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HOMOGENIZATION OF A SYSTEM OF SEMILINEAR DIFFUSION-REACTION EQUATIONS IN AN $H^{1,p}$ SETTING

HARI SHANKAR MAHATO, MICHAEL BÖHM

ABSTRACT. In this article, homogenization of a system of semilinear multi-species diffusion-reaction equations is shown. The presence of highly nonlinear reaction rate terms on the right-hand side of the equations make the model difficult to analyze. We obtain some a-priori estimates of the solution which give the strong and two-scale convergences of the solution. We homogenize this system of diffusion-reaction equations by passing to the limit using two-scale convergence.

1. INTRODUCTION

The existence of a unique global positive weak solution u that belongs the space $[H^{1,p}((0,T);H^{1,q}(\Omega)^*)\cap L^p((0,T);H^{1,p}(\Omega))]^I$ is shown in [15] (by taking $\vec{q}=0$) for a system of semilinear diffusion-reaction equations

$$\frac{\partial u}{\partial t} - \nabla(\, D\nabla u - \vec{q}u) = SR(u) \quad \text{in } (0,T)\times\Omega, \tag{1.1}$$

$$-D\nabla u\cdot\vec{n} = 0 \quad \text{on } (0,T)\times\partial\Omega, \tag{1.2}$$

$$u(0,x) = u_0(x) \quad \text{in } \Omega \tag{1.3}$$

under the assumptions:

(i) $p > n+2$;
(ii) $u_0 \geq 0$, i.e., $u_{0_i} \geq 0$ for $i = 1,2,\dots,I$;
(iii) $u_{0_i} \in (H^{1,q}(\Omega)^*, H^{1,p}(\Omega))_{1-\frac{1}{p},p}$ for $i = 1,2,\dots,I$,
(iv) all reactions are linearly independent such that the stoichiometric matrix $S = (s_{ij})_{1\leq j\leq J,\,1\leq i\leq I}$ has maximal column rank; i.e., $\operatorname{rank}(S) = J$,

where $I \in \mathbb{Z}^+$, $\frac{1}{p}+\frac{1}{q}=1$, $\Omega\subset\mathbb{R}^n$ a bounded domain with sufficiently smooth boundary, $D>0$ a constant (see remark 1.1) and $SR(u)$ the reaction rate vector (see (1.5)). Here $u := (u_1,u_2,\dots,u_I)$ is the concentration vector of I chemical species involved in J reactions given by

$$\tau_{1j}X_1 + \tau_{2j}X_2 + \dots + \tau_{Ij}X_I \rightleftharpoons \nu_{1j}X_1 + \nu_{2j}X_2 + \dots + \nu_{Ij}X_I, \text{ for } 1\leq j\leq J, \tag{1.4}$$

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where X_i, $1 \le i \le I$, denotes the chemical species and the stoichiometric coefficients $-\tau_{ij} \in \mathbb{Z}_0^-$ and $\nu_{ij} \in \mathbb{Z}_0^+$ respectively. Set $s_{ij} = \nu_{ij} - \tau_{ij}$. The reaction rate for the *i-th* species is given by

$$(SR(u))_i = \sum_{j=1}^{J} s_{ij}\Big(k_j^f \prod_{\substack{m=1\\ s_{mj}<0}}^{I} u_m^{-s_{mj}} - k_j^b \prod_{\substack{m=1\\ s_{mj}>0}}^{I} u_m^{s_{mj}}\Big) \quad \text{for } i = 1, 2, \dots, I, \tag{1.5}$$

where k_j^f (> 0) and k_j^b (> 0) are the forward and backward reaction rate factors respectively. cf. [11, 14, 15].

Remark 1.1. The modeling of transport processes in a porous medium very often lead to the equations of type (1.1)-(1.3). In some situations the advective flux dominates diffusion and even though diffusion coefficients actually vary from species to species, we can consider the same value of the diffusion coefficients for all the species. However in this paper (also in [15, 11]), due to mathematical technicality we have considered the same diffusion coefficients for all the species.

In this article, we consider (1.1)-(1.3), assuming $\vec{q} = 0$, in the context of a porous medium and upscale the model via periodic homogenization in an appropriate function space setting (see section 4). The global existence of the solution of (1.1)-(1.3) for $\vec{q} = 0$ considered at the micro scale follows by the techniques used in [15] (see theorem 3.1).

To fix the ideas, let $Y := (0,1)^n \subset \mathbb{R}^n$ be a unit representative cell which is composed of a solid part Y^s with boundary Γ and a pore part Y^p such that $Y = Y^s \cup Y^p$, $\bar{Y}^s \subset Y$ and $\bar{Y}^s \cap \bar{Y}^p = \Gamma$. Suppose that Ω is a porous medium with pore space Ω^p and solid parts Ω^s such that $\Omega := \Omega^p \cup \Omega^s$. The boundary of Ω^s is denoted by Γ^* and the outer boundary of Ω is denoted by $\partial\Omega$. Γ, Γ^* and $\partial\Omega$ are assumed to be sufficiently smooth. Assume further that Ω is periodic (the solid parts in Ω are periodically distributed) and covered by a finite union of the cell Y. To avoid technical difficulties, we postulate:

- solid parts do not touch the boundary $\partial\Omega$,
- solid parts do not touch each other,
- solid parts do not touch the boundary of Y.

We use the standard notation (cf. [16, 20], e.g.). Let $\varepsilon > 0$ be the scale parameter and Ω be covered by a finite union of translated versions of εY_k cells such that $\varepsilon Y_k \subset \Omega$ for $k \in \mathbb{Z}^n$. We also define

$$\Omega_\varepsilon^p := \cup_{k\in\mathbb{Z}^n}\{\varepsilon Y_k^p : \varepsilon Y_k^p \subset \Omega\}, \tag{1.6}$$

$$\Omega_\varepsilon^s := \cup_{k\in\mathbb{Z}^n}\{\varepsilon Y_k^s : \varepsilon Y_k^s \subset \Omega\}, \tag{1.7}$$

$$\Gamma_\varepsilon := \cup_{k\in\mathbb{Z}^n}\{\varepsilon \Gamma_k : \varepsilon \Gamma_k \subset \Omega\}, \tag{1.8}$$

$$\partial\Omega_\varepsilon^p := \partial\Omega \cup \Gamma_\varepsilon. \tag{1.9}$$

We denote by dx and dy the volume elements in Ω and Y, and by $d\sigma_y$ and $d\sigma_x$ the surface elements on Γ and Γ_ε respectively. The characteristic (indicator) function of Ω_ε^p in Ω denoted by

$$\chi^\varepsilon(x) = \chi(\frac{x}{\varepsilon}) \tag{1.10}$$

is defined as

$$\chi^\varepsilon(x) = \begin{cases} 1 & \text{for } x \in \Omega_\varepsilon^p, \\ 0 & \text{for } x \in \Omega - \Omega_\varepsilon^p. \end{cases} \tag{1.11}$$

For $T > 0$, $[0,T)$ denotes the time interval. Following the above notation, the system of diffusion-reaction equations in the pore space at the micro scale is given by

$$\frac{\partial u_\varepsilon}{\partial t} - \nabla \cdot D\nabla u_\varepsilon = SR(u_\varepsilon) \quad \text{in } (0,T)\times\Omega_\varepsilon^p, \tag{1.12}$$
$$u_\varepsilon(0,x) = u_0(x) \quad \text{in } \Omega_\varepsilon^p, \tag{1.13}$$
$$-D\nabla u_\varepsilon \cdot \vec{n} = 0 \quad \text{on } (0,T)\times\partial\Omega, \tag{1.14}$$
$$-D\nabla u_\varepsilon \cdot \vec{n} = 0 \quad \text{on } (0,T)\times\Gamma_\varepsilon. \tag{1.15}$$

We denote the problem (1.12)-(1.15) by (P_ε). The derivation of (P_ε) at the micro scale is motivated from the nondimensionalization of (1.1)-(1.3), for details see [27, 20]. Before we begin with the analysis of (1.12)-(1.15), we make the following assumptions:

$$p > n+2; \tag{1.16}$$
$$u_0 \geq 0, \text{ i.e., } u_{0_i} \geq 0 \text{ for all } i = 1,2,\dots,I; \tag{1.17}$$
$$u_{0_i} \in (H^{1,q}(\Omega_\varepsilon^p)^*, H^{1,p}(\Omega_\varepsilon^p))_{1-\frac{1}{p},p} \text{ for } i = 1,2,\dots,I. \tag{1.18}$$

all reactions are linearly independent such that the stoichiometric matrix

$$S = (s_{ij})_{1\leq j\leq J,\, 1\leq i\leq I} \text{ has maximal column rank, i.e., } \operatorname{rank}(S) = J; \tag{1.19}$$
$$\sup_{\varepsilon>0} \|u_{0_i}\|_{(H^{1,q}(\Omega_\varepsilon^p)^*, H^{1,p}(\Omega_\varepsilon^p))_{1-\frac{1}{p},p}} < \infty \text{ for } i = 1,2,\dots,I. \tag{1.20}$$

2. Mathematical Preliminaries

2.1. **Function Spaces.**

2.1.1. *Function Spaces on Ω.* Let $1 < r, s < \infty$ be such that $\frac{1}{r} + \frac{1}{s} = 1$. Assume that $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$. As usual, $L^r(\Omega)$ is the set of all equivalence classes of real-valued functions $u(.)$ such that $u(x)$ is defined for almost every $x \in \Omega$, is measurable and $|u(\cdot)|^r$ is Lebesgue integrable. $L^r(\Omega)$ is a Banach space with the norm

$$\|u\|_{L^r(\Omega)} = \begin{cases} \big[\int_\Omega |u(x)|^r\,dx\big]^{1/r} & \text{for } 1 \leq r < \infty, \\ \operatorname{ess\,sup}_{x\in\Omega} |u(x)| & \text{for } r = \infty. \end{cases} \tag{2.1}$$

The space $H^{1,r}(\Omega)$ is the usual Sobolev space with the norm

$$\|u\|_{H^{1,r}(\Omega)} = \begin{cases} \big[\|u\|_{L^r(\Omega)}^r + \|\nabla u\|_{L^r(\Omega)}^r\big]^{1/r} & \text{for } 1 \leq r < \infty, \\ \operatorname{ess\,sup}_{x\in\Omega}[|u(x)| + |\nabla u(x)|] & \text{for } r = \infty. \end{cases} \tag{2.2}$$

For a Banach space X, X^* denotes its dual and the duality pairing is denoted by $\langle\cdot,\cdot\rangle_{X^*\times X}$. Let $1 < p, q < \infty$ be such that $p > n+2$ and $\frac{1}{p} + \frac{1}{q} = 1$. We define the continuous embedding $L^p(\Omega) \hookrightarrow H^{1,q}(\Omega)^*$ as

$$\langle f, v\rangle_{H^{1,q}(\Omega)^*\times H^{1,q}(\Omega)} = \langle f, v\rangle_{L^p(\Omega)\times L^q(\Omega)} \quad \text{for } f \in L^p(\Omega),\ v \in H^{1,q}(\Omega). \tag{2.3}$$

