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ASYMPTOTIC BEHAVIOUR OF BRANCHES FOR GROUND STATES OF ELLIPTIC SYSTEMS

VLADIMIR BOBKOV, YAVDAT IL'YASOV

Abstract. We consider the behaviour of solutions to a system of homogeneous equations with indefinite nonlinearity depending on two parameters (λ, μ). Using spectral analysis a critical point (λ^*, μ^*) of the Nehari manifolds and fibering methods is introduced. We study a branch of a ground state and its asymptotic behaviour, including the blow-up phenomenon at (λ^*, μ^*). The differences in the behaviour of similar branches of solutions for the prototype scalar equations are discussed.

1. Introduction

In this article, we discuss the ground state branch to the following system of equations of variational form

$$\begin{gathered} -\Delta_p u = \lambda |u|^{p-2}u + \alpha f(x)|u|^{\alpha-2}|v|^{\beta}u, \quad x \in \Omega, \\ -\Delta_q v = \mu |v|^{q-2}v + \beta f(x)|u|^{\alpha}|v|^{\beta-2}v, \quad x \in \Omega, \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 1$ is a bounded domain with C^1-boundary $\partial\Omega$, $\lambda, \mu \in \mathbb{R}$, $1 < p < +\infty$, $1 < q < +\infty$ and

$$\alpha, \beta > 0, \quad \frac{\alpha}{p} + \frac{\beta}{q} > 1, \quad \frac{\alpha}{p^*} + \frac{\beta}{q^*} < 1. \tag{1.2}$$

Here p^* and q^* are the standard critical Sobolev exponents. We suppose $f \in L^\infty(\Omega)$ and that the function f may change sign on Ω; i.e., the problem (1.1) has indefinite nonlinearity (cf. [1, 7]). Hereinafter we will always assume $f \not\equiv 0$ in Ω.

The problem (1.1) is actually a generalization of the problem with a single equation

$$\begin{gathered} -\Delta_p w = \lambda |w|^{p-2}w + f(x)|w|^{\gamma-2}w, \quad x \in \Omega, \\ w|_{\partial\Omega} = 0, \end{gathered} \tag{1.3}$$

where $\gamma = \alpha + \beta$ and $p < \gamma < p^*$. This and similar problems with indefinite nonlinearities have received a lot of attention that is mainly due to the interesting and complicated structure of its solutions set; see e.g. Alama, Tarantello [1], Bandle, Pozio, Tesei [6], Berestycki, Capuzzo-Dolcetta, Nirenberg [8], Del Pino, Felmer

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[15], Drábek, Pohozaev [17], Ouyang [31, 32]. From these investigations much is known about the existence, nonexistence and multiplicity of solutions of (1.3). Furthermore, the structure of the branches of positive solutions of (1.3), including the existence of turning points and the blow up behaviour of the branches at limit values of λ has been also investigated [1, 22, 24, 27, 31, 32].

The system (1.1) can also be often found in literature relating to the system of elliptic equations; see e.g. [3, 5, 9, 10, 11, 13, 18] and surveys [19, 20]. In these works under different assumptions, including the cases of critical exponents, the existence, nonexistence and multiplicity of solutions to (1.1) have been obtained. However, in the case of systems of equations there are not so many works dedicated to the systematical study of the branches of solutions and the analysis of their behaviour. Such research is particularly difficult if the problem depends on more than one parameter as in the case of system (1.1). The purpose of the present paper is to shed some light in this direction.

It seems that (1.3) and its simplest generalization (1.1) must have a similar properties of solutions and structure of ground states branches. Furthermore, in the particular case $p = q$, $\lambda = \mu$, to any solution w_λ of (1.3) corresponds a solution (u_λ, v_λ) of (1.1) with $u_\lambda = c_1 w_\lambda$, $v_\lambda = c_2 w_\lambda$ for some constants $c_1, c_2 > 0$ (see Remark 10.1 in Section 10). However, in the present paper we show, on the example of (1.3) and (1.1) (in the case $p = q$, $\lambda = \mu$), that there is an essential distinction in geometrical structure for the sets of positive solutions of scalar elliptic equations and its corresponding vector generalizations. To this end we study the ground states of (1.1) and its behaviour with respect to parameter $\lambda = \mu$ and compare it with known results on (1.3). The distinctions can be seen in Figs. 2 and 3 below, where under different assumptions about the function f a level lines of the corresponding energy functional $\mathcal{E}(\lambda, \mu) = E_{\lambda,\mu}(u_{\bar{\lambda}}, v_{\bar{\lambda}})$ are shown for (1.3) and (1.1), respectively. Note that the results for $p = q$ are a corollary of our main Theorems 2.2–2.5, where the common cases are considered.

The ground state of (1.1) will be obtained using Nehari manifolds method [29, 35] with its fibering approach [33, 34]. It should be noted that the direct application of these methods is impossible, since sufficient conditions to this end have to be satisfied. To overcome these difficulties we follow [21, 23, 24], where in the investigation of one-parameter problems it has been proposed to find critical values of parameter λ, which separate intervals where sufficient conditions of Nehari manifold and fibering method are satisfied.

As in the investigation of the one-parameter problem (1.3) (see [21, 24]) it can be shown that one of the critical points of (1.1) is determined by the first eigenvalues λ_1 and μ_1 of the Dirichlet operators $-\Delta_p$ and $-\Delta_q$, respectively. This point (λ_1, μ_1) divides the plane (λ, μ) into four quadrants I-IV (see Fig.1). Actually the existence of ground states in the positive part of quadrant I follows from [10]. However, our research in this quadrant provide a previously unknown properties of the solutions. The main novelty in the present paper is the investigation of the existence and nonexistence of ground states in quadrant IV, that is why special attention is paid to investigation along the line $(\sigma\lambda_1, \sigma\mu_1)$, $\sigma \in \mathbb{R}$ (see Figure 1).

The article is organized as follows. In Section 2 we present the main results of the paper. In Section 3 we give some preliminaries on the Nehari manifold and fibering method of (1.1). In Section 4 we study critical values of Nehari manifold and fibering method. In Section 5 we prove the existence of ground states of (1.1)

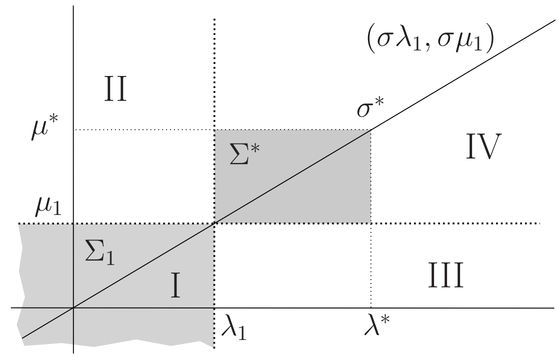

FIGURE 1. The plane (λ, μ)

in quadrant I. In Section 6 we discuss the existence of ground states in quadrant IV. In Section 7 we explore a continuity of the ground states with respect to parameters (λ, μ). In Section 8 we study the behaviour of the energy levels of ground state branches at boundaries of domain of their definition. In Section 9 we obtain some blow-up results for the ground state branches. Section 10 is devoted to final remarks and open problems.

