac{\pi}{4}}, \\ D_2(\rho) = -\frac{1}{2}\csc(\frac{\pi\mu}{2})e^{\rho\int_0^{t_0} |\phi(\tau)|d\tau + i\frac{\pi}{4}}. \end{gathered}$$

The functions $S(x,\lambda)$ and $S'(x,\lambda)$ have zero sets for each fixed point $x \in (0,1)$ referred to as $\lambda_n(x)$ and $\mu_n(x)$, respectively; i.e., $S(x,\lambda_n(x)) = 0$ and $S'(x,\mu_n(x)) =$

0. These two zero sets correspond to the eigenvalues of problems $L_1(\phi^2(t), q(t), x)$ and $L_2(\phi^2(t), q(t), x)$, respectively. So, for fixed x, $x < t_0$, the asymptotic approximation of the infinite sequence of negative eigenvalues for boundary-value problem (1.1) associated with boundary conditions $y(0) = y(x) = 0$ can be obtained from (3.8) of the form

$$\sqrt{-\lambda_n(x)} = \frac{n\pi}{p(x)} + O(\frac{1}{n}), \quad \text{as } n \to \infty,$$

while for boundary value problem (1.1) associated with boundary conditions $y(0) = y'(x) = 0$, the asymptotic form of eigenvalues can be derived similarly from (3.9)

$$\sqrt{-\mu_n(x)} = \frac{n\pi - \frac{\pi}{2}}{p(x)} + O(\frac{1}{n}), \quad \text{as } n \to \infty,$$

where

$$p(x) = \int_0^x |\phi(\tau)| d\tau. \tag{3.10}$$

Note that in the case $x < t_0$, the boundary value problem has only infinitely many negative eigenvalues according to classical results. For fixed x:

$$\cdots < \mu_2(x) < \lambda_2(x) < \mu_1(x) < \lambda_1(x), \quad \lim_{n\to\infty} \lambda_n(x) = \lim_{n\to\infty} \mu_n(x) = -\infty.$$

By applying (2.1) and (2.2), we infer that for $\rho \in S_k$, $t \in D_\epsilon$, $j = 0, 1$,

$$\begin{aligned} S^{(j)}(t, \lambda) = \frac{1}{2}(\pm i\rho)^{j-1} |\phi(0)|^{-1/2} |\phi(t)|^{j-\frac{1}{2}} (e^{\mp i \frac{\pi}{2} \xi(t)})^j e^{\pm i \frac{\pi}{2}} \\ \times e^{\rho \int_0^t |\phi_-(\tau)| d\tau} e^{\pm i\rho \int_0^t |\phi_+(\tau)| d\tau} K_\pm(t) \kappa(t, \rho), \end{aligned} \tag{3.11}$$

and

$$|S^{(j)}(t, \lambda)| \le C |\rho|^{j-1} |e^{\rho \int_0^t |\phi_-(\tau)| d\tau} e^{\pm i\rho \int_0^t |\phi_+(\tau)| d\tau}|. \tag{3.12}$$

It follows from (3.12) that the functions $S^{(j)}(t, .)$ are entire of order 1/2. So, by Hadamard's theorem $S(x, \lambda)$ and $S'(x, \lambda)$ can be represented in the infinite product form

$$S(x, \lambda) = C_{1,0}(x) \prod_{n=1}^{\infty} (1 - \frac{\lambda}{\lambda_n(x)}), \tag{3.13}$$

$$S'(x, \lambda) = C_{2,0}(x) \prod_{n=1}^{\infty} (1 - \frac{\lambda}{\mu_n(x)}), \tag{3.14}$$

where $C_{r,0}$ (r=1,2) are functions of x only. The index r in $C_{r,0}$ is denoted to related problem $L_r(\phi^2(t), q(t), x)$ and the index '0' in $C_{r,0}$ shows that the fixed point x lies before turning point ($x < t_0$). The function $C_{1,0}(x)$ has been calculated in [13]:

$$C_{1,0}(x) = (\phi(0)\phi(x))^{-1/2} p(x) \prod_n \frac{\lambda_n(x)}{w_n^2(x)}, \tag{3.15}$$

where $w_n(x) = n\pi/p(x)$, and $p(x)$ is defined in (3.10).

To estimate $C_{2,0}(x)$ we rewrite the infinite product as

$$S'(x, \lambda) = C_{2,0}(x) \prod_{n=1}^{\infty} \frac{\mu_n(x) - \lambda}{\mu_n(x)} = B_{2,0}(x) \prod_{n=1}^{\infty} \frac{\lambda - \mu_n(x)}{\tilde{w}_n^2(x)} \tag{3.16}$$

with

$$B_{2,0}(x) = C_{2,0}(x) \prod_{n=1}^{\infty} \frac{-\tilde{w}_n^2(x)}{\mu_n(x)}, \tag{3.17}$$

where

$$\tilde{w}_n(x) = \frac{n\pi - \frac{\pi}{2}}{p(x)}.$$

It follows from the asymptotic form of eigenvalues that $\frac{-\tilde{w}_n^2(x)}{\mu_n(x)} = 1+O(\frac{1}{n^2})$, then the infinite product $\prod_{n=1}^{\infty} \frac{-\tilde{w}_n^2(x)}{\mu_n(x)}$ is absolutely convergent on any compact subinterval of $(0, t_0)$.