The symbols $\hookrightarrow$ and $\hookrightarrow\hookrightarrow$ will denote the continuous and compact embeddings respectively. The *Sobolev-Bochner space* is

$$\begin{aligned} F &:= \big\{u \in L^p((0,T); H^{1,p}(\Omega)) : \frac{du}{dt} \in L^p((0,T); H^{1,q}(\Omega)^*)\big\} \\ &= H^{1,p}((0,T); H^{1,q}(\Omega)^*) \cap L^p((0,T); H^{1,p}(\Omega)) \end{aligned} \tag{2.4}$$

and for any $u \in F$,

$$\|u\|_F := \|u\|_{L^p((0,T);H^{1,p}(\Omega))} + \|u\|_{L^p((0,T);H^{1,q}(\Omega)^*)} + \|\frac{du}{dt}\|_{L^p((0,T);H^{1,q}(\Omega)^*)}, \tag{2.5}$$

where $\frac{du}{dt}$ is the distributional time derivative of u. For $0 < \theta < 1$, let

$$\begin{gathered}(H^{1,q}(\Omega)^*, H^{1,p}(\Omega))_{\theta,p} \text{ be the real-interpolation space between} \\ H^{1,q}(\Omega)^* \text{ and } H^{1,p}(\Omega),\end{gathered} \tag{2.6}$$

$$\begin{gathered}\big[H^{1,q}(\Omega)^*, H^{1,p}(\Omega)\big]_\theta \text{ be the complex-interpolation space between} \\ H^{1,q}(\Omega)^* \text{ and } H^{1,p}(\Omega)\end{gathered} \tag{2.7}$$

endowed with one of the usual norms (cf. [4, 26, 13, 9]). Now we introduce the norms on the vector-valued function spaces. Let $I \in \mathbb{N}$ and $u : \Omega \to \mathbb{R}^I$. We define

$$[L^p(\Omega)]^I := \underbrace{L^p(\Omega) \times L^p(\Omega) \times \cdots \times L^p(\Omega)}_{\text{I}-\text{times}} \tag{2.8}$$

and for $u \in [L^p(\Omega)]^I$ the corresponding norm is

$$|||u|||_{[L^p(\Omega)]^I} := \Big[\sum_{i=1}^I \|u_i\|^p_{L^p(\Omega)}\Big]^{1/p}. \tag{2.9}$$

Similarly,

$$|||u|||_{[L^\infty(\Omega)]^I} := \max_{1\le i\le I} \|u_i\|_{L^\infty(\Omega)}, \tag{2.10}$$

$$|||u|||_{[H^{1,p}(\Omega)]^I} = [\sum_{i=1}^I \|u_i\|^p_{H^{1,p}(\Omega)}]^{1/p}, \tag{2.11}$$

$$|||u|||_{[H^{1,\infty}(\Omega)]^I} = \max_{1\le i\le I} \|u_i\|_{H^{1,\infty}(\Omega)}, \tag{2.12}$$

$$|||u|||_{[H^{1,q}(\Omega)^*]^I} = [\sum_{i=1}^I \|u_i\|^p_{H^{1,q}(\Omega)^*}]^{1/p}. \tag{2.13}$$

We also define

$$F^I := [H^{1,p}((0,T); H^{1,q}(\Omega)^*) \cap L^p((0,T); H^{1,p}(\Omega))]^I \tag{2.14}$$

and for $u \in F^I$,

$$|||u|||_{F^I} := [\sum_{i=1}^I \|u_i\|^p_F]^{1/p}. \tag{2.15}$$

Similarly,

$$X_p^I := [(H^{1,q}(\Omega)^*, H^{1,p}(\Omega))_{1-\frac{1}{p},p}]^I \tag{2.16}$$

and for $u \in X_p^I$

$$|||u|||_{X_p^I} := [\sum_{i=1}^I \|u\|^p_{(H^{1,q}(\Omega)^*, H^{1,p}(\Omega))_{1-\frac{1}{p},p}}]^{1/p}. \tag{2.17}$$

Theorem 2.1. *Let $p > n + 2$, then $F \hookrightarrow\hookrightarrow L^\infty((0,T) \times \Omega)$.*

For a proof of the above theorem, see [15, Theorem 2.2].

Theorem 2.2. *Let $p > n + 2$. Then $(H^{1,q}(\Omega)^*, H^{1,p}(\Omega))_{1-\frac{1}{p},p} \hookrightarrow\hookrightarrow L^\infty(\Omega)$.*

For a proof of the above theorem, see [15, Theorem 2.3]. Let V, H and V^* be a *Gelfand triple*, where V a Banach space, H a Hilbert space and V^* is the dual of V. Let H be identified with its own dual ($H \cong H^*$) and $V \overset{d}{\subset} H$, then $H \overset{d}{\subset} V^*$. Denote $\Xi = \{u \in L^p((0,T);V) : \frac{du}{dt} \in L^q((0,T);V^*)\}$. We have the following theorem.

Theorem 2.3. *Let V, H and V^* be as above. Then $\Xi \subset C([0,T];H)$ and the following rule of integration holds for any u, $v \in \Xi$ and any $0 \le t_1 \le t_2 \le T$:*

$$\int_{t_1}^{t_2} \frac{d}{dt}(u(t),v(t))_H \, dt = \int_{t_1}^{t_2} \langle \frac{du}{dt}, v(t)\rangle_{V^*\times V} \, dt + \int_{t_1}^{t_2} \langle u(t), \frac{dv}{dt}\rangle_{V\times V^*} \, dt. \tag{2.18}$$

For a proof of the above theorem, see [24, lemma 7.3].

2.1.2. *Function Spaces on Ω_ε^p.* The function spaces on the domain Ω_ε^p are defined in an analogous way as in section 2.1.1: we replace Ω by Ω_ε^p in the definitions of the function spaces. The spaces on Ω_ε^p are endowed with their usual norms as given in (2.1)-(2.7).

From section 1, we notice that the surface area of Γ_ε increases proportionally to $1/\varepsilon$; i.e., $|\Gamma_\varepsilon| \to \infty$ as $\varepsilon \to 0$. Keeping this in mind, the $L^p - L^q$ duality on Γ_ε can be defined as

$$(u,v)_{L^p(\Gamma_\varepsilon)\times L^q(\Gamma_\varepsilon)} := \varepsilon \int_{\Gamma_\varepsilon} u(x)v(x)\, d\sigma_x \quad \text{for } u \in L^p(\Gamma_\varepsilon) \text{ and } v \in L^q(\Gamma_\varepsilon), \tag{2.19}$$

and the space $L^p(\Gamma_\varepsilon)$ is furnished with the norm

$$\|\cdot\|^p_{L^p(\Gamma_\varepsilon)} = \varepsilon \int_{\Gamma_\varepsilon} |\cdot|^p d\sigma_x \quad \text{and} \quad \|\cdot\|_{L^\infty(\Gamma_\varepsilon)} = \operatorname*{ess\,sup}_{x\in\Gamma_\varepsilon} |\cdot|. \tag{2.20}$$

The vector-valued functions and their respective norms on Ω_ε^p can be defined in the similar way as in (2.8)-(2.17). For the sake of simplicity, we use the following notation:

$$F_\varepsilon^I := [H^{1,p}((0,T);H^{1,q}(\Omega_\varepsilon^p)^*) \cap L^p((0,T);H^{1,p}(\Omega_\varepsilon^p))]^I, \tag{2.21}$$

$$X_{p,\varepsilon}^I := \big[(H^{1,q}(\Omega_\varepsilon^p)^*, H^{1,p}(\Omega_\varepsilon^p))_{1-\frac{1}{p},p}\big]^I, \tag{2.22}$$

C and C_i are generic nonnegative constants which may be different at different steps of the inequalities to come.

2.2. **Weak formulation of (P_ε).**

Definition 2.4. A function $u_\varepsilon \in F_\varepsilon^I$ is said to be a weak solution of the problem (1.12)-(1.15) if it satisfies

$$\begin{aligned} &\langle \frac{\partial u_\varepsilon(t)}{\partial t}, \phi\rangle_{[H^{1,q}(\Omega_\varepsilon^p)^*]^I\times[H^{1,q}(\Omega_\varepsilon^p)]^I} + \int_{\Omega_\varepsilon^p} \langle D\nabla u_\varepsilon(t,x), \nabla\phi(x)\rangle_I \, dx \\ &= \langle SR(u_\varepsilon(t)), \phi\rangle_{[H^{1,q}(\Omega_\varepsilon^p)^*]^I\times[H^{1,q}(\Omega_\varepsilon^p)]^I} \end{aligned} \tag{2.23}$$

for every $\phi \in [H^{1,q}(\Omega_\varepsilon^p)]^I$ and for a.e. t,

$$u_\varepsilon(0,x) = u_0(x) \quad \text{in } \Omega_\varepsilon^p. \tag{2.24}$$

2.3. **Some theorems and lemmas.**

2.3.1. *Trace theorems.*

Lemma 2.5. *Let Γ_ε be as in* (1.8). *Then*

$$\varepsilon|\Gamma_\varepsilon| = |\Gamma|\frac{|\Omega|}{|Y|}. \tag{2.25}$$

The proof of the above lemma can be found in [2, section 2].

Theorem 2.6. *Let $1 \leq p < \infty$. Let Ω^p_ε and Γ_ε be defined as in section 1. Then there exists a bounded linear operator $T^\varepsilon : H^{1,p}(\Omega^p_\varepsilon) \to L^p(\Gamma_\varepsilon)$ such that*

$$T^\varepsilon u = u|_{\Gamma_\varepsilon} \quad \textit{for } u \in H^{1,p}(\Omega^p_\varepsilon) \cap C(\bar{\Omega}^p_\varepsilon) \tag{2.26}$$

$$\varepsilon \int_{\Gamma_\varepsilon} |T^\varepsilon u(x)|^p \, d\sigma_x \leq C\Big(\int_{\Omega^p_\varepsilon} |u(x)|^p \, dx + \varepsilon^p \int_{\Omega^p_\varepsilon} |\nabla_x u(x)|^p \, dx\Big), \tag{2.27}$$

where the constant C is independent of ε and u.

For a proof of the above theorem, see [10, Lemma 5.3 (b)], and [19, Lemma 2.7.2].

2.3.2. *Extension theorems.*

Theorem 2.7. *Let $1 \leq p \leq \infty$. Suppose that Ω^p_ε and Ω are defined as in section 1. For $u \in H^{1,p}(\Omega^p_\varepsilon)$, there exists a bounded linear operator $Q^\varepsilon : H^{1,p}(\Omega^p_\varepsilon) \to H^{1,p}(\Omega)$ such that*

$$Q^\varepsilon u := u \quad \textit{in } \Omega^p_\varepsilon, \tag{2.28}$$

$$\|Q^\varepsilon u\|^p_{H^{1,p}(\Omega)} \leq C\|u\|^p_{H^{1,p}(\Omega^p_\varepsilon)}, \tag{2.29}$$

where the constant C is independent of ε and u but depends on p.

For a proof of the above theorem see [10, Theorem 5.2], also [25].

Now we prove a theorem similar to theorem 2.7 for the functions depending on both t and x. Let $1 \leq p \leq \infty$. For $u \in L^p((0,T); H^{1,p}(\Omega^p_\varepsilon))$, we define an operator $R^\varepsilon : L^p((0,T); H^{1,p}(\Omega^p_\varepsilon)) \to L^p((0,T); H^{1,p}(\Omega))$ such that

$$R^\varepsilon u(t,x) := [Q^\varepsilon u(t,.)](x) \quad \text{for } u \in L^p((0,T); H^{1,p}(\Omega^p_\varepsilon)), \tag{2.30}$$

where Q^ε is the extension operator from theorem 2.7. Then

$$\frac{\partial}{\partial t}[R^\varepsilon u(t,x)] = \frac{\partial}{\partial t}[Q^\varepsilon u(t,.)](x) = [Q^\varepsilon(\frac{\partial u}{\partial t}(t,.))](x) = R^\varepsilon(\frac{\partial u}{\partial t})(t,x).$$

Based on the above definition we have the following extension theorem for the functions depending on t and x.

