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MULTIPLE SOLUTIONS FOR SEMILINEAR ELLIPTIC EQUATIONS WITH SIGN-CHANGING POTENTIAL AND NONLINEARITY

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Abstract. In this article, we study the multiplicity of solutions for the semilinear elliptic equation

$$
\begin{gathered}
-\Delta u + a(x)u = f(x,u), \quad x \in \Omega, \\
u = 0, \quad x \in \partial\Omega,
\end{gathered}
$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$), the potential $a(x)$ satisfies suitable integrability conditions, and the primitive of the nonlinearity f is of super-quadratic growth near infinity and is allowed to change sign. Our super-quadratic conditions are weaker the usual super-quadratic conditions.

1. Introduction

Consider the semilinear elliptic equation

$$
\begin{gathered}
-\Delta u + a(x)u = f(x,u), \quad x \in \Omega, \\
u = 0, \quad x \in \partial\Omega,
\end{gathered}
\tag{1.1}
$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded domain with smooth boundary $\partial\Omega$, $f \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$ and $a(x) \in L^{N/2}(\Omega)$.

Semilinear elliptic equations have found a great deal of interest last years. With the aid of variational methods, the existence and multiplicity of nontrivial solutions for problem (1.1) have been extensively investigated in the literature over the past several decades. See (e.g., [2]-[6], [8]-[11], [21] and the references quoted in them).

In most of the above references, the following condition due to Ambrosetti-Rabinowitz [1] is assumed:

(AR) There exists $\mu > 2$ such that

$$
0 < \mu F(x,u) \leq u f(x,u), \quad u \neq 0;
$$

here and in the sequel, $F(x,u) = \int_0^u f(x,s)ds$.

The role of (AR) is to ensure the boundedness of the Palais-Smale sequences of the energy functional. This is very crucial in applying the critical point theory.

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However, there are many functions which are superlinear at infinity, but do not satisfy the condition (AR) for any $\mu > 2$, for example the superlinear function

$$f(x,u) = u\ln(1+|u|) \tag{1.2}$$

does not satisfy (AR). In fact, (AR) implies that $F(x,u) \geq C|u|^\mu$ for some $C > 0$.

In references [8]-[10] and [15], some new super-quadratic conditions are established instead of (AR), Among them, a few are weaker than (AR), but most complement with it, for example, the monotonicity condition on $f(x,u)/u$. In [8], the authors obtained the infinitely many solutions of (1.1) under some weak super-quadratic conditions, but the conditions there actually imply that $F(x,u)$ is of μ-order ($\mu > 2$) growth near infinity with respect to u.

In a recent paper [21], the authors studied the existence of infinitely many nontrivial solutions of (1.1) under the following assumptions:

(S1) $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$, and there exist constants $c_1 > 0$ and $p \in (2, 2^*)$ such that

$$|f(x,u)| \leq c_1(1+|u|^{p-1}), \quad \forall (x,u) \in \Omega \times \mathbb{R}; \tag{1.3}$$

where $2^* := 2N/(N-2)$, $N \geq 3$.

(S2) $F(x,u) \geq 0$ for all $(x,u) \in \Omega \times \mathbb{R}$, $\lim_{|u|\to\infty} \frac{F(x,u)}{|u|^2} = \infty$, uniformly in $x \in \Omega$;

(S3) There exists constant $\varrho > \max\{2N/(N+2), N(p-2)/2\}$ and $d > 0$ such that

$$\liminf_{|u|\to\infty} \frac{uf(x,u) - 2F(x,u)}{|u|^\varrho} \geq d$$

uniformly for $x \in \Omega$;

(S4) $f(x,-u) = -f(x,u)$ for all $(x,u) \in \Omega \times \mathbb{R}$.

Specifically, the authors established the following theorem in [21].

Theorem 1.1 ([21]). *Assume that f satisfy* (S1)–(S4). *Then problem* (1.1) *possesses infinitely many nontrivial solutions.*

Condition (S3) is just the same as the condition $(F_2)_\mu$ in [4], which plays an important role in proving boundedness of the Palais-Smale sequences.

In the present paper, we will further study multiplicity of solutions for problem (1.1) under the assumptions (S1) and (S4), instead of (S2) and (S3), we give the following more general super-quadratic conditions near infinity.