Lemma 3.1. *Let $\tilde{w}_m(x) = \frac{m\pi - \frac{\pi}{2}}{p(x)}$ and $\mu_m(x), 1 \le m$ be a sequence of continuous functions such that for each x*

$$\mu_m(x) = -\frac{m^2\pi^2}{p^2(x)} + \frac{m\pi^2}{p^2(x)} + O(1), \quad 0 < x < t_0.$$

Then, the infinite product

$$\prod_{m=1}^{\infty} (\frac{\lambda - \mu_m(x)}{\tilde{w}_m^2(x)})$$

is an entire function of λ for fixed x in $(0, t_0)$ whose roots are precisely $\mu_m(x)$, $m \ge 1$. Moreover

$$\prod_{m=1}^{\infty} (\frac{\lambda - \mu_m(x)}{\tilde{w}_m^2(x)}) = \cosh(\sqrt{\lambda}p(x))(1 + O(\frac{\log n}{n})),$$

uniformly on the circles $|\lambda| = \frac{n^2\pi^2}{p^2(x)}$, where $p(x)$ is defined in (3.10).

Proof. Since $\mu_m(x) + \tilde{w}_m^2(x) = \frac{\pi^2}{4p^2(x)} + O(1), m \ge 1$ are uniformly bounded, then

$$\sum_{m=1}^{\infty} \Big|\frac{\lambda - \mu_m(x)}{\tilde{w}_m^2(x)} - 1\Big| = \sum_{m=1}^{\infty} \Big|\frac{\lambda - \mu_m(x) - \tilde{w}_m^2(x)}{\tilde{w}_m^2(x)}\Big| = \sum_{m=1}^{\infty} \Big|\frac{\lambda + O(1)}{\tilde{w}_m^2(x)}\Big|$$

converges uniformly on bounded subsets of complex plane. Therefore, the infinite product converges to an entire function of λ, whose zeroes are precisely $\tilde{w}_m(x), m \ge 1$ (see [6]). By [1, 4.5.69], we have

$$\cosh(p(x)\sqrt{\lambda}) = \prod_{m=1}^{\infty} \big[1 + \frac{4p^2(x)\lambda}{(2m-1)^2\pi^2}\big].$$

On the other hand, since $\frac{4p^2(x)}{(2m-1)^2\pi^2} = \frac{1}{\tilde{w}_m^2(x)}$, we obtain

$$\frac{\prod_{m=1}^{\infty} \frac{\lambda - \mu_m(x)}{\tilde{w}_m^2(x)}}{\prod_{m=1}^{\infty} [1 + \frac{4p^2(x)\lambda}{(2m-1)^2\pi^2}]} = \prod_{m=1}^{\infty} \frac{-\mu_m(x) + \lambda}{\tilde{w}_m^2(x) + \lambda}.$$

Furthermore,

$$\Big|\frac{-\mu_m(x) + \lambda}{\tilde{w}_m^2(x) + \lambda} - 1\Big| \le \frac{|O(1)|}{\big||\lambda| - \frac{(2m-1)^2\pi^2}{4p^2(x)}\big|}.$$

Therefore, on the circles $|\lambda| = \frac{n^2\pi^2}{p^2(x)}$, the uniform estimates

$$\frac{-\mu_m(x) + \lambda}{\tilde{w}_m^2(x) + \lambda} = \begin{cases} 1 + O(\frac{1}{n}) & \text{if } n = m \\ 1 + O(\frac{1}{m^2 - n^2}) & \text{if } n \ne m \end{cases}$$

hold. by [24, page 165], we can write

$$\prod_{1\leq m}\frac{-\mu_m(x)+\lambda}{\lambda+\tilde{w}_m^2(x)}=1+O(\frac{\log n}{n}),$$

uniformly on these circles. Then

$$\prod_{1\leq m}\frac{\lambda-\mu_m(x)}{\tilde{w}_m^2(x)}=\cosh(p(x)\sqrt{\lambda})\Big(1+O(\frac{\log n}{n})\Big).$$

□

Theorem 3.2. *For $0\leq x<t_0$,*

$$S'(x,\lambda)=|\phi(0)|^{-1/2}|\phi(x)|^{1/2}\prod_{n=1}^{\infty}-\frac{\mu_n(x)}{\tilde{w}_n^2(x)}\prod_{n=1}^{\infty}(1-\frac{\lambda}{\mu_n(x)}),$$

where $\tilde{w}_n(x)=\frac{n\pi-\frac{\pi}{2}}{p(x)}$, $p(x)$ is defined in (3.10)*, and $\{\mu_n(x)\}$ is the sequence of eigenvalues for the Dirichlet-Neumann problem associated with* (1.1) *on $[0,x]$.*

Proof. For $0\leq x<t_0$, $\rho\in S_0$ and $|\rho|\to\infty$, by virtue of (3.11) for $j=1$ we calculate

$$S'(x,\lambda)=\frac{1}{2}|\phi(0)|^{-1/2}|\phi(x)|^{1/2}e^{\rho\int_0^x|\phi(\tau)|d\tau}\kappa(x,\rho). \tag{3.18}$$

Now from (3.16), (3.18) and using lemma 3.1 uniformly on the circles $|\lambda|=\frac{n^2\pi^2}{p^2(x)}$, we obtain

$$B_{2,0}(x)=\frac{S'(x,\lambda)}{\prod_{m=1}^{\infty}\frac{\lambda-\mu_m(x)}{\tilde{w}_m^2(x)}}=|\phi(0)|^{-1/2}|\phi(x)|^{1/2},$$

as $|\rho|\to\infty$. So, by (3.17), we obtain

$$C_{2,0}(x)=|\phi(0)|^{-1/2}|\phi(x)|^{1/2}\prod_{n=1}^{\infty}-\frac{\mu_n(x)}{\tilde{w}_n^2(x)}. \tag{3.19}$$