Theorem 2.8. *Let Ω and Ω^p_ε be defined as in section 1 and $1 \leq p, q \leq \infty$. Then there exists a bounded linear operator*

$$\begin{aligned} R^\varepsilon &: L^q((0,T); H^{1,p}(\Omega^p_\varepsilon)) \cap H^{1,q}((0,T); L^p(\Omega^p_\varepsilon)) \\ &\to L^q((0,T); H^{1,p}(\Omega)) \cap H^{1,q}((0,T); L^p(\Omega)) \end{aligned}$$

such that for all $u \in L^q((0,T); H^{1,p}(\Omega^p_\varepsilon)) \cap H^{1,q}((0,T); L^p(\Omega^p_\varepsilon))$,

$$\|R^\varepsilon u\|_{L^q((0,T);H^{1,p}(\Omega))} \leq C\|u\|_{L^q((0,T);H^{1,p}(\Omega^p_\varepsilon))}, \tag{2.31}$$

where the constant C is independent of ε and u.

Proof. Here we only show the measurability of $R^\varepsilon u$. The inequality (2.31) follows by scaling. Since we know that every continuous function is measurable, we show $R^\varepsilon u$ is continuous. But by theorem 2.7 it can be shown $Q^\varepsilon u(t)$ is continuous on $\bar\Omega$. The continuity of $R^\varepsilon u$ on $[0,T]\times\bar\Omega$ follows from the definition (2.30). □

Theorem 2.9. *Let $1<p,q<\infty$ such that $\frac{1}{p}+\frac{1}{q}=1$ and*

$$u\in (H^{1,q}(\Omega_\varepsilon^p)^*, H^{1,p}(\Omega_\varepsilon^p))_{1-\frac{1}{p},p}.$$

Then there exists an extension $\bar u$ of u such that $\bar u\in (H^{1,q}(\Omega)^*, H^{1,p}(\Omega))_{1-\frac{1}{p},p}$.

Proof. Let $\theta = 1-\frac{1}{p}$. We use the K-functional definition for real interpolation space $(H^{1,q}(\Omega_\varepsilon^p)^*, H^{1,p}(\Omega_\varepsilon^p))_{\theta,p}$. To begin with, let $v\in H^{1,q}(\Omega_\varepsilon^p)$, then by theorem 2.11 there exists an extension $Q^\varepsilon v$ of v such that

$$Q^\varepsilon v = v \text{ in } \Omega_\varepsilon^p, \tag{2.32}$$

$$\|Q^\varepsilon v\|_{H^{1,q}(\Omega)} \le C\|v\|_{H^{1,q}(\Omega_\varepsilon^p)}, \tag{2.33}$$

where C is independent of ε and v. Let $a_0\in H^{1,q}(\Omega_\varepsilon^p)^*$, then we define an extension $\bar a_0$ of a_0 as

$$\langle \bar a_0, Q^\varepsilon v\rangle_{H^{1,q}(\Omega)^*\times H^{1,q}(\Omega)} := \langle a_0, v\rangle_{H^{1,q}(\Omega_\varepsilon^p)^*\times H^{1,q}(\Omega_\varepsilon^p)}. \tag{2.34}$$

Therefore,

$$\begin{aligned}
\|\bar a_0\|_{H^{1,q}(\Omega)^*} &= \sup_{\|Q^\varepsilon v\|_{H^{1,q}(\Omega)}\le 1} |\langle \bar a_0, Q^\varepsilon v\rangle_{H^{1,q}(\Omega)^*\times H^{1,q}(\Omega)}| \\
&= \sup_{\|v\|_{H^{1,q}(\Omega_\varepsilon^p)}\le 1} |\langle a_0, v\rangle_{H^{1,q}(\Omega_\varepsilon^p)^*\times H^{1,q}(\Omega_\varepsilon^p)}| \quad \text{by (2.33) and (2.34)} \\
&\le \|a_0\|_{H^{1,q}(\Omega_\varepsilon^p)^*}
\end{aligned}$$

which implies

$$\|\bar a_0\|_{H^{1,q}(\Omega)^*} \le \|a_0\|_{H^{1,q}(\Omega_\varepsilon^p)^*}. \tag{2.35}$$

Again assume that $b_0\in H^{1,p}(\Omega_\varepsilon^p)$. Let $\bar b_0\in H^{1,p}(\Omega)$ denote the extension of b_0 such that

$$\|\bar b_0\|_{H^{1,p}(\Omega)} \le C\|b_0\|_{H^{1,p}(\Omega_\varepsilon^p)} \quad \text{for } b_0\in H^{1,p}(\Omega_\varepsilon^p), \tag{2.36}$$

where C is independent of ε and b_0. Let $t>0$. Then

$$\begin{aligned}
\|\bar a_0\|_{H^{1,q}(\Omega)^*} + t\|\bar b_0\|_{H^{1,p}(\Omega)} &\le \|a_0\|_{H^{1,q}(\Omega_\varepsilon^p)^*} + Ct\|b_0\|_{H^{1,p}(\Omega_\varepsilon^p)} \\
&\le \max(1,C)\big(\|a_0\|_{H^{1,q}(\Omega_\varepsilon^p)^*} + t\|b_0\|_{H^{1,p}(\Omega_\varepsilon^p)}\big)\,.
\end{aligned}$$

Taking the infimum on both sides, we get successively

$$\begin{aligned}
&\inf_{\substack{\bar a_0\in H^{1,q}(\Omega)^* \\ \bar b_0\in H^{1,p}(\Omega)}} \bar u = \bar a_0 + \bar b_0 \Big(\|\bar a_0\|_{H^{1,q}(\Omega)^*} + t\|\bar b_0\|_{H^{1,p}(\Omega)}\Big) \\
&\le \max(1,C) \inf_{\substack{u=a_0+b_0 \\ a_0\in H^{1,q}(\Omega_\varepsilon^p)^* \\ b_0\in H^{1,p}(\Omega_\varepsilon^p)}} \Big(\|a_0\|_{H^{1,q}(\Omega_\varepsilon^p)^*} + t\|b_0\|_{H^{1,p}(\Omega_\varepsilon^p)}\Big),
\end{aligned}$$

$$\underbrace{t^{-\theta} \inf_{\substack{\bar u=\bar a_0+\bar b_0 \\ \bar a_0\in H^{1,q}(\Omega)^* \\ \bar b_0\in H^{1,p}(\Omega)}} \Big(\|\bar a_0\|_{H^{1,q}(\Omega)^*} + t\|\bar b_0\|_{H^{1,p}(\Omega)}\Big)}_{positive}$$

$$\le \max(1,C)\, t^{-\theta} \underbrace{\inf_{\substack{a_0\in H^{1,q}(\Omega_\varepsilon^p)^*\\ b_0\in H^{1,p}(\Omega_\varepsilon^p)}} u = a_0 + b_0 \Big(\|a_0\|_{H^{1,q}(\Omega_\varepsilon^p)^*} + t\|b_0\|_{H^{1,p}(\Omega_\varepsilon^p)}\Big)}_{positive},$$

$$\begin{aligned}
&\Big|t^{-\theta} \inf_{\substack{\bar u=\bar a_0+\bar b_0\\ \bar a_0\in H^{1,q}(\Omega)^*\\ \bar b_0\in H^{1,p}(\Omega)}} (\|\bar a_0\|_{H^{1,q}(\Omega)^*} + t\|\bar b_0\|_{H^{1,p}(\Omega)})\Big|^p\\
&\le [\max(1,C)]^p \Big|t^{-\theta} \inf_{\substack{u=a_0+b_0\\ a_0\in H^{1,q}(\Omega_\varepsilon^p)^*\\ b_0\in H^{1,p}(\Omega_\varepsilon^p)}} \Big(\|a_0\|_{H^{1,q}(\Omega_\varepsilon^p)^*} + t\|b_0\|_{H^{1,p}(\Omega_\varepsilon^p)}\Big)\Big|^p.
\end{aligned}$$

Thus

$$\begin{aligned}
&\int_0^\infty \Big|t^{-\theta} \inf_{\substack{\bar u=\bar a_0+\bar b_0\\ \bar a_0\in H^{1,q}(\Omega)^*\\ \bar b_0\in H^{1,p}(\Omega)}} \Big(\|\bar a_0\|_{H^{1,q}(\Omega)^*} + t\|\bar b_0\|_{H^{1,p}(\Omega)}\Big)\Big|^p \frac{dt}{t}\\
&\le [\max(1,C)]^p \int_0^\infty |t^{-\theta} \inf_{\substack{u=a_0+b_0\\ a_0\in H^{1,q}(\Omega_\varepsilon^p)^*\\ b_0\in H^{1,p}(\Omega_\varepsilon^p)}} \Big(\|a_0\|_{H^{1,q}(\Omega_\varepsilon^p)^*} + t\|b_0\|_{H^{1,p}(\Omega_\varepsilon^p)}\Big)\Big|^p \frac{dt}{t},
\end{aligned}$$

$$\begin{gathered}
\int_0^\infty \big|t^{-\theta} K(t,\bar u, H^{1,q}(\Omega)^*, H^{1,p}(\Omega))\big|^p \frac{dt}{t}\\
\le [\max(1,C)]^p \int_0^\infty \big|t^{-\theta} K(t,u, H^{1,q}(\Omega_\varepsilon^p)^*, H^{1,p}(\Omega_\varepsilon^p))\big|^p \frac{dt}{t},\\
\|\bar u\|_{(H^{1,q}(\Omega)^*, H^{1,p}(\Omega))_{1-\frac1p,p}} \le \max(1,C)\|u\|_{(H^{1,q}(\Omega_\varepsilon^p)^*, H^{1,p}(\Omega_\varepsilon^p))_{1-\frac1p,p}},
\end{gathered}$$

where the constant $\max(1,C)$ is independent of ε and u. □

2.3.3. *Embedding Theorems.*

Theorem 2.10. *Let Ω and Ω_ε^p be as in section 1. Assume that $1 \le p < n$ and $u \in H^{1,p}(\Omega_\varepsilon^p)$. Then $u \in L^{p^*}(\Omega_\varepsilon^p)$ and there is a constant C*

$$\|u\|_{L^{p^*}(\Omega_\varepsilon^p)} \le C\|u\|_{H^{1,p}(\Omega_\varepsilon^p)}, \tag{2.37}$$

where $p^* = np/(n-p)$ and C is independent of ε and u. In other words, $H^{1,p}(\Omega_\varepsilon^p) \hookrightarrow L^{p^*}(\Omega_\varepsilon^p)$ with embedding constant independent of ε.