2. MAIN RESULTS

Henceforth we will use the short notation $\bar{\lambda} = (\lambda, \mu)$ and

$$\Omega^+ := \{x \in \Omega : f(x) > 0\}, \quad \Omega^0 := \{x \in \Omega : f(x) = 0\}.$$

We mean that a subset U of Ω is nonempty if the Lebesgue measure of U is nonzero. By $W_0^{1,p}$ and $W_0^{1,q}$ we denote the standard Sobolev spaces on Ω with the norms

$$\|u\|_p := \Big(\int_\Omega |\nabla u|^p\,dx\Big)^{1/p}, \quad \|v\|_q := \Big(\int_\Omega |\nabla v|^q\,dx\Big)^{1/q},$$

respectively. We will use the symbols (λ_1, φ_1) and (μ_1, ψ_1) for the first eigenpairs of the operators $-\Delta_p$ and $-\Delta_q$ in Ω with zero boundary conditions, respectively. It is known that the eigenvalues λ_1, μ_1 are positive, simple and isolated, and the corresponding eigenfunctions φ_1, ψ_1 are positive and can be normalized so that $\|\varphi_1\|_p = 1$, $\|\psi_1\|_q = 1$ [4, 16, 30].

Problem (1.1) has a variational form with the energy functional

$$E_{\bar{\lambda}}(u, v) = \frac{1}{p}P_\lambda(u) + \frac{1}{q}Q_\mu(v) - F(u, v),$$

which is well defined on $W := W_0^{1,p} \times W_0^{1,q}$ under assumption (1.2), and where

$$\begin{gathered}
P_\lambda(u) := \int_\Omega |\nabla u|^p\,dx - \lambda\int_\Omega |u|^p\,dx, \\
Q_\mu(v) := \int_\Omega |\nabla v|^q\,dx - \mu\int_\Omega |v|^q\,dx, \\
F(u, v) := \int_\Omega f(x)|u|^\alpha|v|^\beta\,dx.
\end{gathered}$$

We obtain solutions of (1.1) using the constrained minimization problem

$$n_{\bar\lambda} := \inf\{E_{\bar\lambda}(u,v) : (u,v) \in \mathcal{N}_{\bar\lambda}\}, \tag{2.1}$$

where $\mathcal{N}_{\bar\lambda}$ is the Nehari manifold

$$\mathcal{N}_{\bar\lambda} := \{(u,v) \in W \setminus \{0\} : P_\lambda(u) - \alpha F(u,v) = 0,\ Q_\mu(v) - \beta F(u,v) = 0\}.$$

In the case $\mathcal{N}_{\bar\lambda} = \emptyset$ we assume $n_{\bar\lambda} = +\infty$.

We say that a weak solution of (1.1) is a *ground state* if it provides minimum in (2.1). Our definition is slightly different from the standard one (see [12]). In particular, we allow the ground state to be zero when $n_{\bar\lambda} = 0$.

In the article following critical value plays a crucial role:

$$\sigma^* = \inf_{u,v} \Big[\max\Big\{ \frac{1}{\lambda_1} \frac{\int |\nabla u|^p \, dx}{\int |u|^p \, dx}, \frac{1}{\mu_1} \frac{\int |\nabla v|^q \, dx}{\int |v|^q \, dx} \Big\} : F(u,v) \ge 0 \Big]. \tag{2.2}$$

The main properties of σ^* are given in the following lemma.

Lemma 2.1. *Assume* (1.2) *is satisfied, $p, q \in (1, +\infty)$ and $f \in L^\infty(\Omega)$. Then*

(I) $1 \le \sigma^* < +\infty$;
(II) $1 < \sigma^*$ *if and only if* $F(\varphi_1, \psi_1) < 0$.

Let us denote $\lambda^* := \sigma^* \lambda_1$, $\mu^* := \sigma^* \mu_1$ and

$$\Sigma_1 := \{\bar\lambda = (\lambda, \mu) \in \mathbb{R}^2 : -\infty < \lambda < \lambda_1,\ -\infty < \mu < \mu_1\},$$
$$\Sigma^* := \{\bar\lambda = (\lambda, \mu) \in \mathbb{R}^2 : \lambda_1 < \lambda < \lambda^*,\ \mu_1 < \mu < \mu^*\}.$$

In fact, Σ_1 is the quadrant I, and Σ^* is a subset of quadrant IV (see Figure 1).

Our first main result is the following.

Theorem 2.2. *Assume* (1.2) *is satisfied, $p, q \in (1, +\infty)$ and $f \in L^\infty(\Omega)$.*

(I) *If $\Omega^+ \neq \emptyset$, then* (1.1) *possesses a ground state $(u_{\bar\lambda}, v_{\bar\lambda})$ for all $\bar\lambda \in \Sigma_1$ such that $E_{\bar\lambda}(u_{\bar\lambda}, v_{\bar\lambda}) > 0$ and $u_{\bar\lambda}, v_{\bar\lambda} > 0$ in Ω;*
(II) *if $F(\varphi_1, \psi_1) < 0$, then* (1.1) *possesses a ground state $(u_{\bar\lambda}, v_{\bar\lambda})$ for all $\bar\lambda \in \Sigma^*$such that $E_{\bar\lambda}(u_{\bar\lambda}, v_{\bar\lambda}) < 0$ and $u_{\bar\lambda}, v_{\bar\lambda} > 0$ in Ω;*

This result can be clarified by obtaining some continuity properties of ground states with respect to $\bar\lambda$ and its asymptotic properties at the boundaries of Σ_1 and Σ^*. To this end we study levels of the energy functional $E_{\bar\lambda}$ on the solutions of (1.1) and consider

$$\mathcal{E}(\bar\lambda) := E_{\bar\lambda}(u_{\bar\lambda}, v_{\bar\lambda}), \quad \bar\lambda \in \mathbb{R}^2.$$

Theorem 2.3. *Assume* (1.2) *is satisfied, $p, q \in (1, +\infty)$ and $f \in L^\infty(\Omega)$.*