The proof is completed by (3.14). □

For $x\in(t_0,1]$, fixed, both of problems $L_1(\phi^2(t),q(t),x)$ and $L_2(\phi^2(t),q(t),x)$ have an infinite number of positive and negative eigenvalues, which we denote by $\{\lambda_n^+(x)\}$, $\{\lambda_n^-(x)\}$ and $\{\mu_n^+(x)\}$, $\{\mu_n^-(x)\}$ respectively. In reference to [4, 11], we derive

$$\sqrt{\lambda_n^+(x)}=\frac{n\pi-\frac{\pi}{4}}{f(x)}+O(\frac{1}{n}),\quad \sqrt{-\lambda_n^-(x)}=\frac{n\pi-\frac{\pi}{4}}{p(t_0)}+O(\frac{1}{n}),$$
$$\sqrt{\mu_n^+(x)}=\frac{n\pi-3\frac{\pi}{4}}{f(x)}+O(\frac{1}{n}),\quad \sqrt{-\mu_n^-(x)}=\frac{n\pi-\frac{\pi}{4}}{p(t_0)}+O(\frac{1}{n}),$$

where

$$f(x)=\int_{t_0}^{x}|\phi(\tau)|d\tau \tag{3.20}$$

and $p(x)$ is defined in (3.10). By Hadamard's theorem, the solution of equation (1.1) and its derivative on $[0,x]$ for $x>t_0$ is of the form

$$S(x,\lambda)=C_{1,1}(x)\prod_{n=1}^{\infty}(1-\frac{\lambda}{\lambda_n^-(x)})\prod_{n=1}^{\infty}(1-\frac{\lambda}{\lambda_n^+(x)}), \tag{3.21}$$

$$S'(x,\lambda)=C_{2,1}(x)\prod_{n=1}^{\infty}(1-\frac{\lambda}{\mu_n^-(x)})\prod_{n=1}^{\infty}(1-\frac{\lambda}{\mu_n^+(x)}). \tag{3.22}$$

Index '1' in $C_{r,1}(r=1,2)$ means that the fixed point x lies after turning point $(x>t_0)$.

The function $C_{1,1}(x)$ has been estimated in [13]:

$$\begin{aligned} C_{1,1}(x)&=\frac{1}{16}\pi|\phi(0)\phi(x)|^{-1/2}\csc(\frac{\pi\mu}{2})p(t_0)^{1/2}f(x)^{1/2}\\ &\quad\times\prod_{n=1}^{\infty}-\frac{\lambda_n^-(x)p^2(t_0)}{\tilde{j}_n^2}\prod_{n=1}^{\infty}\frac{\lambda_n^+(x)f^2(x)}{\tilde{j}_n^2}, \end{aligned} \tag{3.23}$$

where $p(x)$ and $f(x)$ are defined in (3.10) and (3.20) respectively and $\tilde{j}_n(n=1,2,\dots)$ are the positive zeros of the derivative of the Bessel function of first kind $(J_1'(z))$.

Let $J_\nu(z)$ and $J_\nu'(z)$ be the Bessel function of order ν and its derivative, respectively. From [1] we have

$$J_\nu(z)=\frac{(z/2)^\nu}{\Gamma(\nu+1)}\prod_{m=1}^{\infty}(1-\frac{z^2}{j_{\nu,m}^2}),$$

where

$$\begin{gathered} j_{\nu,m}\sim\beta-\frac{\alpha-1}{8\beta}-\frac{4(\alpha-1)(7\alpha-31)}{3(8\beta)^3}-\dots,\\ \beta=(m+\frac{\nu}{2}-\frac{1}{4})\pi,\quad \alpha=4\nu^2. \end{gathered}$$

By inserting $\nu=0$, we can write

$$J_0(z)=\prod_{m=1}^{\infty}(1-\frac{z^2}{j_{0,m}^2}),$$

where

$$j_{0,m}^2=m^2\pi^2-\frac{m\pi^2}{2}+O(1),\quad m=1,2,\dots,$$

are the positive zeros of $J_0(z)$. Also, from [1], We have

$$J_\nu'(z)=\frac{(z/2)^{\nu-1}}{2\Gamma(\nu)}\prod_{m=1}^{\infty}(1-\frac{z^2}{\tilde{j}_{\nu,m}^2}),\quad \nu>0,$$

where

$$\begin{gathered} \tilde{j}_{\nu,m}\sim\beta'-\frac{\alpha+3}{8\beta'}-\frac{4(7\alpha^2+82\alpha-9)}{3(8\beta')^3}-\dots,\\ \beta'=(m+\frac{\nu}{2}-\frac{3}{4})\pi,\quad \alpha=4\nu^2. \end{gathered}$$

In reference to [1], as a result of $J_0'(z)=-J_1(z)$ the zeros of $J_1(z)$ and $J_0'(z)$ are the same, namely $\tilde{j}_{0,m}=j_{1,m}$ for $m=1,2,\dots$. Therefore, we can write

$$J_0'(z)=-J_1(z)=-\frac{z}{2}\prod_{m=1}^{\infty}(1-\frac{z^2}{j_{1,m}^2}),$$

where

$$j_{1,m}=(m+\frac{1}{4})\pi+\dots,\quad m=1,2,\dots.$$

Replacing m by $m-1$ in the previous relation we obtain

$$j_{1,m-1} = (m - \frac{3}{4})\pi + \dots, \quad m = 2, 3, \dots,$$
$$j_{1,m-1}^2 = m^2\pi^2 - \frac{3}{2}m\pi^2 + O(1), \quad m = 2, 3, \dots.$$