Proof. Let $u \in H^{1,p}(\Omega_\varepsilon^p)$. Then from theorem 2.7, there exists an extension $Q^\varepsilon u$ of u from $H^{1,p}(\Omega_\varepsilon^p)$ to $H^{1,p}(\Omega)$ such that

$$\|Q^\varepsilon u\|_{H^{1,p}(\Omega)} \le C\|u\|_{H^{1,p}(\Omega_\varepsilon^p)}. \tag{2.38}$$

Let $v := Q^\varepsilon u$. By assumption Ω is a bounded domain with sufficiently smooth boundary, then from [8, Theorem 2 of section 5.6.1] we obtain

$$\|v\|_{L^{p^*}(\Omega)} \le C\|v\|_{H^{1,p}(\Omega)} \text{ for } v \in H^{1,p}(\Omega), \tag{2.39}$$

where $p^* = \frac{np}{n-p}$ and C depends only on p, n and Ω but is independent of v. Therefore by (2.38) and (2.39) we obtain

$$\|u\|_{L^{p^*}(\Omega_\varepsilon^p)} \le \|v\|_{L^{p^*}(\Omega)} \le C\|v\|_{H^{1,p}(\Omega)} \le C\|u\|_{H^{1,p}(\Omega_\varepsilon^p)},$$

where C is independent of ε and u. □

Theorem 2.11. *Let $1 < p, q < \infty$ be such that $p > n+2$ and $\frac{1}{p} + \frac{1}{q} = 1$. Assume that $u \in (H^{1,q}(\Omega^p_\varepsilon)^*, H^{1,p}(\Omega^p_\varepsilon))_{1-\frac{1}{p},p}$ such that $\sup_{\varepsilon>0} \|u\|_{(H^{1,q}(\Omega^p_\varepsilon)^*, H^{1,p}(\Omega^p_\varepsilon))_{1-\frac{1}{p},p}} < \infty$. Then $u \in L^\infty(\Omega^p_\varepsilon)$ and*

$$\sup_{\varepsilon>0} \|u\|_{L^\infty(\Omega^p_\varepsilon)} < \infty. \tag{2.40}$$

Proof. From theorem 2.2, we know that for $u \in (H^{1,q}(\Omega)^*, H^{1,p}(\Omega))_{1-\frac{1}{p},p}$, $u \in L^\infty(\Omega)$ and

$$\|u\|_{L^\infty(\Omega)} \le C\|u\|_{(H^{1,q}(\Omega)^*, H^{1,p}(\Omega))_{1-\frac{1}{p},p}}, \tag{2.41}$$

where the constant C is independent of u. Let $u \in (H^{1,q}(\Omega^p_\varepsilon)^*, H^{1,p}(\Omega^p_\varepsilon))_{1-\frac{1}{p},p}$, then

$$\begin{aligned}
\|u\|_{L^\infty(\Omega^p_\varepsilon)} &\le \|u\|_{L^\infty(\Omega)} \\
&\le C\|u\|_{(H^{1,q}(\Omega)^*, H^{1,p}(\Omega))_{1-\frac{1}{p},p}} \quad \text{by(2.41)} \\
&\le C\|u\|_{(H^{1,q}(\Omega^p_\varepsilon)^*, H^{1,p}(\Omega^p_\varepsilon))_{1-\frac{1}{p},p}} \quad \text{by theorem 2.9} \\
&\le C \sup_{\varepsilon>0} \|u\|_{(H^{1,q}(\Omega^p_\varepsilon)^*, H^{1,p}(\Omega^p_\varepsilon))_{1-\frac{1}{p},p}} < \infty \quad \forall \varepsilon > 0,
\end{aligned}$$

where the constant C is independent of ε and u. Therefore $\sup_{\varepsilon>0} \|u\|_{L^\infty(\Omega^p_\varepsilon)} < \infty$. $\square$

From this theorem we notice that for $1 \le p < \infty$,

$$\|u\|^p_{L^p(\Omega^p_\varepsilon)} = \int_{\Omega^p_\varepsilon} |u(x)|^p \, dx \le |\Omega^p_\varepsilon| \|u\|^p_{L^\infty(\Omega^p_\varepsilon)} \le |\Omega| \sup_{\varepsilon>0} \|u\|^p_{L^\infty(\Omega^p_\varepsilon)} < \infty \quad \forall \varepsilon. \tag{2.42}$$

2.4. **Two-scale Convergence.**

Definition 2.12. A sequence of functions $(u_\varepsilon)_{\varepsilon>0}$ in $L^p((0,T) \times \Omega)$ is said to two-scale convergent to a limit $u \in L^p((0,T) \times \Omega \times Y)$ if

$$\lim_{\varepsilon\to 0} \int_0^T \int_\Omega u_\varepsilon(t,x)\phi(t,x,\frac{x}{\varepsilon}) \, dx \, dt = \int_0^T \int_\Omega \int_Y u(t,x,y)\phi(t,x,y) \, dx \, dy \, dt \tag{2.43}$$

for all $\phi \in L^q((0,T) \times \Omega; C_{per}(Y))$.

We quote the following theorems whose proofs can be found in [3, 19, 6].

Theorem 2.13. *For every bounded sequence, $(u_\varepsilon)_{\varepsilon>0}$, in $L^p((0,T) \times \Omega)$ there exist a subsequence and a $u \in L^p((0,T) \times \Omega \times Y)$ such that the subsequence two-scale converges to u.*

Theorem 2.14. *Let $(u_\varepsilon)_{\varepsilon>0}$ be strongly convergent to $u \in L^p((0,T) \times \Omega)$, then $(u_\varepsilon)_{\varepsilon>0}$ is two-scale convergent to $u_1(t,x,y) = u(t,x)$.*

Theorem 2.15. *Let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence in $L^p((0,T); H^{1,p}(\Omega))$ such that $u_\varepsilon \to u$ weakly in $L^p((0,T); H^{1,p}(\Omega))$. Then $(u_\varepsilon)_{\varepsilon>0}$ two-scale converges to u and there exist a subsequence ε, still denoted by same symbol, and a $u_1 \in L^p((0,T) \times \Omega; H^{1,p}_{per}(Y))$ such that $\nabla_x u_\varepsilon \xrightarrow{2} \nabla u + \nabla_y u_1$.*

3. Global existence and uniqueness of solution to (P_ε)

The main result of this section is the following existence theorem:

Theorem 3.1. *Suppose that the assumptions* (1.16)-(1.20) *are satisfied. Then there exists a unique positive global weak solution $u_\varepsilon \in F_\varepsilon^I$ of the problem (P_ε).*

Theorem 3.1 is proved in [15, heorem 2.4] or [14, Theorem 4.1.1.1]. The ingredients of the proof are a Lyapunov functional, Schaefer's fixed point theorem and [22, Theorem 2.5] which is based on the maximal regularity of differential operators. In [15] it is also shown that with the help of a Lyapunov functional we can obtain global in time *a-priori* estimates of the type

$$\|u_\varepsilon(t)\|_{L^r(\Omega_\varepsilon^p)^I} \le C_1 < \infty \quad \text{for all } r \ge 2 \text{ and for a.e. } t, \tag{3.1}$$

$$\|u_\varepsilon(t)\|_{L^\infty(\Omega_\varepsilon^p)^I} \le C_2 < \infty \quad \text{for a.e. } t, \tag{3.2}$$

where C_1 and C_2 are independent of i, t, ε and u_{ε_i}. The constant C_1 depends only on $r \in \mathbb{N}$ (cf. [15], see also [14]).

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4. Homogenization of Problem (P_ε)

4.1. **A-priori estimates.** In this section, we obtain ε-independent *a-priori* estimates for the solution u_ε of (P_ε) and extend these estimates to all of $(0,T)\times\Omega$. The major theorem of this section reads as follows.

Theorem 4.1. *Let $r \in \mathbb{N}$ ($r \ge 2$). There exists a constant $C > 0$ independent of ε such that the extension of u_ε (denoted by the same symbol) to all of $(0,T)\times\Omega$ satisfies*

$$\begin{aligned}\sup_{\varepsilon>0}\Big(& |||u_\varepsilon|||_{L^r((0,T);L^r(\Omega))^I} + |||u_\varepsilon|||_{L^\infty((0,T);L^\infty(\Omega))^I} \\ & + |||\nabla u_\varepsilon|||_{L^2((0,T);L^2(\Omega))^I}\Big) \le C < \infty.\end{aligned} \tag{4.1}$$

We start with the following lemma.

Lemma 4.2. *Let $r \in \mathbb{N}$ ($r \ge 2$). There exists a constant $C > 0$ independent of ε such that the solution u_ε of (P_ε) satisfies*

$$\begin{aligned}\sup_{\varepsilon>0}\Big(& |||u_\varepsilon|||_{L^r((0,T);L^r(\Omega_\varepsilon^p))^I} + |||u_\varepsilon|||_{L^\infty((0,T);L^\infty(\Omega_\varepsilon^p))^I} \\ & + |||\nabla u_\varepsilon|||_{L^2((0,T);L^2(\Omega_\varepsilon^p))^I}\Big) \le C < \infty.\end{aligned} \tag{4.2}$$

Proof. By (3.1) we obtain

$$\begin{aligned}|||u_\varepsilon|||^r_{L^r((0,T);L^r(\Omega_\varepsilon^p))^I} &= \sum_{i=1}^I \int_0^T \|u_{\varepsilon_i}(t)\|^r_{L^r(\Omega_\varepsilon^p)}\,dt \le C_1 \sum_{i=1}^I \int_0^T dt \\ &= C_1 IT =: C_3 < \infty \quad \forall \varepsilon.\end{aligned} \tag{4.3}$$

Equation (3.2) gives

$$\begin{aligned}|||u_\varepsilon|||_{L^\infty((0,T);L^\infty(\Omega_\varepsilon^p))^I} &= \operatorname{ess\,sup}_{t\in(0,T)} |||u_\varepsilon(t)|||_{L^\infty(\Omega_\varepsilon^p)^I} \\ &\le \operatorname{ess\,sup}_{t\in(0,T)} C_2 = C_2 \quad \forall \varepsilon.\end{aligned} \tag{4.4}$$

Testing the *i-th* PDE of (1.12) with u_{ε_i}, we obtain[1]

$$\begin{aligned}
&\int_0^T \langle \frac{\partial u_{\varepsilon_i}(t)}{\partial t}, u_{\varepsilon_i}(t)\rangle_{H^{1,q}(\Omega_\varepsilon^p)^*\times H^{1,q}(\Omega_\varepsilon^p)}\,dx\,dt\\
&-\int_0^T \langle \nabla\cdot D\nabla u_{\varepsilon_i}(t), u_{\varepsilon_i}(t)\rangle_{H^{1,q}(\Omega_\varepsilon^p)^*\times H^{1,q}(\Omega_\varepsilon^p)}\,dx\,dt\\
&=\int_0^T \langle SR(u_\varepsilon(t))_i, u_{\varepsilon_i}(t)\rangle_{H^{1,q}(\Omega_\varepsilon^p)^*\times H^{1,q}(\Omega_\varepsilon^p)}\,dt;
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\frac{1}{2}\int_0^T \frac{d}{dt}\|u_{\varepsilon_i}(t)\|^2_{L^2(\Omega_\varepsilon^p)}\,dt + \int_0^T D\|\nabla u_{\varepsilon_i}(t)\|^2_{L^2(\Omega_\varepsilon^p)}\,dt\\
&=\int_0^T \langle SR(u_\varepsilon(t))_i, u_{\varepsilon_i}(t)\rangle_{L^p(\Omega_\varepsilon^p)\times L^q(\Omega_\varepsilon^p)}\,dt\\
&\le \frac{1}{p}\int_0^T \|SR(u_\varepsilon(t))_i\|^p_{L^p(\Omega_\varepsilon^p)}\,dt + \frac{1}{q}\int_0^T \|u_{\varepsilon_i}(t)\|^q_{L^q(\Omega_\varepsilon^p)}\,dt;
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\frac{1}{2}\|u_{\varepsilon_i}(T)\|^2_{L^2(\Omega_\varepsilon^p)} + \int_0^T D\|\nabla u_{\varepsilon_i}(t)\|^2_{L^2(\Omega_\varepsilon^p)}\,dt\\
&\le \frac{1}{2}\|u_{0_i}\|^2_{L^2(\Omega_\varepsilon^p)} + \frac{1}{p}\int_0^T \|SR(u_\varepsilon(t))_i\|^p_{L^p(\Omega_\varepsilon^p)}\,dt + \frac{1}{q}\int_0^T \|u_{\varepsilon_i}(t)\|^q_{L^q(\Omega_\varepsilon^p)}\,dt.
\end{aligned} \tag{4.5}$$

Therefore the right-hand side of (4.5) is bounded by a constant independent of ε, i and t. Let us call this constant by $\bar{C}$. This gives

$$\int_0^T D\|\nabla u_{\varepsilon_i}(t)\|^2_{L^2(\Omega_\varepsilon^p)}\,dt \le \bar{C} \quad \text{for all } \varepsilon,\ i \text{ and for a.e. } t$$

which implies

$$\sum_{i=1}^I \int_0^T \|\nabla u_{\varepsilon_i}(t)\|^2_{L^2(\Omega_\varepsilon^p)}\,dt \le \sum_{i=1}^I \frac{\bar{C}}{D} =: C_4 < \infty \quad \forall\varepsilon. \tag{4.6}$$