(S2') $\lim_{|u|\to\infty} \frac{|F(x,u)|}{|u|^2} = \infty$, a.e. $x \in \Omega$, and there exists $r_0 \geq 0$ such that

$$F(x,u) \geq 0, \quad \forall (x,u) \in \Omega \times \mathbb{R},\ |u| \geq r_0;$$

(S5) $\mathcal{F}(x,u) := \frac{1}{2}uf(x,u) - F(x,u) \geq 0$, and there exists $c_0 > 0$ and $\kappa > N/2$ such that

$$|F(x,u)|^\kappa \leq c_0|u|^{2\kappa}\mathcal{F}(x,u), \quad \forall (x,u) \in \Omega \times \mathbb{R},\ |u| \geq r_0;$$

(S6) There exist $\mu > 2$ and $\lambda > 0$ such that

$$\mu F(x,u) \leq uf(x,u) + \lambda u^2, \quad \forall (x,u) \in \Omega \times \mathbb{R}.$$

Now, we are ready to state the main results of this article.

Theorem 1.2. *Assume that f satisfy* (S1), (S2'), (S4), (S5). *Then problem* (1.1) *possesses infinitely many nontrivial solutions.*

Theorem 1.3. *Assume that f satisfy* (S1), (S2'), (S4), (S6). *Then problem* (1.1) *possesses infinitely many nontrivial solutions.*

Remark 1.4. In our theorems, $F(x,u)$ is allowed change sign. There exists functions, with F sign-changing and satisfying (S5), but not satisfying (S3); for example

$$f(x,u) = u\ln(\frac{1}{2} + |u|).$$

Observe that, if we take $p \in (2+\frac{4}{N}, 2^*)$, condition (S1) is satisfied, and $N(p-2)/2 \geq 2$, $\varrho > 2$ (given in (S3)), but there is no positive d such that

$$\liminf_{|u|\to\infty} \frac{uf(x,u) - 2F(x,u)}{|u|^{\varrho}} \geq d;$$

then condition (S3) can not be satisfied. However, f satisfies (S5). Thus, the assumptions (S2') and (S5) or (S6) are weaker than the super-quadratic conditions obtained in the aforementioned references. It is easy to check that the following nonlinearities f satisfy (S2') and (S5) or (S6):

$$f(x,u) = a(x)u\ln\big(\frac{1}{2} + |u|\big), \tag{1.4}$$

$$f(x,u) = a(x)[4u^4 + 2u^2\sin u - 4u\cos u], \tag{1.5}$$

$$f(x,u) = a(x)\sum_{i=1}^{m} b_i|u|^{\beta_i}u, \tag{1.6}$$

where $b_1 > 0$, $b_i \in \mathbb{R}$, $i = 2,3,\dots,m$, $\beta_1 > \beta_2 > \dots > \beta_m \geq 0$, $a(x) \in C(\Omega,\mathbb{R})$, and $0 < \inf_\Omega a(x) \leq \sup_\Omega a(x) < \infty$.

2. Variational setting and proofs of the main results

Denote by $\Lambda := -\Delta + a$ the associated self-adjoint operator in $L^2(\Omega)$ with domain $D(\Lambda)$. By [7, Theorem VI.1.4] or see [19, paragraph 2.4], we know that $D(\Lambda)$ is dense as a subset of $H_0^1(\Omega)$, and the spectrum of it consists of only eigenvalues numbered in $-\infty < \mu_1 \leq \mu_2 \leq \dots \leq \mu_n \leq 0 < \mu_{n+1} \leq \dots \to +\infty$ (counting multiplicity) with a corresponding system of eigenfunctions $\{e_n\}$ forming an orthogonal basis in $L^2(\Omega)$.