Consequently,

$$\frac{-j_{0,n}^2}{p^2(t_0)\mu_n^-(x)} = 1 + O(\frac{1}{n^2}), \quad \frac{j_{1,n-1}^2}{f^2(x)\mu_n^+(x)} = 1 + O(\frac{1}{n^2}).$$

Therefore, the infinite products $\prod_{n=1}^{\infty} \frac{-j_{0,n}^2}{p^2(t_0)\mu_n^-(x)}$ and $\prod_{n=2}^{\infty} \frac{j_{1,n-1}^2}{f^2(x)\mu_n^+(x)}$ are absolutely convergent for each $x > t_0$. Then, from (3.22), we may write

$$S'(x,\lambda) = B_{2,1}(x)(1 - \frac{\lambda}{\mu_1^+}) \prod_{n=1}^{\infty} \frac{(\lambda - \mu_n^-(x))p^2(t_0)}{j_{0,n}^2} \prod_{n=2}^{\infty} \frac{(\mu_n^+(x) - \lambda)f^2(x)}{j_{1,n-1}^2}, \tag{3.24}$$

where

$$B_{2,1}(x) = C_{2,1}(x) \prod_{n=1}^{\infty} \frac{-j_{0,n}^2}{p^2(t_0)\mu_n^-(x)} \prod_{n=2}^{\infty} \frac{j_{1,n-1}^2}{f^2(x)\mu_n^+(x)}. \tag{3.25}$$

Lemma 3.3. *Let $j_{0,m}$ be the positive zeros of $J_0(z)$ and for fixed x in $(t_0, 1)$*

$$\mu_m^-(x) = -\frac{m^2\pi^2}{p^2(t_0)} + \frac{3}{2}\frac{m\pi^2}{p^2(t_0)} + O(1), m \geq 1,$$

be a negative sequence of continuous functions. The infinite product

$$\prod_{m=1}^{\infty} \frac{(\lambda - \mu_m^-(x))p^2(t_0)}{j_{0,m}^2}$$

is an entire function of λ for fixed x, whose roots are precisely $\mu_m^-(x)$, $m \geq 1$. Moreover,

$$\prod_{m=1}^{\infty} \frac{(\lambda - \mu_m^-(x))p^2(t_0)}{j_{0,m}^2} = J_0(i\sqrt{\lambda}p(t_0))(1 + O(\frac{\log n}{n})),$$

uniformly on the circles $|\lambda| = \frac{n^2\pi^2}{p^2(t_0)}$.

Proof. This follows from using the method of the proof of lemma 3.1. For more details, see [9]. □

Lemma 3.4. *Let $j_{1,m}$ be the positive zeros of $J_1(z)$ and for fixed x in $(t_0, 1)$*

$$\mu_m^+(x) = \frac{m^2\pi^2}{f^2(x)} - \frac{m\pi^2}{2f^2(x)} + O(1), \quad m \geq 1,$$

be a positive sequence of continuous functions. Then, the infinite product

$$\prod_{m=2}^{\infty} \frac{(\mu_m^+(x) - \lambda)f^2(x)}{j_{1,m-1}^2}$$

is an entire function of λ for fixed x, whose roots are precisely $\mu_m^+(x)$, $m \geq 1$. Moreover,

$$\prod_{m=2}^{\infty} \frac{(\mu_m^+(x) - \lambda)f^2(x)}{j_{1,m-1}^2} = -\frac{2}{\sqrt{\lambda}f(x)} J_0'(\sqrt{\lambda}f(x))(1 + O(\frac{\log n}{n})),$$

uniformly on the circles $|\lambda| = \frac{n^2\pi^2}{p^2(t_0)}$.

Proof. This follows from using the method of the proof of lemma 3.1. For more details, see [9]. □

Theorem 3.5. *Let $S'(t,\lambda)$ be the derivative of the solution of problem* (1.1) *in association with initial condition* (3.1)*. Then, for each fixed $x > t_0$,*

$$\begin{aligned} S'(x,\lambda) &= -\frac{1}{4}|\phi(0)|^{-1/2}|\phi(x)|^{1/2} i^{1/2}\pi\mu_1^+(x)e^{i\frac{\pi}{4}}\csc(\frac{\pi\mu}{2})f^{3/2}(x)p^{1/2}(t_0) \\ &\quad\times\prod_{n=1}^{\infty} -\frac{\mu_n^-(x)p^2(t_0)}{j_{0,n}^2}\prod_{n=2}^{\infty}\frac{\mu_n^+(x)f^2(x)}{j_{1,n-1}^2}\prod_{n=1}^{\infty}(1-\frac{\lambda}{\mu_n^-(x)})\prod_{n=2}^{\infty}(1-\frac{\lambda}{\mu_n^+(x)}), \end{aligned}$$

where $p(x)$ and $f(x)$ is defined in (3.10) *and* (3.20)*. Sequences $\{\mu_n^+(x)\}$ and $\{\mu_n^-(x)\}$ represent the positive and negative eigenvalues of $L_2(\phi^2(t), q(t), x)$, respectively and $j_{\nu,n}(\nu = 0,1)$ are the positive zeros of $J_\nu(z)$.*

Proof. Using (3.9) for $t_0 < x < 1$ it is obtained that

$$\begin{aligned} S'(x,\lambda) = \frac{|\phi(0)|^{1/2}|\phi(x)|^{1/2}}{-2}\big\{&(2\sin(\frac{\pi\mu}{2})e^{-\sqrt{\lambda}p(t_0)-i\frac{\pi}{4}} \\ &-\frac{1}{2}\csc(\frac{\pi\mu}{2})e^{\sqrt{\lambda}p(t_0)-i\frac{\pi}{4}})e^{i\sqrt{\lambda}f(x)}[1] \\ &+\frac{1}{2}\csc(\frac{\pi\mu}{2})e^{\sqrt{\lambda}p(t_0)+i\frac{\pi}{4}}e^{-i\sqrt{\lambda}f(x)}[1]\big\}. \end{aligned}$$