Note that $D > 0$ is constant in (4.6). Adding (4.3), (4.4) and (4.6) yields

$$\begin{aligned}
&|||u_\varepsilon|||_{L^r((0,T);L^r(\Omega_\varepsilon^p))^I} + |||u_\varepsilon|||_{L^\infty((0,T);L^\infty(\Omega_\varepsilon^p))^I} + |||\nabla u_\varepsilon|||_{L^2((0,T);L^2(\Omega_\varepsilon^p))^I}\\
&\le C_3^{1/r} + C_2 + C_4^{1/2} = C < \infty \quad \text{for all } \varepsilon.
\end{aligned}$$

This completes the proof. □

Proof of theorem 4.1. The estimate (4.2) from lemma 4.2 and theorem 2.8 accomplish the proof. □

[1] From (3.1), we have $\|u_{\varepsilon_i}(t)\|_{L^r(\Omega_\varepsilon^p)} \le C_1$ for all i and for a.e. t, where C_1 is independent of ε. This gives $\sup_{\varepsilon>0}\|SR(u_\varepsilon)_i\|_{L^p(\Omega_\varepsilon^p)} \le C$. Since $L^p(\Omega_\varepsilon^p) \hookrightarrow H^{1,q}(\Omega_\varepsilon^p)^*$, from the definition (2.3) we get $\langle SR(u_\varepsilon)_i, \phi_i\rangle_{H^{1,q}(\Omega_\varepsilon^p)^*\times H^{1,q}(\Omega_\varepsilon^p)} = \langle SR(u_\varepsilon)_i, \phi_i\rangle_{L^p(\Omega_\varepsilon^p)\times L^q(\Omega_\varepsilon^p)}$ for $\phi_i \in H^{1,q}(\Omega_\varepsilon^p)$.

4.2. **Convergence of u_ε.** In this section, we show the *weak*, *strong* and *two-scale* convergences of the solution of (P_ε).

Theorem 4.3. *There exists a constant C independent of ε such that the solution u_ε of the problem (P_ε) satisfies the estimate*

$$\begin{aligned}&\sup_{\varepsilon>0}\Big(|||u_\varepsilon|||_{L^\infty((0,T);L^2(\Omega))^I}+|||u_\varepsilon|||_{L^2((0,T);H^{1,2}(\Omega))^I}\\&+|||\chi^\varepsilon\frac{\partial u_\varepsilon}{\partial t}|||_{L^2((0,T);H^{1,2}(\Omega)^*)^I}\Big)\le C<\infty,\end{aligned}\tag{4.7}$$

where the function χ^ε is defined in (1.11).

Proof. From (4.1), we have

$$\begin{aligned}&|||u_\varepsilon|||^2_{L^\infty((0,T);L^2(\Omega))^I}\\&\le|\Omega||||u_\varepsilon|||^2_{L^\infty((0,T);L^\infty(\Omega))^I}\quad\le|\Omega|\sup_{\varepsilon>0}|||u_\varepsilon|||^2_{L^\infty((0,T);L^\infty(\Omega))^I}\\&=:C_5<\infty\quad\forall\varepsilon.\end{aligned}\tag{4.8}$$

Again,

$$\begin{aligned}&|||u_\varepsilon|||^2_{L^2((0,T);H^{1,2}(\Omega))^I}\\&=\sum_{i=1}^I\|u_{\varepsilon_i}\|^2_{L^2((0,T);H^{1,2}(\Omega))}\\&=\sum_{i=1}^I\Big(\|\nabla u_{\varepsilon_i}\|^2_{L^2((0,T);L^2(\Omega))}+\|u_{\varepsilon_i}\|^2_{L^2((0,T);L^2(\Omega))}\Big)\\&\le\sup_{\varepsilon>0}\sum_{i=1}^I(\|\nabla u_{\varepsilon_i}\|^2_{L^2((0,T);L^2(\Omega))}+(T|\Omega|)^{1-\frac{2}{r}}\|u_{\varepsilon_i}\|^2_{L^r((0,T);L^r(\Omega))})<\infty,\end{aligned}$$

by (4.1); i.e.,

$$|||u_\varepsilon|||_{L^2((0,T);H^{1,2}(\Omega))^I}\le C_6<\infty\quad\forall\varepsilon.\tag{4.9}$$

Now, let $\phi\in H_0^{1,2}(0,T)$ and $\psi\in H^{1,2}(\Omega)$. Then the weak formulation of the *i-th* PDE of the problem (1.12)-(1.15) is

$$\begin{aligned}&\int_0^T\langle\chi^\varepsilon\frac{\partial u_{\varepsilon_i}(t)}{\partial t},\phi(t)\psi\rangle_{H^{1,2}(\Omega)^*\times H^{1,2}(\Omega)}\,dt+\int_0^T\int_\Omega\phi(t)\chi^\varepsilon(x)\nabla u_{\varepsilon_i}(t,x)\nabla\psi(x)\,dx\,dt\\&=\int_0^T\langle\chi^\varepsilon SR(u_\varepsilon(t))_i,\phi(t)\psi\rangle_{H^{1,2}(\Omega)^*\times H^{1,2}(\Omega)}\,dt\,dt;\end{aligned}$$

i.e.,

$$\begin{aligned}&\Big|\int_0^T\langle\chi^\varepsilon\frac{\partial u_{\varepsilon_i}(t)}{\partial t},\phi(t)\psi\rangle_{H^{1,2}(\Omega)^*\times H^{1,2}(\Omega)}\,dt\Big|\\&\le\int_0^T\int_\Omega|\chi^\varepsilon(x)||\nabla u_{\varepsilon_i}(t,x)||\nabla\psi(x)||\phi(t)|\,dx\,dt\\&\quad+\frac12\int_0^T[\|\chi^\varepsilon SR(u_\varepsilon(t))_i\|^2_{L^2(\Omega)}+\|\phi(t)\psi\|^2_{L^2(\Omega)}]\,dt.\end{aligned}$$

Note that $|\chi^\varepsilon(x)| \leq 1$. From (4.1) the terms $\sup_{\varepsilon>0} \|\nabla u_{\varepsilon_i}\|^2_{L^2((0,T);L^2(\Omega))}$ and $\sup_{\varepsilon>0} \|SR(u_\varepsilon)_i\|^2_{L^2((0,T);L^2(\Omega))}$ are finite. This gives

$$
\begin{aligned}
&\Big|\int_0^T \langle \chi^\varepsilon \frac{\partial u_{\varepsilon_i}(t)}{\partial t}, \phi(t)\psi\rangle_{H^{1,2}(\Omega)^*\times H^{1,2}(\Omega)}\,dt\Big| \\
&\leq C + \frac{1}{2}\|\phi(t)\|^2_{L^2(0,T)}[\|\nabla\psi\|^2_{L^2(\Omega)} + \|\psi\|^2_{L^2(\Omega)}] \\
&= C + \|\phi\|^2_{L^2(0,T)}\|\psi\|^2_{H^{1,2}(\Omega)}.
\end{aligned}
$$

Note that $\phi \in H_0^{1,2}(0,T)$ implies $\|\phi\|_{L^2(0,T)} \leq \bar{C}\|\phi\|_{H_0^{1,2}(0,T)}$; i.e., $\|\frac{\phi}{\bar{C}}\|_{L^2(0,T)} \leq \|\phi\|_{H_0^{1,2}(0,T)}$, where $\bar{C}>0$ is the embedding constant. Taking the supremum, over all $\|\psi\|^2_{H^{1,2}(\Omega)} \leq 1$, $\|\frac{\phi}{\bar{C}}\|^2_{L^2(0,T)} \leq 1$, $\psi \in H^{1,2}(\Omega)$, $\frac{\phi}{\bar{C}} \in L^2(0,T)$, on both sides of the above inequality yields

$$
\begin{aligned}
&\bar{C}\sup\Big|\int_0^T \langle \chi^\varepsilon \frac{\partial u_{\varepsilon_i}(t)}{\partial t}, \frac{\phi(t)}{\bar{C}}\psi\rangle_{H^{1,2}(\Omega)^*\times H^{1,2}(\Omega)}\,dt\Big| \\
&\leq C + \bar{C}^2\sup\|\psi\|^2_{H^{1,2}(\Omega)}\|\frac{\phi}{\bar{C}}\|^2_{L^2(0,T)}.
\end{aligned}
$$

This implies

$$
\|\chi^\varepsilon \frac{\partial u_{\varepsilon_i}}{\partial t}\|_{L^2((0,T);H^{1,2}(\Omega)^*)} \leq C_7
$$

which implies

$$
\sum_{i=1}^I \|\chi^\varepsilon \frac{\partial u_{\varepsilon_i}}{\partial t}\|^2_{L^2((0,T);H^{1,2}(\Omega)^*)} \leq I\,C_7^2
$$

which implies

$$
\||\chi^\varepsilon \frac{\partial u_\varepsilon}{\partial t}\||_{L^2((0,T);H^{1,2}(\Omega)^*)^I} \leq (I\,C_7^2)^{1/2} \quad \forall\varepsilon. \tag{4.10}
$$

Adding (4.8), (4.9) and (4.10), we obtain

$$
\begin{aligned}
&\|u_\varepsilon\|_{L^\infty((0,T);L^2(\Omega))^I} + \|u_\varepsilon\|_{L^2((0,T);H^{1,2}(\Omega))^I} + \|\chi^\varepsilon \frac{\partial u_\varepsilon}{\partial t}\|_{L^2((0,T);H^{1,2}(\Omega)^*)^I} \\
&\leq C_5 + C_6 + (I\,C_7^2)^{1/2} =: C < \infty \quad \forall\varepsilon.
\end{aligned}
$$

Hence the proof is complete. □

The next statement is crucial. It gives the strong convergence of the subsequence of the sequence $(u_{\varepsilon_i})_{\varepsilon>0}$. This is the main result of Meirmanov & Zimin in [18].

Lemma 4.4 ([18, Theorem 2.1])**.** *Let $(c_\varepsilon)_{\varepsilon>0}$ be a bounded sequence in the space $L^\infty((0,T);L^2(\Omega)) \cap L^2((0,T);H^{1,2}(\Omega))$ and weakly convergent in $L^2((0,T);L^2(\Omega)) \cap L^2((0,T);H^{1,2}(\Omega))$ to a function c. Suppose further that the sequence $(\chi^\varepsilon\frac{\partial}{\partial t}c_\varepsilon)_{\varepsilon>0}$ is bounded in $L^2((0,T);H^{1,2}(\Omega)^*)$. Then the sequence $(c_\varepsilon)_{\varepsilon>0}$ is strongly convergent to the function c in $L^2((0,T);L^2(\Omega))$.*

Theorem 4.5. *Let $(u_\varepsilon)_{\varepsilon>0}$ satisfy the estimates (4.1) and (4.7). Then there exists a function $u \in L^2((0,T);H^{1,2}(\Omega))^I$ and a function $u^1 \in L^2((0,T)\times\Omega;H^{1,2}_{per}(Y)/\mathbb{R})^I$ such that up to a subsequence, still denoted by same subscript, the following convergence results hold:*

$$
(u_\varepsilon)_{\varepsilon>0} \text{ is weakly convergent to } u \text{ in } L^2((0,T);H^{1,2}(\Omega))^I. \tag{4.11}
$$

$$
(u_\varepsilon)_{\varepsilon>0} \text{ is strongly convergent to } u \text{ in } L^2((0,T);L^2(\Omega))^I. \tag{4.12}
$$

$$(u_\varepsilon)_{\varepsilon>0} \text{ and } (\nabla_x u_\varepsilon)_{\varepsilon>0} \text{ are two-scale convergent to } u \text{ and } \nabla_x u + \nabla_y u^1 \text{ in the sense of (2.43) respectively.} \tag{4.13}$$

Proof. Statement (4.11) follows from estimate (4.7), we note that the sequence $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $L^2((0,T); H^{1,2}(\Omega))^I$. This implies that, up to a subsequnce, still indexed by the same subscript, $(u_\varepsilon)_{\varepsilon>0}$ is weakly convergent to a function u in $L^2((0,T); H^{1,2}(\Omega))^I$.