(I) *If $\Omega_+ \neq \emptyset$, then*
  (a) *the function $\mathcal{E}(\bar\lambda)$ in Σ_1 is continuous;*
  (b) *$\mathcal{E}(\bar\lambda) \to 0$ as $\lambda \uparrow \lambda_1$ and $\mu \uparrow \mu_1$,*
  (c) *$\mathcal{E}(\bar\lambda) \to 0$ as $\bar\lambda \to (\lambda_1, \mu_0)$ for any $\mu_0 < \mu_1$, and $\mathcal{E}(\bar\lambda) \to 0$ as $\bar\lambda \to (\lambda_0, \mu_1)$ for any $\lambda_0 < \lambda_1$;*
(II) *If $F(\varphi_1, \psi_1) < 0$, then*
  (a) *the function $\mathcal{E}(\bar\lambda)$ in Σ^* is continuous;*
  (b) *$\mathcal{E}(\bar\lambda) \to 0$ as $\lambda \downarrow \lambda_1$ and $\mu \downarrow \mu_1$;*
  (c) *$\mathcal{E}(\bar\lambda) \to 0$ as $\bar\lambda \to (\lambda_1, \mu_0)$ for any $\mu_0 \in (\mu_1, \mu^*)$, and $\mathcal{E}(\bar\lambda) \to 0$ as $\bar\lambda \to (\lambda_0, \mu_1)$ for any $\lambda_0 \in (\lambda_1, \lambda^*)$;*
(III) *If $f(x) \le 0$, $p, q \ge 2$ and $\max\{p, q\} > 2$, then $\mathcal{E}(\bar\lambda) \to -\infty$ as $\bar\lambda \to (\lambda^*, \mu^*)$.*

The statements of Theorems 2.2, 2.3 should be supplemented by the fact that in quadrants II and III, i.e. for $\lambda < \lambda_1$, $\mu > \mu_1$ and $\lambda > \lambda_1$, $\mu < \mu_1$, we have $n_{\bar\lambda} = 0$ (see Remark 10.2 in Section 10).

Our next results deal with the blow-up behaviour of ground states $(u_{\bar\lambda}, v_{\bar\lambda})$ at the boundaries of Σ_1 and Σ^*.

Theorem 2.4. *Assume* (1.2) *is satisfied,* $p, q \in (1, +\infty)$, $f \in L^\infty(\Omega)$ *and* $\Omega_+ \neq \emptyset$.

(1) *Let* $q < \beta$. *Then* $\|u_{\bar\lambda}\|_p \to +\infty$ *and* $\|v_{\bar\lambda}\|_q \to 0$ *as* $\lambda \uparrow \lambda_1$, $\mu \to \mu_0$ *for any* $\mu_0 < \mu_1$.
(2) *Let* $p < \alpha$. *Then* $\|u_{\bar\lambda}\|_p \to 0$ *and* $\|v_{\bar\lambda}\|_q \to +\infty$ *as* $\lambda \to \lambda_0$, $\mu \uparrow \mu_1$ *for any* $\lambda_0 < \lambda_1$.
(3) *Let* $p < \alpha$, $q < \beta$, $F(\varphi_1, \psi_1) < 0$ *and* $(\bar\lambda_m)$, $m \in \mathbb{N}$ *be a sequence in* Σ_1 *such that* $\lambda_m \to \lambda_1$ *and* $\mu_m \to \mu_1$ *as* $m \to \infty$. *Then up to a subsequence one of the following convergences hold:*
   - $\|u_{\bar\lambda_m}\|_p \to \infty$ *and* $\|v_{\bar\lambda_m}\|_q \to 0$ *as* $m \to \infty$, *or*
   - $\|u_{\bar\lambda_m}\|_p \to 0$ *and* $\|v_{\bar\lambda_m}\|_q \to \infty$ *as* $m \to \infty$.

Theorem 2.5. *Assume* (1.2) *is satisfied,* $f \in L^\infty(\Omega)$, $f(x) \le 0$, $p, q \ge 2$ *and* $\max\{p, q\} > 2$. *Then* $\|u_{\bar\lambda}\|_p \to +\infty$ *and* $\|v_{\bar\lambda}\|_q \to +\infty$ *as* $\bar\lambda \to (\lambda^*, \mu^*)$.

It is relevant to remark that in this theorem and in statement III of Theorem 2.3 the assumption $f(x) \le 0$ includes the case $f(x) < 0$ in Ω. Notice that for the scalar problem (1.3) in the case $f(x) < 0$ the corresponding ground states w_λ cannot blow up at any finite value of parameter λ (see Section 10).

It is interesting to consider a special case of (1.1), with $p = q$ and $\lambda = \mu$. Note that in this case $\lambda^* = \mu^*$. From Lemma 2.1 and Theorem 2.2 we have the following result.

Corollary 2.6. *Assume* $p = q$, $\lambda = \mu$, $p < \alpha + \beta < p^*$ *and* $f \in L^\infty(\Omega)$. *Then*

(I) $\lambda^* < +\infty$;
(II) $\lambda_1 < \lambda^*$ *if and only if* $F(\varphi_1, \varphi_1) < 0$;
(III) *the problem* (1.1) *has two sets of ground states:*
   (1) (u_λ, v_λ) *for* $\lambda \in (-\infty, \lambda_1)$ *in the case* $\Omega^+ \neq \emptyset$;
   (2) (u_λ, v_λ) *for* $\lambda \in (\lambda_1, \lambda^*)$ *in the case* $F(\varphi_1, \varphi_1) < 0$.

From Theorem 2.3 we have he following corollary.

Corollary 2.7. *Assume* $p = q$, $\lambda = \mu$, $p < \alpha + \beta < p^*$ *and* $f \in L^\infty(\Omega)$.

(I) *If* $\Omega_+ \neq \emptyset$, *then*
   (a) *the function* $\mathcal{E}(\lambda)$ *on* $(-\infty, \lambda_1)$ *is continuous;*
   (b) $\mathcal{E}(\lambda) \to 0$ *as* $\lambda \to \lambda_1$;
(II) *If* $F(\varphi_1, \varphi_1) < 0$, *then the function* $\mathcal{E}(\lambda)$ *on* (λ_1, λ^*) *is continuous and* $\mathcal{E}(\lambda) \to 0$ *as* $\lambda \downarrow \lambda_1$.
(III) *If* $f(x) \le 0$ *and* $p > 2$, *then* $\mathcal{E}(\lambda) \to -\infty$ *as* $\lambda \to \lambda^*$.

3. Nehari manifold and fibering method

According to the fibering method [33, 34] consider

$$E_{\bar\lambda}(tu, sv) = \frac{t^p}{p} P_\lambda(u) + \frac{s^q}{q} Q_\mu(v) - t^\alpha s^\beta F(u, v), \quad t, s > 0,$$

for $(u,v) \in W$, and the system of equations

$$\begin{aligned} \frac{\partial}{\partial t} E_{\bar\lambda}(tu,sv) \equiv t^{p-1}P_\lambda(u) - \alpha t^{\alpha-1}s^{\beta}F(u,v) = 0,\\ \frac{\partial}{\partial s} E_{\bar\lambda}(tu,sv) \equiv s^{q-1}Q_\mu(v) - \beta t^{\alpha}s^{\beta-1}F(u,v) = 0. \end{aligned} \tag{3.1}$$