In the following, let $|\Lambda|$ be the absolute value of Λ, and $|\Lambda|^{1/2}$ be the square root of $|\Lambda|$ with domain $D(|\Lambda|^{1/2})$, we know that $E := D(|\Lambda|^{1/2}) = H_0^1(\Omega)$. Let θ be a positive constant with $\mu_1 > -\theta$, where μ_1 is the smallest eigenvalue of Λ, then $\Lambda + \theta I > 0$. We introduce a new inner product on E by

$$(u,v) = ((\Lambda + \theta I)^{1/2}u, (\Lambda + \theta I)^{1/2}v)_2 = \int_\Omega [\nabla u \cdot \nabla v + (a(x) + \theta)uv]dx \tag{2.1}$$

for $u, v \in E$, and the associated norm

$$\|u\| = (u,u)^{1/2} = \Big(\int_\Omega [|\nabla u|^2 + (a(x) + \theta)|u|^2]dx\Big)^{1/2}, \quad u \in E, \tag{2.2}$$

where $(\cdot,\cdot)_2$ denote the inner product of $L_2(\Omega)$. Then $\|\cdot\|$ is equivalent to the usual Soblev norm $\|\cdot\|_{1,2}$.

Let $V(x) = a(x) + \theta$ and $g(x,u) = f(x,u) + \theta u$. It is easy to check that the hypotheses (S1), (S2'), (S5) and (S6) still hold for $g(x,u)$ provided that those hold for $f(x,u)$. Hence we have the following lemma.

Lemma 2.1. *Problem* (1.1) *is equivalent to the problem*

$$\begin{gathered} -\Delta u + V(x)u = g(x,u), \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{gathered} \tag{2.3}$$

Since $\|\cdot\|$ is equivalent to the usual Sobloev norm $\|\cdot\|_{1,2}$, we obtain the following lemma.

Lemma 2.2. *The space E is compactly embedded in $L^s(\Omega)$ for $1 \le s < 2^*$, and continuously embedded in $L^{2^*}(\Omega)$, hence there exists $\gamma_s > 0$ such that*

$$\|u\|_s \le \gamma_s \|u\|, \quad \forall u \in E, \tag{2.4}$$

where $\|u\|_s$ denotes the usual norm in $L^s(\Omega)$ for all $1 \le s \le 2^*$.

Now, we define a function Φ on E by

$$\Phi(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + V(x)u^2)dx - \Psi(u), \tag{2.5}$$

where $\Psi(u) = \int_\Omega G(x,u)dx$, by (S1) we have

$$|G(x,u)| \le c_1|u| + \frac{c_1}{p}|u|^p \quad \forall (x,u) \in \Omega \times \mathbb{R}; \tag{2.6}$$

here and in the sequel, $G(x,u) = \int_0^u g(x,s)ds$. In view of (2.6) and Lemma 2.2, Φ and Ψ are well defined, furthermore, we have the following statement.

Proposition 2.3. *Suppose* (S1) *is satisfied. Then $\Psi \in C^1(E,\mathbb{R})$ and $\Psi' : E \to E^*$ is compact and hence the functional Φ is of class $C^1(E,\mathbb{R})$. Moreover,*

$$\Phi(u) = \frac{1}{2}\|u\|^2 - \int_\Omega G(x,u)dx, \quad \forall u \in E, \tag{2.7}$$

$$\langle \Phi'(u), v\rangle = (u,v) - \int_\Omega g(x,u)vdx, \quad \forall u,v \in E. \tag{2.8}$$

By Lemma 2.2, the proof of the above proposition is standard; we refer the reader to to [16, 19].

Lemma 2.4 ([16]). *Let X be an infinite dimensional Banach space, $X = Y \oplus Z$, where Y is finite dimensional. If $I \in C^1(X,\mathbb{R})$ satisfies (C) $_c$-condition for all $c > 0$, and*

- (I1) *$I(0) = 0$, $I(-u) = I(u)$ for all $u \in X$;*
- (I2) *there exist constants $\rho, \alpha > 0$ such that $\Phi|_{\partial B_\rho \cap Z} \ge \alpha$;*
- (I3) *for any finite dimensional subspace $\tilde{X} \subset X$, there is $R = R(\tilde{X}) > 0$ such that $I(u) \le 0$ on $\tilde{X} \setminus B_R$;*

then I possesses an unbounded sequence of critical values.

Lemma 2.5. *Under assumptions* (S1), (S2'), (S5)*, any sequence $\{u_n\} \subset E$ satisfying*

$$\Phi(u_n) \to c > 0, \quad \langle \Phi'(u_n), u_n\rangle \to 0 \tag{2.9}$$

is bounded in E.