For statement (4.12), from (4.7), it follows that, up to a subsequence, still denoted by the same subscript, $(u_\varepsilon)_{\varepsilon>0}$ is weakly convergent to u in the space $L^2((0,T); L^2(\Omega))^I \cap L^2((0,T); H^{1,2}(\Omega))^I$ and is bounded in $L^\infty((0,T); L^2(\Omega))^I \cap L^2((0,T); H^{1,2}(\Omega))^I$. Also note that from (4.7) the function $(\frac{\partial}{\partial t}\chi^\varepsilon u_\varepsilon)_{\varepsilon>0}$ is bounded in $L^2((0,T); H^{1,2}(\Omega)^*)^I$. Therefore the subsequence $(u_\varepsilon)_{\varepsilon>0}$, still denoted by the same subscript, is strongly convergent to u in $L^2((0,T); L^2(\Omega))^I$.

Statement (4.12)follows from estimate (4.7) and theorem 2.15. $\square$

Theorem 4.6. *The limit function u belongs to $L^\infty((0,T)\times\Omega\times Y)^I$. (Note that the function u is independent of the variable y.)*

Proof. Since $(u_\varepsilon)_{\varepsilon>0}$ is strongly convergent to u in $L^2((0,T); L^2(\Omega))^I$, there exists a subsequence $(u_{\varepsilon'})_{\varepsilon'>0}$ which is pointwise convergent [28, Corollary on page 53] to u almost everywhere in $(0,T)\times\Omega$; i.e.,

$$\lim_{\varepsilon'\to 0} u_{\varepsilon'}(t,x) = u(t,x) \quad \text{a.e.} \quad (t,x)\in(0,T)\times\Omega.$$

By theorem 4.1, we have $\|u_{\varepsilon_i}\|_{L^\infty((0,T);L^\infty(\Omega))} \le C$ for all i, therefore

$$\begin{aligned}
|u_i(t,x)|^2 \le \sum_{i=1}^I |u_i(t,x)|^2 &= \lim_{\varepsilon'\to 0}\sum_{i=1}^I |u_{\varepsilon'_i}(t,x)|^2 \\
&\le \sum_{i=1}^I \limsup_{\varepsilon'\to 0} \operatorname{ess\,sup}_{t\in(0,T)} \operatorname{ess\,sup}_{x\in\Omega} |u_{\varepsilon'_i}(t,x)|^2 \\
&\le \sum_{i=1}^I \limsup_{\varepsilon'\to 0} C^2 = C^2 I \quad \text{for a.e. } t \text{ and } x
\end{aligned}$$

which implies

$$\operatorname{ess\,sup}_{t\in(0,T)} \operatorname{ess\,sup}_{x\in\Omega} |u_i(t,x)|^2 \le C^2 I < \infty \quad \text{for all } i.$$

This gives

$$\begin{aligned}
\||u|\|^2_{L^\infty((0,T)\times\Omega\times Y)^I} &\le \max_{1\le i\le I} \operatorname{ess\,sup}_{y\in Y} \operatorname{ess\,sup}_{t\in(0,T)} \operatorname{ess\,sup}_{x\in\Omega} |u_i(t,x)|^2 \\
&\le \operatorname{ess\,sup}_{y\in Y} C^2 I < \infty.
\end{aligned}$$

$\square$

Corollary 4.7. *For all $1\le p\le\infty$, the sequence $(u_\varepsilon)_{\varepsilon>0}$ is strongly convergent to u in $L^p((0,T)\times\Omega)^I$.*

Proof. This follows from the straightforward application of Lyapunov's interpolation inequality (cf. lemma 5.2) and L^∞-estimates of u_ε and u (cf. Lemma 4.1 and Theorem 4.6). $\square$

Theorem 4.8. *The sequence $(SR(u_\varepsilon))_{\varepsilon>0}$ is strongly convergent to $SR(u)$ in space $L^2((0,T)\times\Omega)^I$ as $\varepsilon\to 0$.*

Proof. Note that

$$\||SR(u_\varepsilon) - SR(u)\||^2_{L^2((0,T)\times\Omega)^I} = \sum_{i=1}^{I} \|SR(u_\varepsilon)_i - SR(u)_i\|^2_{L^2((0,T)\times\Omega)} \tag{4.14}$$

From (1.5), we have

$$SR(u_\varepsilon)_i = \sum_{j=1}^{J} s_{ij}\Big(k_j^f \prod_{\substack{m=1\\ s_{mj}<0}}^{I} u_{\varepsilon_m}^{-s_{mj}} - k_j^b \prod_{\substack{m=1\\ s_{mj}>0}}^{I} u_{\varepsilon_m}^{s_{mj}}\Big) \tag{4.15}$$

$$SR(u)_i = \sum_{j=1}^{J} s_{ij}\Big(k_j^f \prod_{\substack{m=1\\ s_{mj}<0}}^{I} u_m^{-s_{mj}} - k_j^b \prod_{\substack{m=1\\ s_{mj}>0}}^{I} u_m^{s_{mj}}\Big). \tag{4.16}$$

From (4.15) and (4.16),

$$\begin{aligned}
&\|SR(u_\varepsilon)_i - SR(u)_i\|_{L^2((0,T)\times\Omega)}\\
&= \Big\| \sum_{j=1}^{J} s_{ij}\Big(k_j^f \prod_{\substack{m=1\\ s_{mj}<0}}^{I} u_{\varepsilon_m}^{-s_{mj}} - k_j^f \prod_{\substack{m=1\\ s_{mj}<0}}^{I} u_m^{-s_{mj}}\Big)\\
&\quad - \sum_{j=1}^{J} s_{ij}\Big(k_j^b \prod_{\substack{m=1\\ s_{mj}>0}}^{I} u_{\varepsilon_m}^{s_{mj}} - k_j^b \prod_{\substack{m=1\\ s_{mj}>0}}^{I} u_m^{s_{mj}}\Big)\Big\|_{L^2((0,T)\times\Omega)}\\
&\le \sum_{j=1}^{J} s_{ij}k_j^f \Big\| \prod_{\substack{m=1\\ s_{mj}<0}}^{I} u_{\varepsilon_m}^{-s_{mj}} - \prod_{\substack{m=1\\ s_{mj}<0}}^{I} u_m^{-s_{mj}}\Big\|_{L^2((0,T)\times\Omega)}\\
&\quad + \sum_{j=1}^{J} s_{ij}k_j^b \Big\| \prod_{\substack{m=1\\ s_{mj}>0}}^{I} u_{\varepsilon_m}^{s_{mj}} - \prod_{\substack{m=1\\ s_{mj}>0}}^{I} u_m^{s_{mj}}\Big\|_{L^2((0,T)\times\Omega)}.
\end{aligned} \tag{4.17}$$

By using the strong convergence of u_ε and L^∞-estimates of u_ε and u, it follows that

$$\| \prod_{\substack{m=1\\ s_{mj}<0}}^{I} u_{\varepsilon_m}^{-s_{mj}} - \prod_{\substack{m=1\\ s_{mj}<0}}^{I} u_m^{-s_{mj}}\|_{L^2((0,T)\times\Omega)} \quad \text{and} \quad \| \prod_{\substack{m=1\\ s_{mj}>0}}^{I} u_{\varepsilon_m}^{s_{mj}} - \prod_{\substack{m=1\\ s_{mj}>0}}^{I} u_m^{s_{mj}}\|_{L^2((0,T)\times\Omega)}$$

are convergent to 0 as $\varepsilon \to 0$. Therefore $\|SR(u_\varepsilon)_i - SR(u)_i\|_{L^2((0,T)\times\Omega)} \to 0$ as $\varepsilon \to 0$. From (4.14), the theorem follows. □

Remark 4.9. The strong convergence of $(SR(u_\varepsilon))_{\varepsilon>0}$ implies that it is *two-scale* convergent to $SR(u)$ in the sense of (2.43).

4.3. **Passage to the limit as** $\varepsilon \to 0$**.** Let us consider the two functions $\phi_0 \in C_0^\infty((0,T)\times\Omega)^I$ and $\phi_1 \in C_0^\infty(((0,T)\times\Omega); C_{per}^\infty(Y))^I$ such that $\phi(t,x,\frac{x}{\varepsilon}) := \phi_0(t,x) + \varepsilon\phi_1(t,x,\frac{x}{\varepsilon}) \in C_0^\infty(((0,T)\times\Omega); C_{per}^\infty(Y))^I$. Using ϕ as test function in the weak formulation of (1.12) one obtains

$$\int_0^T \langle \frac{\partial u_\varepsilon(t)}{\partial t}, \phi(t)\rangle_{[H^{1,2}(\Omega_\varepsilon^p)^*]^I \times [H^{1,2}(\Omega_\varepsilon^p)]^I}\, dt$$

$$
\begin{aligned}
&- \int_0^T \langle \nabla \cdot D\nabla u_\varepsilon(t), \phi(t)\rangle_{[H^{1,2}(\Omega_\varepsilon^p)^*]^I \times [H^{1,2}(\Omega_\varepsilon^p)]^I}\, dt \\
&= \int_0^T \langle SR(u_\varepsilon(t)), \phi(t)\rangle_{[H^{1,2}(\Omega_\varepsilon^p)^*]^I \times [H^{1,2}(\Omega_\varepsilon^p)]^I}\, dt;
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
&\sum_{i=1}^I \int_0^T \langle \frac{\partial u_{\varepsilon_i}(t)}{\partial t}, \phi_i(t)\rangle_{H^{1,2}(\Omega_\varepsilon^p)^* \times H^{1,2}(\Omega_\varepsilon^p)}\, dt \\
&- \sum_{i=1}^I \int_0^T \langle \nabla \cdot D\nabla u_{\varepsilon_i}(t), \phi_i(t)\rangle_{H^{1,2}(\Omega_\varepsilon^p)^* \times H^{1,2}(\Omega_\varepsilon^p)}\, dt \\
&= \sum_{i=1}^I \int_0^T \langle SR(u_\varepsilon(t))_i, \phi_i(t)\rangle_{H^{1,2}(\Omega_\varepsilon^p)^* \times H^{1,2}(\Omega_\varepsilon^p)}\, dt;
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
&\sum_{i=1}^I \int_0^T \langle \frac{\partial u_{\varepsilon_i}(t)}{\partial t}, \phi_i(t)\rangle_{H^{1,2}(\Omega_\varepsilon^p)^* \times H^{1,2}(\Omega_\varepsilon^p)}\, dt \\
&+ \sum_{i=1}^I \int_0^T \int_{\Omega_\varepsilon^p} D\nabla u_{\varepsilon_i}(t,x) \nabla \phi_i(t,x,\frac{x}{\varepsilon})\, dx\, dt \\
&= \sum_{i=1}^I \int_0^T \langle SR(u_\varepsilon(t))_i, \phi_i(t)\rangle_{H^{1,2}(\Omega_\varepsilon^p)^* \times H^{1,2}(\Omega_\varepsilon^p)}\, dx\, dt.
\end{aligned}
\tag{4.18}
$$

Now we pass to the *two-scale* limit in (4.18) term by term.