Simple analysis shows that only if (u,v) belongs to one of the following sets

$$\mathcal{A} := \{(u,v) \in W : P_\lambda(u) > 0,\ Q_\mu(v) > 0,\ F(u,v) > 0\},$$
$$\mathcal{B} := \{(u,v) \in W : P_\lambda(u) < 0,\ Q_\mu(v) < 0,\ F(u,v) < 0\},$$

the system (3.1) has a unique nontrivial solution $s = s(u,v)$, $t = t(u,v)$ and

$$t^{pqd} = \frac{\alpha^{\beta-q}}{\beta^{\beta}} \frac{|P_\lambda(u)|^{q-\beta}|Q_\mu(v)|^{\beta}}{|F(u,v)|^{q}}, \tag{3.2}$$

$$s^{pqd} = \frac{\beta^{\alpha-p}}{\alpha^{\alpha}} \frac{|P_\lambda(u)|^{\alpha}|Q_\mu(v)|^{p-\alpha}}{|F(u,v)|^{p}}, \tag{3.3}$$

where we denote

$$d := \frac{\alpha}{p} + \frac{\beta}{q} - 1.$$

Substituting these roots to $E_{\bar\lambda}(tu,sv)$ we obtain the function

$$\mathcal{J}_{\bar\lambda}(u,v) := E_{\bar\lambda}(t(u,v)u, s(u,v)v) = C\frac{|P_\lambda(u)|^{\alpha/(pd)}|Q_\mu(v)|^{\beta/(qd)}}{|F(u,v)|^{1/d}} \operatorname{sign}(F(u,v)), \tag{3.4}$$

where

$$C = \Big(\frac{1}{\alpha^{\alpha q}\beta^{\beta p}}\Big)^{1/(pqd)} d.$$

Observe that $\mathcal{J}_{\bar\lambda}(u,v)$ is zero-homogeneous and weak lower semicontinuous function on $W \setminus \{0\}$.

Consider the Nehari manifold corresponding to (1.1),

$$\mathcal{N}_{\bar\lambda} := \{(u,v) \in W \setminus \{0\} : P_\lambda(u) - \alpha F(u,v) = 0,\ Q_\mu(v) - \beta F(u,v) = 0\},$$

and the Hessian of $E_{\bar\lambda}(u,v)$,

$$\Gamma_{\bar\lambda}(u,v) = \begin{pmatrix} D_{uu}E_{\bar\lambda}(u,v) & D_{uv}E_{\bar\lambda}(u,v) \\ D_{uv}E_{\bar\lambda}(u,v) & D_{vv}E_{\bar\lambda}(u,v) \end{pmatrix},$$

where

$$D_{uu}E_{\bar\lambda}(u,v) = (p-1)P_\lambda(u) - \alpha(\alpha-1)F(u,v), \quad D_{uv}E_{\bar\lambda}(u,v) = -\alpha\beta F(u,v),$$
$$D_{vv}E_{\bar\lambda}(u,v) = (q-1)Q_\mu(v) - \beta(\beta-1)F(u,v).$$

Lemma 3.1. *Let $\bar\lambda \in \mathbb{R}^2$ and (u_0,v_0) be a solution of (2.1) such that*

$$\det \Gamma_{\bar\lambda}(u_0,v_0) \neq 0.$$

Then (u_0,v_0) is a critical point of $E_{\bar\lambda}(u,v)$; i.e., a weak solution of (1.1).

Proof. Let $\bar\lambda \in \mathbb{R}^2$ and (u_0,v_0) be a solution of (2.1). Then by the Lagrange multiplier rule there exist μ_0, μ_1, μ_2 such that $|\mu_0| + |\mu_1| + |\mu_2| \neq 0$ and

$$\begin{aligned} &\mu_0 D_u E_{\bar\lambda}(u_0,v_0)(\xi) + \mu_1[D_u E_{\bar\lambda}(u_0,v_0)(\xi) + D_{uu}E_{\bar\lambda}(u_0,v_0)(u_0,\xi)]\\ &+ \mu_2 D_{uv}E_{\bar\lambda}(u_0,v_0)(v_0,\xi) = 0;\\ &\mu_0 D_v E_{\bar\lambda}(u_0,v_0)(\zeta) + \mu_2[D_{vv}E_{\bar\lambda}(u_0,v_0)(\zeta,v_0) + D_v E_{\bar\lambda}(u_0,v_0)(\zeta)] \end{aligned}$$

$$+\mu_1 D_{uv}E_{\bar\lambda}(u_0,v_0)(u_0,\zeta)=0,$$

for all $\xi\in W_0^{1,p}$ and $\zeta\in W_0^{1,q}$.

The proof will be obtained if we show that $\mu_1=\mu_2=0$. Let $\xi=u_0$ and $\zeta=v_0$. Then taking into account that $(u_0,v_0)\in\mathcal{N}_{\bar\lambda}$ we obtain

$$\mu_1 D_u E_\lambda(u_0,v_0)(u_0,u_0)+\mu_2 D_{uv}E_\lambda(u_0,v_0)(v_0,u_0)=0,$$
$$\mu_1 D_{uv}E_\lambda(u_0,v_0)(v_0,u_0)+\mu_2 D_{vv}E_\lambda(u_0,v_0)(v_0,v_0)=0.$$

But under the assumption $\det\Gamma_\lambda(u_0,v_0)\neq 0$ this is possible if and only if $\mu_1=\mu_2=0$. □

It is not hard to see that on $\mathcal{N}_{\bar\lambda}$ one has

$$\Gamma_{\bar\lambda}(u,v)=\begin{pmatrix}\alpha(p-\alpha)F(u,v) & -\alpha\beta F(u,v)\\ -\alpha\beta F(u,v) & \beta(q-\beta)F(u,v)\end{pmatrix}$$

Hence

$$\det\Gamma_\lambda(u,v)=\alpha\beta pq\big(1-\frac{\alpha}{p}-\frac{\beta}{q}\big)F^2(u,v). \tag{3.5}$$

Under assumptions $\alpha,\beta>0$ and $F(u,v)\neq 0$ it can be highlighted three cases:

$$\frac{\alpha}{p}+\frac{\beta}{q}=1,\quad \frac{\alpha}{p}+\frac{\beta}{q}<1,\quad \frac{\alpha}{p}+\frac{\beta}{q}>1,$$

that corresponds to the cases when $\det\Gamma_\lambda(u,v)$ is zero, positive and negative, respectively. Note that in the present paper we deal only with the last case with negative determinant $\det\Gamma_\lambda(u,v)<0$ that corresponds in the case $p=q$, $\lambda=\mu$ to the super-linear problem (1.3).

4. On critical values of Nehari manifolds and fibering methods

As it has been noted above we study (1.1) using constrained minimization method with Nehari manifold $\mathcal{N}_{\bar\lambda}$ as a constraint. However, the application of this method is restricted by the assumption of Lemma 3.1. The critical value σ^* in (2.2) is introduced to separate the domains in the plane (λ,μ) where this assumption is satisfied. A general approach of finding such kind of values has been introduced in [23, 24] and has been developed in [25, 26, 28] with applications to different problems. However, the direct application of this theory to the system (1.1) is complicated. Moreover, the full theoretical introduction to this approach would lead us away from the main aims of the present paper. We leave it to our forthcoming paper.