Proof. To prove the boundedness of $\{u_n\}$, arguing by contradiction, suppose that $\|u_n\| \to \infty$. Let $v_n = u_n/\|u_n\|$. Then $\|v_n\| = 1$ and $\|v_n\|_s \le \gamma_s\|v_n\| = \gamma_s$ for $1 \le s \le 2^*$. Observe that for n large

$$c + 1 \ge \Phi(u_n) - \frac{1}{2}\langle \Phi'(u_n), u_n\rangle = \int_\Omega \mathcal{G}(x,u_n)dx. \tag{2.10}$$

Here and in the sequel $\mathcal{G} = \frac{1}{2}ug(x,u) - G(x,u)$. It follows from (2.7) and (2.9) that

$$\frac{1}{2} \le \limsup_{n\to\infty} \int_\Omega \frac{|G(x,u_n)|}{\|u_n\|^2} dx. \tag{2.11}$$

For $0 \le a < b$, let

$$\Omega_n(a,b) = \{x \in \Omega : a \le |u_n(x)| < b\}. \tag{2.12}$$

Passing to a subsequence, we may assume that $v_n \rightharpoonup v$ in E, then by Lemma 2.2, $v_n \to v$ in $L^s(\Omega)$, $1 \le s < 2^*$, and $v_n \to v$ a.e. on Ω.

If $v = 0$, then $v_n \to 0$ in $L^s(\Omega)$, $1 \le s < 2^*$, $v_n \to 0$ a.e. on Ω. Hence, it follows from (2.6) that

$$\begin{aligned} \int_{\Omega_n(0,r_0)} \frac{|G(x,u_n)|}{|u_n|^2} |v_n|^2 dx &\le \big(c_1 + \frac{c_1}{p} r_0^{p-1}\big) \int_{\Omega_n(0,r_0)} \frac{|v_n|}{\|u_n\|} dx \\ &\le \big(c_1 + \frac{c_1}{p} r_0^{p-1}\big) \int_\Omega \frac{|v_n|}{\|u_n\|} dx \to 0. \end{aligned} \tag{2.13}$$

Set $\kappa' = \kappa/(\kappa-1)$. Since $\kappa > N/2$, one sees that $2\kappa' \in (2, 2^*)$. Hence, from (S5) and (2.10), one has

$$\begin{aligned} &\int_{\Omega_n(r_0,\infty)} \frac{|G(x,u_n)|}{|u_n|^2} |v_n|^2 dx \\ &\le \Big[\int_{\Omega_n(r_0,\infty)} \Big(\frac{|G(x,u_n)|}{|u_n|^2}\Big)^\kappa dx\Big]^{1/\kappa} \Big[\int_{\Omega_n(r_0,\infty)} |v_n|^{2\kappa'} dx\Big]^{1/\kappa'} \\ &\le c_0^{1/\kappa} \Big[\int_{\Omega_n(r_0,\infty)} \mathcal{G}(x,u_n) dx\Big]^{1/\kappa} \Big(\int_\Omega |v_n|^{2\kappa'} dx\Big)^{1/\kappa'} \\ &\le [c_0(c+1)]^{1/\kappa} \Big(\int_\Omega |v_n|^{2\kappa'} dx\Big)^{1/\kappa'} \to 0. \end{aligned} \tag{2.14}$$

Combining (2.13) with (2.14), we have

$$\int_\Omega \frac{|G(x,u_n)|}{\|u_n\|^2} dx = \int_{\Omega_n(0,r_0)} \frac{|G(x,u_n)|}{|u_n|^2} |v_n|^2 dx + \int_{\Omega_n(r_0,\infty)} \frac{|G(x,u_n)|}{|u_n|^2} |v_n|^2 dx \to 0,$$

which contradicts (2.11).

Set $A := \{x \in \Omega : v(x) \neq 0\}$. If $v \neq 0$, then $\mathrm{meas}(A) > 0$. For a.e. $x \in A$, we have $\lim_{n\to\infty} |u_n(x)| = \infty$. Hence $A \subset \Omega_n(r_0,\infty)$ for large $n \in \mathbb{N}$, it follows from