$$
\begin{aligned}
&\lim_{\varepsilon \to 0 \sum_{i=1}^I} \int_0^T \langle \frac{\partial u_\varepsilon(t)}{\partial t}, \phi_i(t)\rangle_{H^{1,2}(\Omega_\varepsilon^p)^* \times H^{1,2}(\Omega_\varepsilon^p)}\, dt \\
&= -\lim_{\varepsilon \to 0} \sum_{i=1}^I \int_0^T \int_{\Omega_\varepsilon^p} u_{\varepsilon_i}(t,x) \Big(\frac{\partial \phi_{0_i}(t,x)}{\partial t} + \varepsilon \frac{\partial \phi_{1_i}(t,x,\frac{x}{\varepsilon})}{\partial t}\Big)\, dx\, dt \\
&= -\lim_{\varepsilon \to 0} \sum_{i=1}^I \int_0^T \int_\Omega \chi(\frac{x}{\varepsilon}) u_{\varepsilon_i}(t,x) \frac{\partial \phi_{0_i}}{\partial t}\, dx\, dt \\
&\quad \underbrace{- \lim_{\varepsilon \to 0} \varepsilon \sum_{i=1}^I \int_0^T \int_\Omega \chi(\frac{x}{\varepsilon}) u_{\varepsilon_i}(t,x) \frac{\partial \phi_{1_i}}{\partial t}\, dx\, dt}_{=0} \\
&= -\sum_{i=1}^I \int_0^T \int_\Omega \int_Y \chi(y) u_i(t,x) \frac{\partial \phi_{0_i}(t,x)}{\partial t}\, dx\, dy\, dt \\
&= -\sum_{i=1}^I \int_0^T \int_\Omega \int_{Y^p} u_i(t,x) \frac{\partial \phi_{0_i}(t,x)}{\partial t}\, dx\, dy\, dt, \quad \text{since } \chi(y) = 1 \text{ in } Y^p \\
&= |Y^p| \sum_{i=1}^I \int_0^T \langle \frac{\partial u_i(t)}{\partial t}, \phi_{0_i}(t)\rangle_{H^{1,2}(\Omega_\varepsilon^p)^* \times H^{1,2}(\Omega_\varepsilon^p)}\, dt
\end{aligned}
$$

$$= |Y^p| \int_0^T \langle \frac{\partial u(t)}{\partial t}, \phi_0(t) \rangle_{[H^{1,2}(\Omega_\varepsilon^p)^*]^I \times [H^{1,2}(\Omega_\varepsilon^p)]^I} \, dt. \tag{4.19}$$

Again,

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \sum_{i=1}^I \int_0^T \int_{\Omega_\varepsilon^p} D\nabla_x u_{\varepsilon_i}(t,x) \nabla_x \phi_i(t,x,\frac{x}{\varepsilon}) \, dx \, dt \\
&= \lim_{\varepsilon\to 0} \sum_{i=1}^I \int_0^T \int_{\Omega_\varepsilon^p} D\nabla_x u_{\varepsilon_i}(t,x) \nabla_x (\phi_{0_i}(t,x) + \varepsilon \phi_{1_i}(t,x,\frac{x}{\varepsilon})) \, dx \, dt \\
&= \lim_{\varepsilon\to 0} \Big[\sum_{i=1}^I \int_0^T \int_{\Omega} \chi(\frac{x}{\varepsilon}) D\nabla_x u_{\varepsilon_i}(t,x) (\nabla_x \phi_{0_i}(t,x) + \nabla_y \phi_{1_i}(t,x,\frac{x}{\varepsilon})) \, dx \, dt \\
&\quad + \varepsilon \sum_{i=1}^I \int_0^T \int_{\Omega} \chi(\frac{x}{\varepsilon}) D\nabla_x u_{\varepsilon_i}(t,x) \nabla_x \phi_{1_i}(t,x,\frac{x}{\varepsilon}) \, dx \, dt \Big] \\
&= \lim_{\varepsilon\to 0} \sum_{i=1}^I \int_0^T \int_{\Omega} \chi(\frac{x}{\varepsilon}) D\nabla_x u_{\varepsilon_i}(t,x) (\nabla_x \phi_{0_i}(t,x) + \nabla_y \phi_{1_i}(t,x,\frac{x}{\varepsilon})) \, dx \, dt \qquad (4.20) \\
&\quad \underbrace{+ \lim_{\varepsilon\to 0} \varepsilon \sum_{i=1}^I \int_0^T \int_{\Omega} \chi(\frac{x}{\varepsilon}) D\nabla_x u_{\varepsilon_i}(t,x) \nabla_x \phi_{1_i}(t,x,\frac{x}{\varepsilon}) \, dx \, dt}_{=0} \\
&= \sum_{i=1}^I \int_0^T \int_\Omega \int_Y \chi(y) D(\nabla_x u_i(t,x) + \nabla_y u_{1_i}(t,x,y))(\nabla_x \phi_{0_i}(t,x) \\
&\quad + \nabla_y \phi_{1_i}(t,x,y)) \, dx \, dy \, dt \\
&= \sum_{i=1}^I \int_0^T \int_\Omega \int_{Y^p} D(\nabla_x u_i(t,x) + \nabla_y u_{1_i}(t,x,y))(\nabla_x \phi_{0_i}(t,x) \\
&\quad + \nabla_y \phi_{1_i}(t,x,y)) \, dx \, dy \, dt
\end{aligned}$$

By (3.1), $\sup_{\varepsilon>0} \|u_{\varepsilon_i}(t)\|_{L^r(\Omega_\varepsilon^p)} \le C_1$ for all i and for a.e. t. Then we have $\|SR(u_\varepsilon)_i\|_{L^2(\Omega_\varepsilon^p)} \le C$. Since $L^2(\Omega_\varepsilon^p) \hookrightarrow H^{1,2}(\Omega_\varepsilon^p)^*$, from (2.3),

$$\langle SR(u_\varepsilon)_i, \phi_i \rangle_{H^{1,2}(\Omega_\varepsilon^p)^* \times H^{1,2}(\Omega_\varepsilon^p)} = \langle SR(u_\varepsilon)_i, \phi_i \rangle_{L^2(\Omega_\varepsilon^p) \times L^2(\Omega_\varepsilon^p)}, \quad \phi_i \in H^{1,2}(\Omega_\varepsilon^p).$$

Thus

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \sum_{i=1}^I \int_0^T \langle SR(u_\varepsilon(t))_i, \phi_i(t) \rangle_{H^{1,2}(\Omega_\varepsilon^p)^* \times H^{1,2}(\Omega_\varepsilon^p)} \, dt \\
&= \lim_{\varepsilon\to 0} \sum_{i=1}^I \int_0^T \langle SR(u_\varepsilon(t))_i, \phi_i(t) \rangle_{L^2(\Omega_\varepsilon^p) \times L^2(\Omega_\varepsilon^p)} \, dt \\
&= \lim_{\varepsilon\to 0} \sum_{i=1}^I \int_0^T \int_{\Omega_\varepsilon^p} SR(u_\varepsilon(t,x))_i \phi_i(t,x) \, dx \, dt \\
&= \lim_{\varepsilon\to 0} \sum_{i=1}^I \int_0^T \int_{\Omega} \chi(\frac{x}{\varepsilon}) SR(u_\varepsilon)_i \phi_{0_i}(t,x) \, dx \, dt
\end{aligned}$$

$$
\begin{aligned}
&+ \underbrace{\lim_{\varepsilon\to 0} \varepsilon \sum_{i=1}^{I} \int_0^T \int_\Omega \chi(\frac{x}{\varepsilon}) SR(u_\varepsilon)_i \phi_{1_i}(t,x,\frac{x}{\varepsilon})\,dx\,dt}_{=0} \\
&= \sum_{i=1}^{I} \int_0^T \int_\Omega \int_Y \chi(y) SR(u(t,x))_i \phi_{0_i}(t,x)\,dx\,dy\,dt \\
&= \sum_{i=1}^{I} \int_0^T \int_\Omega \int_{Y^p} SR(u(t,x))_i \phi_{0_i}(t,x)\,dx\,dy\,dt \\
&= |Y^p| \int_0^T \langle SR(u(t)), \phi_0(t)\rangle_{[H^{1,2}(\Omega)^*]^I\times[H^{1,2}(\Omega)]^I}\,dt. \qquad (4.21)
\end{aligned}
$$

Combining (4.19)-(4.21), we obtain

$$
\begin{aligned}
&|Y^p| \int_0^T \langle \frac{\partial u(t)}{\partial t}, \phi_0(t)\rangle_{[H^{1,2}(\Omega)^*]^I\times[H^{1,2}(\Omega)]^I}\,dt + \sum_{i=1}^{I} \int_0^T \int_\Omega \int_{Y^p} D(\nabla_x u_i(t,x) \\
&+ \nabla_y u_{1_i}(t,x,y))(\nabla_x \phi_{0_i}(t,x) + \nabla_y \phi_{1_i}(t,x,y))\,dx\,dy\,dt \\
&= |Y^p| \int_0^T \langle SR(u(t)), \phi_0(t)\rangle_{[H^{1,2}(\Omega)^*]^I\times[H^{1,2}(\Omega)]^I}\,dt.
\end{aligned} \tag{4.22}
$$

Now choosing $\phi_0(t,x)\equiv 0$, i.e., $\phi_{0_i}(t,x)\equiv 0$ for all $i=1,2,\dots,I$, then $\phi(t,x) = \phi_1(t,x,\frac{x}{\varepsilon})$ and the equation (4.22) reduces to

$$
\sum_{i=1}^{I} \int_0^T \int_\Omega \int_{Y^p} D(\nabla_x u_i(t,x) + \nabla_y u_{1_i}(t,x,y)) \nabla_y \phi_{1_i}(t,x,y)\,dx\,dy\,dt = 0. \tag{4.23}
$$

Let us choose $u_{1_i}(t,x,y) = \sum_{j=1}^n \frac{\partial u_i(t,x)}{\partial x_j} a_j(t,x,y) + c_i(x)$, for all $i = 1,2,\dots,I$, where $c(x)$ is any arbitrary function of x. The equation (4.23) is satisfied by each u_{1_i} if a_j, for $j=1,2,\dots,n$, is the solution of the *Cell-Problem*

$$
\begin{aligned}
-\nabla_y \cdot (D(\nabla_y a_j(t,x,y) + e_j)) = 0 \quad &\text{for } (t,x,y)\in(0,T)\times\Omega\times Y^P, &(4.24)\\
-D(\nabla_y a_j(t,x,y) + e_j)\cdot \vec{n} = 0 \quad &\text{for } (t,x,y)\in(0,T)\times\Omega\times\Gamma, &(4.25)\\
y \mapsto a_j(y) \text{ is } Y-\text{periodic}.& &(4.26)
\end{aligned}
$$

On the other hand, if a_j is the solution of the cell-problem (4.24)-(4.26), the equation (4.23) is satisfied if $u_{1_i}(t,x,y) = \sum_{j=1}^n \frac{\partial u_i(t,x)}{\partial x_j} a_j(t,x,y) + c_i(x)$. Setting $\phi_1(t,x,\frac{x}{\varepsilon})\equiv 0$; i.e., $\phi_{1_i}(t,x,\frac{x}{\varepsilon})\equiv 0$ for all i. Then the equation (4.22) reduces to

$$
\begin{aligned}
&|Y^p| \int_0^T \langle \frac{\partial u(t)}{\partial t}, \phi_0(t)\rangle_{[H^{1,2}(\Omega)^*]^I\times[H^{1,2}(\Omega)]^I}\,dt + \sum_{i=1}^{I} \int_0^T \int_\Omega \int_{Y^p} D(\nabla_x u_i(t,x) \\
&+ \nabla_y u_{1_i}(t,x,y))(\nabla_x \phi_{0_i}(t,x) + \nabla_y \phi_{1_i}(t,x,y))\,dx\,dy\,dt \\
&= |Y^p| \int_0^T \langle SR(u(t)), \phi_0(t)\rangle_{[H^{1,2}(\Omega)^*]^I\times[H^{1,2}(\Omega)]^I}\,dt;
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
&\sum_{i=1}^{I}\int_0^T\langle\frac{\partial u_i(t)}{\partial t},\phi_{0_i}(t)\rangle_{H^{1,2}(\Omega)^*\times H^{1,2}(\Omega)}\,dt\\
&+\sum_{i=1}^{I}\int_0^T\int_\Omega\int_{Y^p}\frac{D}{|Y^p|}(\nabla_x u_i(t,x)+\nabla_y u_{1_i}(t,x,y))\nabla_x\phi_{0_i}(t,x)\,dx\,dy\,dt\\
&=\sum_{i=1}^{I}\int_0^T\langle SR(u(t))_i,\phi_{0_i}(t)\rangle_{H^{1,2}(\Omega)^*\times H^{1,2}(\Omega)}\,dt.
\end{aligned}
\tag{4.27}
$$