The proof of Lemma 2.1 will be a consequence of the next three propositions.

Proposition 4.1. *$1\le\sigma^*<+\infty$.*

Proof. The first estimate easily follows from observation

$$\sigma^*\ge\inf_{u,v}\Big[\max\Big\{\frac{1}{\lambda_1}\frac{\int|\nabla u|^p\,dx}{\int|u|^p\,dx},\ \frac{1}{\mu_1}\frac{\int|\nabla v|^q\,dx}{\int|v|^q\,dx}\Big\}\Big]=1,$$

which holds for (φ_1,ψ_1).

The second estimate is also true, since one can find $u\in W_0^{1,p}\setminus\{0\}$ and $v\in W_0^{1,q}\setminus\{0\}$ such that $\operatorname{supp}u\cap\operatorname{supp}v=\emptyset$. Then $F(u,v)=0$ and

$$\sigma^*\le\max\Big\{\frac{1}{\lambda_1}\frac{\int|\nabla u|^p\,dx}{\int|u|^p\,dx},\ \frac{1}{\mu_1}\frac{\int|\nabla v|^q\,dx}{\int|v|^q\,dx}\Big\}<+\infty.$$

□

Proposition 4.2. *There exists a nonzero minimizer* $(u^*, v^*) \in W$ *of* (2.2) *such that* $u^*, v^* \geq 0$ *in* Ω.

Proof. Let (u_k, v_k) be a minimizing sequence for (2.2). We may assume that $\|u_k\| = 1$ and $\|v_k\| = 1$ for all $k \in \mathbb{N}$, since (2.2) is zero-homogeneous. Hence by the Eberlein-Shmulyan theorem and the Sobolev embedding theorem there exists a subsequence of (u_k, v_k) (which we denote again (u_k, v_k)) and $(u^*, v^*) \in W$ such that

$$u_k \rightharpoonup u^* \text{ weakly in } W_0^{1,p}, \quad v_k \rightharpoonup v^* \text{ weakly in } W_0^{1,q},$$
$$u_k \to u^* \text{ in } L^r,\ r < p^*, \quad v_k \to v^* \text{ in } L^r,\ r < q^*.$$

This implies $F(u^*, v^*) \geq 0$. From Proposition 4.1 we know that $\sigma^* < c < +\infty$ for some $c > 0$. Consequently

$$\int_\Omega |u^*|^p\,dx = \lim_{k\to+\infty} \int_\Omega |u_k|^p\,dx > 1/c > 0,$$
$$\int_\Omega |v^*|^q\,dx = \lim_{k\to+\infty} \int_\Omega |v_k|^q\,dx > 1/c > 0,$$

and therefore, $u^*, v^* \not\equiv 0$. By the weak lower semicontinuity we have

$$\frac{\int_\Omega |\nabla u^*|^p}{\int_\Omega |u^*|^p} \leq \liminf_{k\to+\infty} \frac{\int_\Omega |\nabla u_k|^p}{\int_\Omega |u_k|^p}, \quad \frac{\int_\Omega |\nabla v^*|^q}{\int_\Omega |v^*|^q} \leq \liminf_{k\to+\infty} \frac{\int_\Omega |\nabla v_k|^q}{\int_\Omega |v_k|^q}.$$

Now arguing by contradiction we conclude that (u^*, v^*) is a minimizer of (2.2). Since the functionals in (2.2) are even, we may assume that $u^*, v^* \geq 0$. □

Proposition 4.3. $\sigma^* > 1$ *if and only if* $F(\varphi_1, \psi_1) < 0$.

Proof. Assume first $\sigma^* > 1$. Conversely, suppose that $F(\varphi_1, \psi_1) \geq 0$. Then (φ_1, ψ_1) is an admissible point for minimization problem (2.2). But then $\sigma^* = 1$, which contradicts the assumption $\sigma^* > 1$.

Assume now $F(\varphi_1, \psi_1) < 0$. Suppose, contrary to our claim, that $\sigma^* = 1$. By Proposition 4.2 there exists a nonzero minimizer $(u^*, v^*) \in W$ of (2.2). Then it follows easily that

$$\frac{1}{\lambda_1} \frac{\int |\nabla u^*|^p\,dx}{\int |u^*|^p\,dx} = 1, \quad \frac{1}{\mu_1} \frac{\int |\nabla v^*|^q\,dx}{\int |v^*|^q\,dx} = 1.$$

These equalities are true only if $u^* = \varphi_1$ and $v^* = \psi_1$ up to multipliers, but it is impossible, since $F(\varphi_1, \psi_1) < 0$. □

Proposition 4.4. *Assume* $F(\varphi_1, \psi_1) < 0$. *Let* $\lambda < \lambda^*$ *and* $\mu < \mu^*$. *Then for any* $(u, v) \in W \setminus \{0\}$ *the following implication is true:*

$$P_\lambda(u) \leq 0,\ Q_\mu(v) \leq 0 \Longrightarrow F(u, v) < 0.$$

Proof. Let $\lambda < \lambda^*$ and $\mu < \mu^*$. On the contrary, suppose that $P_\lambda(u) \leq 0$, $Q_\mu(v) \leq 0$ and $F(u, v) \geq 0$. Then there exist $\lambda_0 \leq \lambda < \lambda^*$ and $\mu_0 \leq \mu < \mu^*$ such that $P_{\lambda_0}(u) = 0$ and $Q_{\mu_0}(v) = 0$. From here it follows

$$\frac{1}{\lambda_1} \frac{\int |\nabla u|^p\,dx}{\int |u|^p\,dx} = \frac{\lambda_0}{\lambda_1} < \frac{\lambda^*}{\lambda_1} = \sigma^*, \quad \frac{1}{\mu_1} \frac{\int |\nabla v|^q\,dx}{\int |v|^q\,dx} = \frac{\mu_0}{\mu_1} < \frac{\mu^*}{\mu_1} = \sigma^*.$$

Hence we obtain

$$\max\Big\{\frac{1}{\lambda_1}\frac{\int|\nabla u|^p\,dx}{\int|u|^p\,dx},\frac{1}{\mu_1}\frac{\int|\nabla v|^q\,dx}{\int|v|^q\,dx}\Big\}<\sigma^*,$$

which contradicts the definition of σ^*. $\square$

For further applications of Lemma 3.1 we need the next result.

Corollary 4.5. *Assume* (1.2) *is satisfied,* $p,q\in(1,+\infty)$ *and* $f\in L^\infty(\Omega)$*. Let* $\lambda<\lambda^*$ *and* $\mu<\mu^*$*. Then* $\det\Gamma_{\bar\lambda}(u,v)\neq 0$*, for any* $(u,v)\in\mathcal{N}_{\bar\lambda}$*.*

Proof. By (3.5) we know that

$$\det\Gamma_\lambda(u,v)=\alpha\beta dpqF^2(u,v).$$

Hence, the proof will be obtained if we show that $F(u,v)\neq 0$. Suppose, contrary to our claim, that $F(u,v)=0$ for $(u,v)\in\mathcal{N}_{\bar\lambda}$ and $\lambda<\lambda^*$ and $\mu<\mu^*$. Then the constraints of the Nehari manifold entail $P_\lambda(u)=0$ and $Q_\mu(v)=0$. However, this contradicts the definition of σ^*. $\square$

5. Solutions of (1.1) in Σ_1

In this section we prove statement (I) of Theorem 2.2. First we prove the following proposition.