As $|\lambda| \to \infty$, the first expression in the accolade tends to zero, resulting in

$$S'(x,\lambda) = \frac{|\phi(0)|^{-1/2}|\phi(x)|^{1/2}i}{2}\csc(\frac{\pi\mu}{2})e^{\sqrt{\lambda}p(t_0)}\sin(\sqrt{\lambda}f(x)-\frac{\pi}{4})[1]. \tag{3.26}$$

On the other hand, by use of (3.24) and lemma 3.3, lemma 3.4, on the circles $|\lambda| = \min\{\frac{n^2\pi^2}{p^2(t_0)}, \frac{n^2\pi^2}{f^2(x)}\}$, we obtain

$$S'(x,\lambda) = -\frac{2}{\sqrt{\lambda}f(x)}B_{2,1}(x)(1-\frac{\lambda}{\mu_1^+})J_0(i\sqrt{\lambda}p(t_0))J_0'(\sqrt{\lambda}f(x))1+O(\frac{\log n}{n})).$$

Using the asymptotic form of the Bessel function and its derivative in the previous relation, we have

$$\begin{aligned} S'(x,\lambda) &= \frac{2}{\sqrt{\lambda}f(x)}B_{2,1}(x)(1-\frac{\lambda}{\mu_1^+})(\frac{2}{i\sqrt{\lambda}p(t_0)\pi})^{1/2}\cos(i\sqrt{\lambda}p(t_0)-\frac{\pi}{4}) \\ &\quad\times(\frac{2}{\sqrt{\lambda}f(x)\pi})^{1/2}\sin(\sqrt{\lambda}f(x)-\frac{\pi}{4})[1](1+O(\frac{\log n}{n})). \end{aligned}$$

After some calculations, we obtain

$$\begin{aligned} S'(x,\lambda) &= \frac{2}{\lambda\pi i^{1/2}f^{3/2}(x)p^{1/2}(t_0)}B_{2,1}(x)(1-\frac{\lambda}{\mu_1^+})(e^{\sqrt{\lambda}p(t_0)+i\frac{\pi}{4}}+e^{-\sqrt{\lambda}p(t_0)-i\frac{\pi}{4}}) \\ &\quad\times\sin(\sqrt{\lambda}f(x)-\frac{\pi}{4})[1](1+O(\frac{\log n}{n})). \end{aligned} \tag{3.27}$$

We know that in the above relation, the expression $\exp\big(-\sqrt{\lambda}p(t_0)-i\frac{\pi}{4}\big)$ vanishes as $|\lambda| \to \infty$. Comparing (3.26) and (3.27) and considering $|\lambda| \to \infty$ we obtain

$$B_{2,1}(x) = -\frac{1}{4}|\phi(0)|^{-1/2}|\phi(x)|^{1/2}i^{1/2}\pi\mu_1^+e^{i\frac{\pi}{4}}\csc(\frac{\pi\mu}{2})f^{3/2}(x)p^{1/2}(t_0).$$

So, by (3.25), we obtain

$$\begin{aligned} C_{2,1}(x) = &-\frac{1}{4}|\phi(0)|^{-1/2}|\phi(x)|^{1/2} i^{1/2}\pi\mu_1^+ e^{i\frac{\pi}{4}} \csc(\frac{\pi\mu}{2}) \\ &\times f^{3/2}(x)p^{1/2}(t_0)\prod_{n=1}^{\infty} -\frac{\mu_n^- p^2(t_0)}{j_{0,n}^2}\prod_{n=2}^{\infty}\frac{\mu_n^+ f^2(x)}{j_{1,n-1}^2}, \end{aligned} \tag{3.28}$$

which depends only on x. By(3.22) the proof is complete. □

4. Dual equations

In this section, we derive the dual equations associated with problem (1.1) by using the infinite product representation. First we prove some lemmas which are necessary to present the main theorem.

Lemma 4.1. *$C_{2,j}(x) = C'_{1,j}(x)(j = 0, 1)$, where $C_{1,0}(x)$, $C_{2,0}(x)$, $C_{1,1}(x)$ and $C_{2,1}(x)$ are defined in* (3.15), (3.19), (3.23) *and* (3.28)*, respectively.*

Proof. If one inserts $\lambda = 0$ in (3.13), (3.14), (3.21) and (3.22), the proof becomes trivial. □

Theorem 4.2. *The functions $q(x)$ and $\phi^2(x)$ in problem* (1.1) *satisfy the following relations:*

$$q(x) = \begin{cases} \frac{C'_{2,0}(x)}{C_{1,0}(x)} = \frac{C''_{1,0}(x)}{C_{1,0}(x)} & 0 \le x < t_0, \\ \frac{C'_{2,1}(x)}{C_{1,1}(x)} = \frac{C''_{1,1}(x)}{C_{1,1}(x)} & t_0 < x \le 1, \end{cases} \tag{4.1}$$

$$\phi^2(x) = \begin{cases} \frac{C'_{2,0}(x)}{C_{1,0}(x)}\sum_i(\frac{1}{\mu_i(x)} - \frac{1}{\lambda_i(x)}) - \frac{C_{2,0}(x)}{C_{1,0}(x)}\sum_i\frac{\mu'_i(x)}{\mu_i^2(x)} & 0 \le x < t_0, \\ \frac{C'_{2,1}(x)}{C_{1,1}(x)}\sum_i(\frac{1}{\mu_i^-(x)} + \frac{1}{\mu_i^+(x)} - \frac{1}{\lambda_i^-(x)} - \frac{1}{\lambda_i^+(x)}) & \\ -\frac{C_{2,1}(x)}{C_{1,1}(x)}\sum_i(\frac{{\mu_i^-}'(x)}{{\mu_i^-}^2(x)} + \frac{{\mu_i^+}'(x)}{{\mu_i^+}^2(x)}) & t_0 < x \le 1, \end{cases} \tag{4.2}$$

where $C_{1,0}(x)$, $C_{2,0}(x)$, $C_{1,1}(x)$ and $C_{2,1}(x)$ are defined in (3.15), (3.19), (3.23) *and* (3.28)*.*