(2.6), (2.7), (S2'), Lemma 2.2 and Fadou's Lemma that

$$
\begin{aligned}
0 &= \lim_{n\to\infty} \frac{c+o(1)}{\|u_n\|^2} = \lim_{n\to\infty} \frac{\Phi(u_n)}{\|u_n\|^2} \\
&= \lim_{n\to\infty} \Big[\frac{1}{2} - \int_\Omega \frac{G(x,u_n)}{|u_n|^2}|v_n|^2 dx\Big] \\
&= \lim_{n\to\infty} \Big[\frac{1}{2} - \int_{\Omega_n(0,r_0)} \frac{G(x,u_n)}{|u_n|^2}|v_n|^2 dx - \int_{\Omega_n(r_0,\infty)} \frac{G(x,u_n)}{|u_n|^2}|v_n|^2 dx\Big] \\
&\le \limsup_{n\to\infty} \Big[\frac{1}{2} + \big(c_1 + \frac{c_1}{p} r_0^{p-1}\big) \int_\Omega \frac{|v_n|}{\|u_n\|} dx - \int_{\Omega_n(r_0,\infty)} \frac{G(x,u_n)}{|u_n|^2}|v_n|^2 dx\Big] \\
&\le \frac{1}{2} + (c_1 + \frac{c_1}{p} r_0^{p-1}) \limsup_{n\to\infty} \frac{\|v_n\|_1}{\|u_n\|} - \liminf_{n\to\infty} \int_\Omega \frac{|G(x,u_n)|}{u_n^2} [\chi_{\Omega_n(r_0,\infty)}(x)] v_n^2 dx \\
&\le \frac{1}{2} + (c_1 + \frac{c_1}{p} r_0^{p-1}) \limsup_{n\to\infty} \frac{\gamma_1}{\|u_n\|} - \int_\Omega \liminf_{n\to\infty} \frac{G(x,u_n)}{|u_n|^2} [\chi_{\Omega_n(r_0,\infty)}(x)] |v_n|^2 dx \\
&= -\infty,
\end{aligned} \tag{2.15}
$$

which is a contradiction. Thus $\{u_n\}$ is bounded in E. □

Lemma 2.6. *Under assumptions* (S1), (S2'), (S5)*, any sequence* $\{u_n\} \subset E$ *satisfying* (2.9) *has a convergent subsequence in* E*.*

Proof. Lemma 2.5 implies that $\{u_n\}$ is bounded in E. Going if necessary to a subsequence, we can assume that $u_n \rightharpoonup u$ in E. By Lemma 2.2, $u_n \to u$ in $L^s(\Omega)$ for $1 \le s < 2^*$ and $u_n \to u$ a.e. on Ω. By (S1), Hölder inequality and Lemma 2.2 again, one can easily gets that

$$
\int_\Omega [g(x,u_n) - g(x,u)](u_n - u) dx \to 0. \tag{2.16}
$$

Observe that

$$
\|u_n - u\|^2 = \langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle + \int_\Omega [g(x,u_n) - g(x,u)](u_n - u) dx, \tag{2.17}
$$

it is clear that

$$
\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle \to 0, \quad n \to \infty. \tag{2.18}
$$

By (2.16)–(2.18), we have $\|u_n - u\| \to 0$ as $n \to \infty$. □

Lemma 2.7. *Under assumptions* (S1), (S2'), (S6)*, any sequence* $\{u_n\} \subset E$ *satisfying* (2.9) *has a convergent subsequence in* E*.*

Proof. First, we prove that $\{u_n\}$ is bounded in E. To prove the boundedness of $\{u_n\}$, arguing by contradiction, suppose that $\|u_n\| \to \infty$. Let $v_n = u_n/\|u_n\|$. Then $\|v_n\| = 1$ and $\|v_n\|_s \le \gamma_s \|v_n\| = \gamma_s$ for $1 \le s < 2^*$. By (2.7)–(2.9) and (S6), one has

$$
\begin{aligned}
c + 1 &\ge \Phi(u_n) - \frac{1}{\mu}\langle \Phi'(u_n), u_n\rangle \\
&= \frac{\mu-2}{2\mu}\|u_n\|^2 + \int_\Omega \Big[\frac{1}{\mu} g(x,u_n)u_n - G(x,u_n)\Big] dx \\
&\ge \frac{\mu-2}{2\mu}\|u_n\|^2 - \frac{\lambda}{\mu}\|u_n\|_2^2, \quad \text{for large } n \in \mathbb{N},
\end{aligned}
$$

which implies

$$1 \le \frac{2\lambda}{\mu-2}\limsup_{n\to\infty}\|v_n\|_2^2. \tag{2.19}$$

Passing to a subsequence, we may assume that $v_n \rightharpoonup v$ in E, then by Lemma 2.2, $v_n \to v$ in $L^s(\Omega)$, $1 \le s < 2^*$, and $v_n \to v$ a.e. on Ω. Hence, it follows from (2.19) that $v \neq 0$. By a similar fashion as (2.15), we can conclude a contradiction. Thus, $\{u_n\}$ is bounded in E. The rest proof is the same as that in Lemma 2.6. $\square$