Proposition 5.1. *Assume that* $\Omega_+\neq\emptyset$*. Then* $\mathcal{N}_{\bar\lambda}\neq\emptyset$ *for all* $\bar\lambda\in\mathbb{R}^2$*.*

Proof. Let $\bar\lambda\in\mathbb{R}^2$. Consider the first eigenpairs $(\lambda_1(B),\varphi_1(B))$ and $(\mu_1(B),\psi_1(B))$ of operators $-\Delta_p$ and $-\Delta_q$ in a ball $B\subset\Omega_+$ with zero boundary conditions, respectively. Evidently, we can choose B such that $\lambda<\lambda_1(B)$ and $\mu<\mu_1(B)$. Then

$$P_\lambda(\varphi_1(B))>0,\quad Q_\mu(\psi_1(B))>0,\quad F(\varphi_1(B),\psi_1(B))>0.$$

Thus $(\varphi_1(B),\psi_1(B))\in\mathcal{A}$ and $(t\varphi_1(B),s\psi_1(B))\in\mathcal{N}_{\bar\lambda}$, where t,s are given by (3.2) and (3.3). $\square$

Lemma 5.2. *Let* $\Omega_+\neq\emptyset$ *and* $\bar\lambda\in\Sigma_1$*. Then problem* (1.1) *has a ground state* $(u_{\bar\lambda},v_{\bar\lambda})$ *such that* $E_{\bar\lambda}(u_{\bar\lambda},v_{\bar\lambda})>0$ *and* $u_{\bar\lambda},v_{\bar\lambda}>0$*.*

Proof. Assume $\Omega_+\neq\emptyset$ and $\bar\lambda\in\Sigma_1$. Let us consider constrained minimization problem (2.1). By Proposition 5.1 we know that $\mathcal{N}_{\bar\lambda}\neq\emptyset$. This yields $n_{\bar\lambda}<+\infty$. Notice that $\mathcal{N}_{\bar\lambda}\subset\mathcal{A}$ for $\bar\lambda\in\Sigma_1$, since $P_\lambda(u)>0$, $Q_\mu(v)>0$ for $(u,v)\in W\setminus\{0\}$ in case $\lambda<\lambda_1$ and $\mu<\mu_1$. Then

$$E_{\bar\lambda}(u,v)=\frac{\alpha}{p}F(u,v)+\frac{\beta}{q}F(u,v)-F(u,v)=\Big(\frac{\alpha}{p}+\frac{\beta}{q}-1\Big)F(u,v)>0. \tag{5.1}$$

for any $(u,v)\in\mathcal{N}_{\bar\lambda}$. This implies that $n_{\bar\lambda}\geq 0$.

Let (u_n,v_n) be a minimizing sequence of (2.1). Let us verify the boundedness of (u_n,v_n). Suppose, contrary to our claim, that for instance $\|u_n\|_p\to+\infty$. Then $P_\lambda(u_n)\to+\infty$, because for $\lambda<\lambda_1$ we have

$$P_\lambda(u_n)=\|u_n\|_p^p-\lambda\int|u_n|^p\,dx\geq C_0(\lambda)\|u_n\|_p^p, \tag{5.2}$$

where $C_0(\lambda)=1-\lambda/\lambda_1$ if $\lambda\in(0,\lambda_1)$, and $C_0(\lambda)=1$ if $\lambda\leq 0$. Since $P_\lambda(u_n)-\alpha F(u_n,v_n)=0$, it follows $F(u_n,v_n)\to+\infty$. Thus by (5.1) we have $E_{\bar\lambda}(u_n,v_n)\to+\infty$. But it is impossible, since $n_{\bar\lambda}<+\infty$. Thus, (u_n,v_n) is bounded.

This implies the existence of subsequence of (u_n, v_n) (which we denote again (u_n, v_n)) and $(u, v) \in W$ such that

$$u_n \rightharpoonup u \text{ weakly in } W_0^{1,p}, \quad v_n \rightharpoonup v \text{ weakly in } W_0^{1,q},$$
$$u_n \to u \text{ in } L^r,\ r < p^*, \quad v_n \to v \text{ in } L^r,\ r < q^*.$$

Let us show that $u, v \not\equiv 0$. Note first that similar to (5.2) we have the following estimate

$$P_\lambda(u_n) \le C_1(\lambda)\|u_n\|_p^p, \tag{5.3}$$

where $C_1(\lambda) = 1$ if $\lambda > 0$ and $C_1(\lambda) = 1 - \lambda/\lambda_1$ if $\lambda \le 0$. Further, from assumptions (1.2) it follows the existence of $p' \in (p, p^*)$ and $q' \in (q, q^*)$ such that $\frac{\alpha}{p'} + \frac{\beta}{q'} = 1$. Hence, applying Hölder's inequality and the Sobolev embedding theorem we obtain

$$F(u_n, v_n) \le C_2\Big(\int |u_n|^{p'}\,dx\Big)^{\alpha/p'}\Big(\int |v_n|^{q'}\,dx\Big)^{\beta/q'} \le C_3\|u_n\|_p^\alpha \|v_n\|_q^\beta, \tag{5.4}$$

where C_2 and C_3 are some constants in $(0, +\infty)$ which do not depend on $n \in \mathbb{N}$ and λ, μ. Since $(u_n, v_n) \in \mathcal{N}_{\bar\lambda}$, we have $t_n, s_n = 1$. Let $q \ge \beta$. Substituting estimates (5.2) and (5.4) to (3.2) we obtain

$$1 = \frac{\alpha^{\beta-q}}{\beta^\beta}\frac{P_\lambda(u_n)^{q-\beta}Q_\mu(v_n)^\beta}{F(u_n, v_n)^q} \ge \frac{\alpha^{\beta-q}}{\beta^\beta}\frac{C_0(\lambda)^{q-\beta}C_0(\mu)^\beta}{C_3^q}\frac{\|u_n\|_p^{qp-\beta p}\|v_n\|_q^{\beta q}}{\|u_n\|_p^{\alpha q}\|v_n\|_q^{\beta q}},$$

and consequently

$$\|u_n\|_p^{pq(\frac{\alpha}{p}+\frac{\beta}{q}-1)} \ge \frac{\alpha^{\beta-q}}{\beta^\beta}\frac{C_0(\lambda)^{q-\beta}C_0(\mu)^\beta}{C_3^q} > 0,$$

for all $n \in \mathbb{N}$. Let $q < \beta$. Using (5.3) instead of (5.2), for all $n \in \mathbb{N}$ we derive

$$\|u_n\|_p^{pq(\frac{\alpha}{p}+\frac{\beta}{q}-1)} \ge \frac{\alpha^{\beta-q}}{\beta^\beta}\frac{C_1(\lambda)^{q-\beta}C_0(\mu)^\beta}{C_3^q} > 0.$$

At the same time, we have

$$\begin{aligned} F(u, v) &= \lim_{n\to+\infty} F(u_n, v_n) \\ &= \frac{1}{\alpha}\lim_{n\to+\infty} P_\lambda(u_n) \\ &\ge \frac{C_0(\lambda)}{\alpha}\lim_{n\to+\infty}\|u_n\|_p^p > C_4(\lambda, \mu) > 0, \end{aligned} \tag{5.5}$$

for some constant $C_4(\lambda, \mu)$ which does not depend on n. But this is possible if and only if $u, v \not\equiv 0$.