Proof. We prove it for the case $0 \le x < t_0$. There is a similar proof for $t_0 < x \le 1$. We know that $S(x, \lambda)$ satisfies the original problem (1.1), so

$$\frac{\partial}{\partial x}S'(x,\lambda) + (\lambda\phi^2(x) - q(x))S(x,\lambda) = 0.$$

Using (3.13) and (3.14) in the previous relation we obtain

$$\begin{aligned} &C'_{2,0}(x)\prod_{k=1}(1 - \frac{\lambda}{\mu_k(x)}) + C_{2,0}(x)\sum_i\frac{\mu'_i(x)}{\mu_i^2(x)}\cdot\lambda\prod_{k\ne i,1\le k}(1 - \frac{\lambda}{\mu_k(x)}) \\ &+ [\lambda\phi^2(x) - q(x)]C_{1,0}(x)\prod_{k=1}(1 - \frac{\lambda}{\lambda_k(x)}) = 0. \end{aligned} \tag{4.3}$$

By putting the coefficients of different powers of λ equal to zero in the previous relation, we obtain

Coefficient of λ^0: $C'_{2,0}(x) - q(x).C_{1,0}(x) = 0$ which implies

$$q(x) = \frac{C'_{2,0}(x)}{C_{1,0}(x)},$$

and from Lemma 4.1,

$$q(x) = \frac{C''_{1,0}(x)}{C_{1,0}(x)}.$$

From the coefficient of λ^1,

$$-C'_{2,0}(x)\sum_i \frac{1}{\mu_i(x)} + C_{2,0}(x)\sum_i \frac{\mu'_i(x)}{\mu_i^2(x)} + \phi^2(x)C_{1,0}(x) + q(x)C_{1,0}(x)\sum_i \frac{1}{\lambda_i(x)} = 0,$$

we obtain

$$\phi^2(x) = \frac{C'_{2,0}(x)}{C_{1,0}(x)}\sum_i\Big(\frac{1}{\mu_i(x)} - \frac{1}{\lambda_i(x)}\Big) - \frac{C_{2,0}(x)}{C_{1,0}(x)}\sum_i \frac{\mu'_i(x)}{\mu_i^2(x)}.$$

The proof is complete. □

Theorem 4.3. *(a) For fixed value x in $[0, t_0)$, the sequences $\{\lambda_n(x)\}_{n=1}^{\infty}$ and $\{\mu_n(x)\}_{n=1}^{\infty}$ which are the negative eigenvalues of problems $L_1(\phi^2(t), q(t), x)$ and $L_2(\phi^2(t), q(t), x)$, respectively, satisfy the system of equations:*

$$\begin{aligned}
\frac{d\lambda_n(x)}{dx} &= \frac{C_{2,0}(x)}{C_{1,0}(x)}\lambda_n(x)\frac{\prod_{k=1}^{\infty}(1-\frac{\lambda_n(x)}{\mu_k(x)})}{\prod_{k\neq n}^{\infty}(1-\frac{\lambda_n(x)}{\lambda_k(x)})},\\
\frac{d\mu_n(x)}{dx} &= -\frac{C_{1,0}(x)}{C_{2,0}(x)}[\mu_n^2(x)\phi^2(x) - \mu_n(x)\frac{C'_{2,0}(x)}{C_{1,0}(x)}]\frac{\prod_{k=1}^{\infty}(1-\frac{\mu_n(x)}{\lambda_k(x)})}{\prod_{k\neq n}^{\infty}(1-\frac{\mu_n(x)}{\mu_k(x)})}.
\end{aligned}\tag{4.4}$$

(b) For fixed value x in $(t_0, 1]$, the sequences $\{\lambda_n^-(x)\}_{n=1}^{\infty}$, $\{\lambda_n^+(x)\}_{n=1}^{\infty}$ and $\{\mu_n^-(x)\}_{n=1}^{\infty}$, $\{\mu_n^+(x)\}_{n=1}^{\infty}$ which are the negative, positive eigenvalues of problems $L_1(\phi^2(t), q(t), x)$ and $L_2(\phi^2(t), q(t), x)$, respectively, satisfy the system of equations:

$$\begin{aligned}
\frac{d\lambda_n^-(x)}{dx} &= \frac{C_{2,1}(x)}{C_{1,1}(x)}\lambda_n^-(x)\frac{\prod_{k=1}^{\infty}(1-\frac{\lambda_n^-(x)}{\mu_k^-(x)})\prod_{k=1}^{\infty}(1-\frac{\lambda_n^-(x)}{\mu_k^+(x)})}{\prod_{k\neq n}^{\infty}(1-\frac{\lambda_n^-(x)}{\lambda_k^-(x)})\prod_{k=1}^{\infty}(1-\frac{\lambda_n^-(x)}{\lambda_k^+(x)})},\\
\frac{d\lambda_n^+(x)}{dx} &= \frac{C_{2,1}(x)}{C_{1,1}(x)}\lambda_n^+(x)\frac{\prod_{k=1}^{\infty}(1-\frac{\lambda_n^+(x)}{\mu_k^-(x)})\prod_{k=1}^{\infty}(1-\frac{\lambda_n^+(x)}{\mu_k^+(x)})}{\prod_{k=1}^{\infty}(1-\frac{\lambda_n^+(x)}{\lambda_k^-(x)})\prod_{k\neq n}^{\infty}(1-\frac{\lambda_n^+(x)}{\lambda_k^+(x)})},\\
\frac{d\mu_n^-(x)}{dx} &= -\frac{C_{1,1}(x)}{C_{2,1}(x)}[{\mu_n^-}^2(x)\phi^2(x) - \mu_n^-(x)\frac{C'_{2,1}}{C_{1,1}}]\\
&\quad\times\frac{\prod_{k=1}^{\infty}(1-\frac{\mu_n^-(x)}{\lambda_k^-(x)})\prod_{k=1}^{\infty}(1-\frac{\mu_n^-(x)}{\lambda_k^+(x)})}{\prod_{k\neq n}^{\infty}(1-\frac{\mu_n^-(x)}{\mu_k^-(x)})\prod_{k=1}^{\infty}(1-\frac{\mu_n^-(x)}{\mu_k^+(x)})},\\
\frac{d\mu_n^+(x)}{dx} &= -\frac{C_{1,1}(x)}{C_{2,1}(x)}[{\mu_n^+}^2(x)\phi^2(x) - \mu_n^+(x)\frac{C'_{2,1}(x)}{C_{1,1}(x)}]\\
&\quad\times\frac{\prod_{k=1}^{\infty}(1-\frac{\mu_n^+(x)}{\lambda_k^-(x)})\prod_{k=1}^{\infty}(1-\frac{\mu_n^+(x)}{\lambda_k^+(x)})}{\prod_{k=1}^{\infty}(1-\frac{\mu_n^+(x)}{\mu_k^-(x)})\prod_{k\neq n}^{\infty}(1-\frac{\mu_n^+(x)}{\mu_k^+(x)})}.
\end{aligned}\tag{4.5}$$

Proof. We prove it for the case $0 \le x < t_0$. There is a similar proof for $t_0 < x \le 1$. Since $\{\lambda_n(x)\}$ is the eigenvalues of problem (1.1) associated with the boundary condition (1.2), we have

$$S(x, \lambda_n(x)) = 0.$$

By differentiating,

$$\frac{\partial}{\partial x}S(x,\lambda)+\frac{\partial}{\partial\lambda}S(x,\lambda)\lambda_n'(x)=0.$$

Therefore, at the points $(x,\lambda_n(x))$, we obtain

$$\lambda_n'(x)=-\frac{\frac{\partial}{\partial x}S(x,\lambda_n(x))}{\frac{\partial}{\partial\lambda}S(x,\lambda_n(x))}=-\frac{S'(x,\lambda_n(x)}{\frac{\partial}{\partial\lambda}S(x,\lambda_n(x))}. \tag{4.6}$$

We calculate $\frac{\partial S}{\partial\lambda}$ at the points $(x,\lambda_n(x))$. Using (3.13), we reach

$$\frac{\partial S}{\partial\lambda}=C_{1,0}(x)\sum_{i=1}^{\infty}\frac{-1}{\lambda_i(x)}\prod_{k\neq i,1\leq k}(1-\frac{\lambda}{\lambda_k(x)}).$$

So, we have

$$\frac{\partial}{\partial\lambda}S(x,\lambda_n(x))=-\frac{C_{1,0}(x)}{\lambda_n(x)}\prod_{k\neq n,1\leq k}(1-\frac{\lambda_n(x)}{\lambda_k(x)}). \tag{4.7}$$

Therefore, substituting (4.7) and (3.14) in (4.6) we obtain

$$\lambda_n'(x)=\frac{C_{2,0}(x)}{C_{1,0}(x)}\lambda_n(x)\frac{\prod_{k=1}^{\infty}(1-\frac{\lambda_n(x)}{\mu_k(x)})}{\prod_{k\neq n}^{\infty}(1-\frac{\lambda_n(x)}{\lambda_k(x)})}.$$

On the other hand, replacing λ by $\mu_n(x)$ in (4.3), the first statement in the relation vanishes and we have

$$\begin{aligned}&C_{2,0}(x)\frac{\mu_n'(x)}{\mu_n^2(x)}\cdot\mu_n(x)\prod_{k\neq n,1\leq k}(1-\frac{\mu_n(x)}{\mu_k(x)})\\&+[\mu_n(x)\phi^2(x)-q(x)]C_{1,0}(x)\prod_{k\geq 1}(1-\frac{\mu_n(x)}{\lambda_k(x)})=0,\end{aligned}$$

so, we obtain

$$\mu_n'(x)=-\frac{C_{1,0}(x)}{C_{2,0}(x)}[\mu_n^2(x)\phi^2(x)-\mu_n(x)q(x)]\frac{\prod_{k=1}^{\infty}(1-\frac{\mu_n(x)}{\lambda_k(x)})}{\prod_{k\neq n}^{\infty}(1-\frac{\mu_n(x)}{\mu_k(x)})}.$$

By inserting $q(x)=\frac{C_{2,0}'(x)}{C_{1,0}(x)}$ from theorem 4.2, the proof is complete. □

Theorem 4.3 which is the main result of this article gives us an algorithm for the solution of the inverse problem, i.e., retrieving $q(x)$ in $(0,1)$.