Substituting $u_{1_i}(t,x,y)=\vec{a}(t,x,y)\cdot\nabla_x u_i(t,x)+c(x)$; i.e., $\nabla_y u_{1_i}=\sum_{j=1}^n\nabla_y a_j\frac{\partial u_i}{\partial x_j}$ in (4.27), then we obtain

$$
\begin{aligned}
&\sum_{i=1}^{I}\int_0^T\langle\frac{\partial u_i(t)}{\partial t},\phi_{0_i}(t)\rangle_{H^{1,2}(\Omega)^*\times H^{1,2}(\Omega)}\,dt\\
&+\sum_{i=1}^{I}\int_0^T\int_\Omega\int_{Y^p}\frac{D}{|Y^p|}\Big(\nabla_x u_i(t,x)+\sum_{j=1}^{n}\nabla_y a_j\frac{\partial u_i(t,x,y)}{\partial x_j}\Big)\nabla_x\phi_{0_i}(t,x)\,dx\,dy\,dt\\
&=\sum_{i=1}^{I}\int_0^T\langle SR(u(t))_i,\phi_{0_i}(t)\rangle_{H^{1,2}(\Omega)^*\times H^{1,2}(\Omega)}\,dt,
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
&\sum_{i=1}^{I}\int_0^T\langle\frac{\partial u_i(t)}{\partial t},\phi_{0_i}(t)\rangle_{H^{1,2}(\Omega)^*\times H^{1,2}(\Omega)}\,dt\\
&+\sum_{i=1}^{I}\int_0^T\int_\Omega\sum_{j,k=1}^{n}\Big\{\frac{D}{|Y^p|}\int_{Y^p}(\delta_{jk}+\frac{\partial a_j}{\partial y_k})\,dy\Big\}\frac{\partial u_i(t,x)}{\partial x_j}\frac{\partial\phi_{0_i}(t,x)}{\partial x_k}\,dx\,dt\\
&=\sum_{i=1}^{I}\int_0^T\langle SR(u(t))_i,\phi_{0_i}(t)\rangle_{H^{1,2}(\Omega)^*\times H^{1,2}(\Omega)}\,dt;
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
&\sum_{i=1}^{I}\int_0^T\langle\frac{\partial u_i(t)}{\partial t},\phi_{0_i}(t)\rangle_{H^{1,2}(\Omega)^*\times H^{1,2}(\Omega)}\,dt+\sum_{i=1}^{I}\int_0^T\int_\Omega\Upsilon\,\nabla_x u_i(t,x)\nabla\phi_{0_i}(t,x)\,dx\,dt\\
&=\sum_{i=1}^{I}\int_0^T\langle SR(u(t))_i,\phi_{0_i}(t)\rangle_{H^{1,2}(\Omega)^*\times H^{1,2}(\Omega)}\,dt,
\end{aligned}
\tag{4.28}
$$

where Υ is a second order tensor whose components are given as

$$
\rho_{jk}=\int_{Y^p}\frac{D}{|Y^p|}(\delta_{jk}+\frac{\partial a_j}{\partial y_k})\,dy\quad\text{for all } j,k=1,2,\dots,n,
\tag{4.29}
$$

where a_j is the solution of the cell-problem (4.24)-(4.26). Similarly the boundary condition simplifies to

$$
\Upsilon\nabla u\cdot\vec{n}=0\quad\text{on } (0,T)\times\partial\Omega.
\tag{4.30}
$$

Therefore the strong form of the complete homogenized problem is

$$\frac{\partial u}{\partial t} - \nabla \Upsilon \nabla u = SR(u) \quad \text{in } (0,T) \times \Omega, \tag{4.31}$$

$$-\Upsilon \nabla u \cdot \vec{n} = 0 \quad \text{on } (0,T) \times \partial\Omega, \tag{4.32}$$

$$u(0,x) = u_0(x) \quad \text{in } \Omega. \tag{4.33}$$

Let us denote this problem by (P).

Proposition 4.10 ([10]). *The tensor* $\Upsilon = (\rho_{jk})_{\substack{1\le j\le n\\ 1\le k\le n}}$ *is a second-order positive definite symmetric tensor.*

Theorem 4.11. *There exists a unique solution* $u \in F^I \cap L^\infty((0,T); L^\infty(\Omega))^I$ *of the homogenized problem* (4.31)–(4.33).

Proof. From (4.1) and (4.7), it follows that the *two-scale* limit u belongs to the space $[H^{1,2}((0,T); H^{1,2}(\Omega)^*) \cap L^2((0,T); H^{1,2}(\Omega)) \cap L^\infty((0,T)\times\Omega)]^I$. It remains to prove

- Uniqueness of the solution of (4.31)–(4.33) and
- $u \in F^I$.

Given the positive definiteness of Υ and the L^∞-estimate of u, the uniqueness follows by a straightforward application of Gronwall's inequality. Now, the reformulation of problem (4.31)-(4.33) is given by

$$\frac{du(t)}{dt} + Au(t) = f(t), \tag{4.34}$$

$$u(0,x) = u_0(x), \tag{4.35}$$

where $f(t) = SR(u(t)) + \kappa u(t)$, $\kappa > 0$ and the operator $A : H^{1,p}(\Omega)^I \to [H^{1,q}(\Omega)^*]^I$ is defined as $Au_\varepsilon := (A_1 u_{\varepsilon_1}, A_2 u_{\varepsilon_2}, \dots, A_I u_{\varepsilon_I})$ such that for $1 \le i \le I$,

$$\langle A_i u_i, w_i\rangle := \int_\Omega \nabla u_i(x) \Upsilon \nabla w_i(x)\,dx + \kappa \int_\Omega u_i(x) w_i(x)\,dx$$

for $u_i \in H^{1,p}(\Omega)$ and $w_i \in H^{1,q}(\Omega)$. Operator A has maximal parabolic regularity on $[H^{1,q}(\Omega)^*]^I$ by section 5.1. Since $u \in L^\infty((0,T)\times\Omega)^I$, $SR(u) + \kappa u \in L^p((0,T); L^p(\Omega))^I$. The embedding $L^p(\Omega) \hookrightarrow H^{1,q}(\Omega)^*$ implies $SR(u) + \kappa u \in L^p((0,T); H^{1,q}(\Omega)^*)^I$. Furthermore, theorem 2.9 shows that u_0 is in X_p^I. Therefore, by [22, Theorem 2.5], there exists a unique solution u in F^I of the problem (4.34)-(4.35) such that

$$|||u|||_{F^I} \le \tilde{C}\Big(|||u_0|||_{X_p^I} + |||f|||_{L^p((0,T); H^{1,q}(\Omega)^*)^I}\Big), \tag{4.36}$$

where $\tilde{C} > 0$ depends only on p but independent of u, u_0 and f. In other words, the problem (P) has a unique positive global weak solution u in F^I. $\square$

5. Appendix

5.1. Maximal regularity.

Definition 5.1. Let $1 < p < \infty$, X be a Banach space and $A : X \to X$ be a closed, not necessarily bounded, operator, where the domain $D(A)$ of A is dense in X. A

is said to have the maximal L^p-regularity if for every $f \in L^p((0,T);X)$ there exists a unique solution $u \in L^p((0,T);D(A)) \cap H^{1,p}((0,T);X)$ of the problem

$$\frac{du(t)}{dt} + Au(t) = f(t) \quad \text{for } t > 0, \tag{5.1}$$

$$u(0) = 0, \tag{5.2}$$

which satisfies

$$\|u\|_{L^p((0,T);X)} + \|u_t\|_{L^p((0,T);X)} + \|u\|_{L^p((0,T);D(A))} \le C\|f\|_{L^p((0,T);X)}, \tag{5.3}$$

where $C > 0$ is a constant independent of f.

For a detailed overview on maximal regularity, we refer to [1, 17, 21, 23, 12] and references therein. Now we set $D(A) := H^{1,p}(\Omega)$ and $X := H^{1,q}(\Omega)^*$. Clearly, $D(A) \overset{d}{\subseteq} X$. ($A \overset{d}{\subseteq} B$ means that A is dense in B.) Let $\mu = (\mu_{ij})_{1\le j\le n,\, 1\le i\le n}$ be a positive definite symmetric matrix-field, where $\mu_{ij} \in C(\bar\Omega)$ and there is a constant $C > 0$

$$\sum_{i,j=1}^{n} \mu_{ij}(x)\zeta_i\zeta_j \ge C|\zeta|^2 \quad \text{for all } \zeta \in \mathbb{R}^n \text{ and } x \in \Omega. \tag{5.4}$$

We define a sesquilinear form $a(u,v) : H^{1,p}(\Omega) \times H^{1,q}(\Omega) \to \mathbb{R}$ by

$$a(u,v) := \int_\Omega \mu\nabla u \cdot \nabla v\, dx + \kappa \int_\Omega uv\, dx \quad \text{for } u \in H^{1,p}(\Omega) \text{ and } v \in H^{1,q}(\Omega), \tag{5.5}$$

where $\kappa > 0$. We further define an operator $A : H^{1,p}(\Omega) \to H^{1,q}(\Omega)^*$ associated with the form $a(u,v)$ by

$$\langle Au, v\rangle := a(u,v) \quad \text{for } u \in H^{1,p}(\Omega) \text{ and } v \in H^{1,q}(\Omega). \tag{5.6}$$

In [5] and [23], it is shown that: (i) $\|A^{is}\|_{L(X)} \le Ke^{\theta|s|}$ for some $0 < \theta < \frac{\pi}{2}$, $s \in \mathbb{R}$, where $K > 0$, and (ii) $(-\infty, 0] \subset \rho(A)$ (resolvent of A) and $\|(\lambda + A)^{-1}\|_{L(X)} \le \frac{C}{1+|\lambda|}$ for every $\lambda \in [0,\infty)$, where $C > 0$. By a theorem of Dore and Venni (cf. [7]), A has maximal L^p-regularity on $H^{1,q}(\Omega)^*$.

5.2. **Some Inequalities.**

Lemma 5.2 (Lyapunov's interpolation inequality)**.** *Let $1 \le p \le q \le r \le \infty$ and $0 < \theta < 1$ be such that $\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{r}$. Assume also that $u \in L^p(\Omega) \cap L^r(\Omega)$. Then $u \in L^q(\Omega)$ and satisfies*

$$\|u\|_{L^q(\Omega)} \le \|u\|^\theta_{L^p(\Omega)} \|u\|^{1-\theta}_{L^r(\Omega)}. \tag{5.7}$$

The prof of the above lemma can be found in [8, Inequality B.2.h].

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Hari Shankar Mahato
Center of Industrial Mathematics, University of Bremen, D-28359, Bremen, Germany
E-mail address: `mahato@math.uni-bremen.de`

Michael Böhm
Center of Industrial Mathematics, University of Bremen, D-28359, Bremen, Germany
E-mail address: `mbohm@math.uni-bremen.de`