From here and using (3.2), (3.3) we can find $s, t > 0$ such that $(tu, sv) \in \mathcal{N}_{\bar\lambda}$. Since $\mathcal{J}_{\bar\lambda}(u, v)$ is a weak lower semicontinuous and zero-homogeneous functional, we have

$$E_{\bar\lambda}(tu, sv) = \mathcal{J}_{\bar\lambda}(u, v) \le \liminf_{n\to+\infty}\mathcal{J}_{\bar\lambda}(u_n, v_n) = \liminf_{n\to+\infty} E_{\bar\lambda}(u_n, v_n) = n_{\bar\lambda}.$$

By the definition of $n_{\bar\lambda}$ here is only equality possible. Thus (tu, sv) is a solution of (2.1). Applying Lemma 3.1 and Corollary 4.5 we deduce that $(u_{\bar\lambda} := tu, v_{\bar\lambda} := sv)$ is a weak solution of (1.1). Furthermore, $(u_{\bar\lambda}, v_{\bar\lambda})$ is a ground state, since it is a minimizer of (2.1). Notice that $E_{\bar\lambda}(u_{\bar\lambda}, v_{\bar\lambda})$ is even function with respect to both variables. Therefore, we may assume that $u_{\bar\lambda}, v_{\bar\lambda} \ge 0$. Applying the arguments of [17] we obtain that $u_{\bar\lambda}, v_{\bar\lambda} > 0$ in Ω. Finally, (5.5) and (5.1) entail $E_{\bar\lambda}(u_{\bar\lambda}, v_{\bar\lambda}) > 0$. □

6. Solutions of (1.1) in Σ^*

In this section we prove statement (II) of Theorem 2.2. Below we always suppose that $F(\varphi_1, \psi_1) < 0$.

Proposition 6.1. *Let $\lambda > \lambda_1$ and $\mu > \mu_1$. Then $\mathcal{N}_{\bar\lambda} \neq \emptyset$. Moreover, there is $(u, v) \in \mathcal{N}_{\bar\lambda}$ such that $E_{\bar\lambda}(u, v) < 0$.*

Proof. Consider (φ_1, ψ_1). Since $\lambda > \lambda_1$ and $\mu > \mu_1$, we have $P_\lambda(\varphi_1) < 0$, $Q_\lambda(\psi_1) < 0$, and by the assumption $F(\varphi_1, \psi_1) < 0$. Thus $(\varphi_1, \psi_1) \in \mathcal{B}$ and $(t\varphi_1, s\psi_1) \in \mathcal{N}_{\bar\lambda}$, where t, s are given by (3.2) and (3.3). Furthermore by (3.4) we have $E_{\bar\lambda}(t\varphi_1, s\psi_1) = \mathcal{J}_{\bar\lambda}(\varphi_1, \psi_1) < 0$. □

Lemma 6.2. *Let $\bar\lambda \in \Sigma^*$. Then problem* (1.1) *has a ground state $(u_{\bar\lambda}, v_{\bar\lambda})$ such that $E_{\bar\lambda}(u_{\bar\lambda}, v_{\bar\lambda}) < 0$ and $u_{\bar\lambda}, v_{\bar\lambda} > 0$.*

Proof. Let $\bar\lambda \in \Sigma^*$. As above in the proof of Lemma 5.2, we will obtain the ground state by the finding of a minimizer of (2.1). By Proposition 6.1 we know that $\mathcal{N}_{\bar\lambda} \neq \emptyset$ and $n_{\bar\lambda} < 0$. Let $(u_n, v_n) \in \mathcal{N}_{\bar\lambda}$ be a minimizing sequence for (2.1). Then $E_{\bar\lambda}(u_n, v_n) < 0$ and by (3.1), (3.4) it easy follows that

$$P_\lambda(u_n) < 0, \quad Q_\mu(v_n) < 0, \quad F(u_n, v_n) < 0.$$

Let us consider $u_n = t_n \hat u_n$ and $v_n = s_n \hat v_n$, where $\|\hat u_n\|_p = 1$ and $\|\hat v_n\|_q = 1$. Then the boundedness of $(\hat u_n)$ and $(\hat v_n)$ in W implies the existence of $(\hat u, \hat v) \in W$ and a subsequence of $(\hat u_n, \hat v_n)$ (which we denote again $(\hat u_n, \hat v_n)$) such that

$$\begin{gathered}\hat u_n \rightharpoonup \hat u \text{ weakly in } W_0^{1,p}, \quad \hat v_n \rightharpoonup \hat v \text{ weakly in } W_0^{1,q},\\ \hat u_n \to \hat u \text{ in } L^r, r < p^*, \quad \hat v_n \to \hat v \text{ in } L^r, r < q^*.\end{gathered}$$

Observe that

$$F(\hat u_n, \hat v_n) < C_0 < 0, \tag{6.1}$$

where C_0 does not depend on $n \in \mathbb{N}$. Indeed, suppose conversely that $F(\hat u, \hat v) = \lim_{n\to+\infty} F(\hat u_n, \hat v_n) = 0$. Note that by the weak lower semicontinuity we have

$$P_\lambda(\hat u) \le \liminf_{n\to+\infty} P_\lambda(\hat u_n) \le 0, \quad Q_\lambda(\hat v) \le \liminf_{n\to+\infty} Q_\lambda(\hat v_n) \le 0.$$

Hence applying Proposition 4.4 we obtain a contradiction.

Let us show that $n_{\bar\lambda} > -\infty$. By (3.4) we have

$$E_{\bar\lambda}(t_n \hat u_n, s_n \hat v_n) = -C\Big(\frac{|P_\lambda(\hat u_n)|^{\alpha q} |Q_\mu(\hat v_n)|^{\beta p}}{|F(\hat u_n, \hat v_n)|^{pq}}\Big)^{1/(pqd)}. \tag{6.2}$$

Taking into account that $\|\hat u_n\|_p = 1$ and $\|\hat v_n\|_q = 1$ we see that $E_{\bar\lambda}(t_n \hat u_n, s_n \hat v_n) \to -\infty$ if and only if $F(\hat u_n, \hat v_n) \to 0$. By (6.1) the last is impossible and therefore, $n_{\bar\lambda} > -\infty$.