Algorithm. Suppose that $\phi^2(t)=(t-t_0)^{l_0}\phi_0(t)$ is given where l_0 is odd and $\phi^2(t)(t-t_0)^{-l_0}>0$ in $[0,t_0)\cup(t_0,1]$; i.e., t_0 is a turning point of type IV and the sequences $\{\lambda_n^-\}$, $\{\lambda_n^+\}$, $\{\mu_n^-\}$ and $\{\mu_n^+\}$ satisfy the following relations:

$$\begin{aligned}\sqrt{\lambda_n^+}&=\frac{n\pi-\frac{\pi}{4}}{\int_{t_0}^1|\phi(\tau)|d\tau}+O(\frac{1}{n}),\quad \sqrt{-\lambda_n^-}=\frac{n\pi-\frac{\pi}{4}}{\int_0^{t_0}|\phi(\tau)|d\tau}+O(\frac{1}{n}),\\ \sqrt{\mu_n^+}&=\frac{n\pi-3\frac{\pi}{4}}{\int_{t_0}^1|\phi(\tau)|d\tau}+O(\frac{1}{n}),\quad \sqrt{-\mu_n^-}=\frac{n\pi-\frac{\pi}{4}}{\int_0^{t_0}|\phi(\tau)|d\tau)}+O(\frac{1}{n}).\end{aligned}$$

(1) By solving the dual equation (4.5) with initial conditions

$$\lambda_n^-(1)=\lambda_n^-,\quad \lambda_n^+(1)=\lambda_n^+,\quad \mu_n^-(1)=\mu_n^-,\quad \mu_n^+(1)=\mu_n^+, \tag{4.8}$$

we find $\lambda_n^-(x)$, $\lambda_n^+(x)$, $\mu_n^-(x)$ and $\mu_n^+(x)$ for $x\in(t_0,1)$.

(2) Calculate $q(x) = \frac{C'_{2,1}(x)}{C_{1,1}(x)}$ where $C_{1,1}(x)$ and $C_{2,1}(x)$ are defined in (3.23) and (3.28), respectively.
(3) By solving the dual equation (4.4) with initial conditions

$$\lambda_n(t_0) = \lim_{x\to t_0^+} \lambda_n^-(x), \quad \mu_n(t_0) = \lim_{x\to t_0^+} \mu_n^-(x), \tag{4.9}$$

we find $\lambda_n(x)$ and $\mu_n(x)$ for $x \in (0, t_0)$.
(4) Calculate $q(x) = \frac{C'_{2,0}(x)}{C_{1,0}(x)}$ where $C_{1,0}(x)$ and $C_{2,0}(x)$ are defined in (3.15) and (3.19), respectively.

Remark 4.4. It is obvious that the system of equations (4.4) are dual equations for indefinite Sturm-Liouville equation (1.1), corresponds to the system of equations (1.5) in the classic Sturm-Liouville case (vibrating string). It means that the classical result is a particular case of our result; i.e., by inserting $q(x) \equiv 0$, $C_{1,0}(x) = -(L-x)$ and $C_{2,0}(x) = 1$ in (4.4), one can obtain (1.5). We can use the method stated in [2] to show that the systems of equations (4.4) and (4.5) with initial conditions (4.9) and (4.8), respectively, satisfy the Lipschitz condition which guarantees the existence of a unique solution to the initial value problem.

Proposition 4.5. *Putting* (4.1) *in* (4.2) *for* $0 < x < t_0$, *we obtain*

$$\phi^2(x) = q(x)\sum_i \Big(\frac{1}{\mu_i(x)} - \frac{1}{\lambda_i(x)}\Big) - \frac{C_{2,0}(x)}{C_{1,0}(x)}\sum_i \frac{\mu_i'(x)}{\mu_i^2(x)},$$

which shows the relationship between weight function $\phi^2(x)$ *and potential function* $q(x)$ *by means of eigenvalues* $\{\lambda_n(x)\}$ *and* $\{\mu_n(x)\}$. *The same relation can be written for* $t_0 < x < 1$.

Proposition 4.6. *By differentiating relation* (3.13) *with respect to* x, $0 < x < t_0$, *and then replacing* λ *by* $\mu_n(x)$ *for each* $n \in \mathbf{N}$, *we can write*

$$S'(x,\mu_n(x)) = C'_{1,0}(x)\prod_{k=1}^{\infty}\Big(1-\frac{\mu_n(x)}{\lambda_k(x)}\Big)+\mu_n(x)\cdot C_{1,0}(x)\sum_i \frac{\lambda_i'(x)}{\lambda_i^2(x)}\prod_{k\neq i,1\le k}\Big(1-\frac{\mu_n(x)}{\lambda_k(x)}\Big).$$

On the other hand, $S'(x,\mu_n(x)) = 0$, *so*

$$\prod_k\Big(1-\frac{\mu_n(x)}{\lambda_k(x)}\Big)\Big\{C'_{1,0}(x) + \mu_n(x)\cdot C_{1,0}(x)\sum_i \frac{\lambda_i'(x)}{\lambda_i(x)}\cdot\frac{1}{\lambda_i(x)-\mu_n(x)}\Big\} = 0.$$

From lemma 4.1 this implies

$$C_{2,0}(x) + \mu_n(x)\cdot C_{1,0}(x)\sum_i \frac{\lambda_i'(x)}{\lambda_i(x)}\frac{1}{\lambda_i(x)-\mu_n(x)} = 0, \quad \forall n \in \mathbf{N},$$

which represents the relationship between eigenvalues and coefficients $C_{1,0}(x)$ *and* $C_{2,0}(x)$. *The same relation can be written for* $t_0 < x < 1$.

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