Lemma 2.8. *Under assumptions* (S1), (S2'), *for any finite dimensional subspace $\tilde{E} \subset E$, there holds*

$$\Phi(u) \to -\infty, \quad \|u\| \to \infty,\ u \in \tilde{E}. \tag{2.20}$$

Proof. Arguing indirectly, assume that for some sequence $\{u_n\} \subset \tilde{E}$ with $\|u_n\| \to \infty$, there is $M > 0$ such that $\Phi(u_n) \ge -M$ for all $n \in \mathbb{N}$. Set $v_n = u_n/\|u_n\|$, then $\|v_n\| = 1$. Passing to a subsequence, we may assume that $v_n \rightharpoonup v$ in E. Since $\tilde{E}$ is finite dimensional, then $v_n \to v \in \tilde{E} \subset E$, $v_n \to v$ a.e. on Ω, and so $\|v\| = 1$. Hence, we can conclude a contradiction by a similar fashion as (2.15). $\square$

Corollary 2.9. *Under assumptions (S1), (S2'), for any finite dimensional subspace $\tilde{E} \subset E$, there is $R = R(\tilde{E}) > 0$ such that*

$$\Phi(u) \le 0, \quad \forall u \in \tilde{E},\ \|u\| \ge R. \tag{2.21}$$

Let $\{e_j\}$ be a total orthonormal basis of E and $X_j = \mathbb{R}e_j$,

$$Y_k = \oplus_{j=1}^{k} X_j, \quad Z_k = \oplus_{j=k+1}^{\infty} X_j, \quad k \in \mathbb{Z}\,. \tag{2.22}$$

Lemma 2.10. *If $1 \le s < 2^*$, then*

$$\beta_k(s) := \sup_{u\in Z_k, \|u\|=1} \|u\|_s \to 0, \quad k \to \infty.$$

Since the embedding from E into $L^s(\Omega)$ is compact, then the above can be proved by a similar fashion as [19, Lemma 3.8].

By Lemma 2.10, we can choose an integer m big enough such that

$$\|u\|_1 \le \frac{1}{4c_1}\|u\|^2, \quad \|u\|_p^p \le \frac{p}{4c_1}\|u\|^p, \quad \forall u \in Z_m. \tag{2.23}$$

Lemma 2.11. *Under assumption* (S1), *there exist constants $\rho, \alpha > 0$ such that $\Phi|_{\partial B_\rho \cap Z_m} \ge \alpha$.*

Proof. By (2.6), (2.7) and (2.23), we have

$$\begin{aligned}
\Phi(u) &= \frac{1}{2}\|u\|^2 - \int_\Omega G(x,u)dx \\
&\ge \frac{1}{2}\|u\|^2 - c_1\|u\|_1 - \frac{c_1}{p}\|u\|_p^p \\
&\ge \frac{1}{4}(\|u\|^2 - \|u\|^p) \\
&= \frac{2^{p-2}-1}{2^{p+2}} := \alpha, \quad \forall u \in Z_m, \quad \|u\| = \frac{1}{2} := \rho.
\end{aligned}$$

$\square$

Proof of Theorem 1.2. Let $X = E$, $Y = Y_m$ and $Z = Z_m$. By Lemmas 2.5, 2.6, 2.11 and Corollary 2.9, all conditions of Lemma 2.4 are satisfied. Thus, problem (2.3) possesses infinitely many nontrivial solutions. By Lemma 2.1, problem (1.1) also possesses infinitely many nontrivial solutions. $\square$

Proof of Theorem 1.3. Let $X = E$, $Y = Y_m$ and $Z = Z_m$. The rest proof is the same as that of Theorem 1.2, by using Lemma 2.7 instead of Lemmas 2.5 and 2.6. $\square$

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