Let us prove the boundedness of (t_n, s_n). Assume for example that $t_n \to +\infty$. In view of (3.2) and by (6.1) it is possible only in case $q < \beta$ when $P_\lambda(\hat u_n) \to 0$. However by (6.2) this implies that $E_{\bar\lambda}(t_n \hat u_n, s_n \hat v_n) \to 0$, which contradicts $n_{\bar\lambda} < 0$. By the similar arguments it can be shown that $s_n, t_n \not\to 0$.

Thus, (u_n, v_n) is bounded and up to subsequence we have

$$\begin{gathered}u_n \rightharpoonup u \text{ weakly in } W_0^{1,p}, \quad v_n \rightharpoonup v \text{ weakly in } W_0^{1,q},\\ u_n \to u \text{ in } L^r,\ r < p^*, \quad v_n \to v \text{ in } L^r,\ r < q^*.\end{gathered}$$

Since $u, v \not\equiv 0$, by (3.2) and (3.3), we can find $s, t > 0$ such that $(tu, sv) \in \mathcal{N}_{\bar\lambda}$. Since $\mathcal{J}_{\bar\lambda}(u, v)$ is a weak lower semicontinuous and zero-homogeneous functional, we have

$$E_{\bar\lambda}(tu, sv) = \mathcal{J}_{\bar\lambda}(u, v) \le \liminf_{n\to+\infty} \mathcal{J}_{\bar\lambda}(u_n, v_n) = n_{\bar\lambda}.$$

But by the definition of $n_{\bar\lambda}$ here is only equality possible. Thus $(u_{\bar\lambda} := tu, v_{\bar\lambda} := sv)$ is a solution of (2.1) and by Lemma 3.1 and Corollary 4.5 it follows that $(u_{\bar\lambda}, v_{\bar\lambda})$ is a weak solution of (1.1). Arguing as in the proof of Lemma 5.2 we derive that $u_{\bar\lambda}, v_{\bar\lambda} > 0$ and $(u_{\bar\lambda}, v_{\bar\lambda})$ is a ground state. Obviously $E_{\bar\lambda}(u_{\bar\lambda}, v_{\bar\lambda}) < 0$. □

7. Continuity of the set of ground states

In this section we prove statements (I):(a) and (II):(a) of Theorem 2.3. Observe first that by the construction of $(u_{\bar\lambda}, v_{\bar\lambda})$ we have

$$n_{\bar\lambda} = E_{\bar\lambda}(u_{\bar\lambda}, v_{\bar\lambda}) = \mathcal{J}_{\bar\lambda}(u_{\bar\lambda}, v_{\bar\lambda}),$$

for $\bar\lambda \in \Sigma_1$ and Σ^*. Therefore to confirm (I):(a) and (II):(a) in Theorem 2.3 it is sufficient to establish the following result.

Lemma 7.1. *The function $\mathcal{J}_{\bar\lambda}(u_{\bar\lambda}, v_{\bar\lambda})$ is continuous in Σ_1 and Σ^*.*

Proof. Fix $\bar\lambda_0$ in Σ_1 or Σ^* and the ball B correspondingly in Σ_1 or Σ^* with the center $\bar\lambda_0$. Let $\bar\lambda \in B$. Denote $\Delta\lambda := \lambda - \lambda_0$, $\Delta\mu := \mu - \mu_0$. The proof of the lemma will follows from the next proposition. □

Proposition 7.2. *For sufficiently small $|\bar\lambda - \bar\lambda_0|$ the following inequalities are satisfied*

$$\begin{aligned} &-\Delta\lambda \frac{1}{p} G_p(u_{\bar\lambda}) - \Delta\mu \frac{1}{q} G_q(v_{\bar\lambda}) + r_1(\Delta\lambda, \Delta\mu) \\ &\le \mathcal{J}_{\bar\lambda}(u_{\bar\lambda}, v_{\bar\lambda}) - \mathcal{J}_{\bar\lambda_0}(u_{\bar\lambda_0}, v_{\bar\lambda_0}) \\ &\le -\Delta\lambda \frac{1}{p} G_p(u_{\bar\lambda_0}) - \Delta\mu \frac{1}{q} G_q(v_{\bar\lambda_0}) + r_2(\Delta\lambda, \Delta\mu), \end{aligned} \tag{7.1}$$

where $r_i(\Delta\lambda, \Delta\mu) = o(|\bar\lambda - \bar\lambda_0|)$, $i = 1, 2$; i.e., $\frac{r_i(\Delta\lambda,\Delta\mu)}{|\bar\lambda - \bar\lambda_0|} \to 0$ as $|\bar\lambda - \bar\lambda_0| \to 0$ uniformly on B, and

$$G_p(u) := \int_\Omega |u|^p\,dx, \quad G_q(v) := \int_\Omega |v|^q\,dx, \quad (u, v) \in W.$$

Proof. Note first that

$$\mathcal{J}_{\bar\lambda}(u_{\bar\lambda_0}, v_{\bar\lambda_0}) \ge \mathcal{J}_{\bar\lambda}(u_{\bar\lambda}, v_{\bar\lambda}), \tag{7.2}$$

since $\mathcal{J}_{\bar\lambda}(u_{\bar\lambda}, v_{\bar\lambda}) = n_{\bar\lambda}$. Furthermore, by (3.4)

$$\mathcal{J}_{\bar\lambda}(u_{\bar\lambda_0}, v_{\bar\lambda_0}) = C\frac{(P_{\lambda_0}(u_{\bar\lambda_0}) - \Delta\lambda G_p(u_{\bar\lambda_0}))^{\alpha/(pd)} (Q_{\mu_0}(v_{\bar\lambda_0}) - \Delta\mu G_q(v_{\bar\lambda_0}))^{\beta/(qd)}}{F(u_{\bar\lambda_0}, v_{\bar\lambda_0})^{1/d}}. \tag{7.3}$$

Using Taylor's theorem with Lagrange form of the remainder we obtain

$$\begin{aligned} &(Q_{\mu_0}(v_{\bar\lambda_0}) - \Delta\mu G_q(v_{\bar\lambda_0}))^{\beta/(qd)} \\ &= Q_{\mu_0}(v_{\bar\lambda_0})^{\beta/(qd)} - \Delta\mu \frac{\beta}{qd} Q_{\mu_0}(v_{\bar\lambda_0})^{\frac{\beta}{qd}-1} G_q(v_{\bar\lambda_0}) \\ &\quad - \frac{(\Delta\mu)^2}{2!} \frac{\beta}{qd}\big(\frac{\beta}{qd} - 1\big)(Q_{\mu_0}(v_{\bar\lambda_0}) - \Delta\mu\theta G_q(v_{\bar\lambda_0}))^{\frac{\beta}{qd}-2} G_q(v_{\bar\lambda_0})^2, \end{aligned} \tag{7.4}$$

where $\theta \in (0,1)$. Similar formula is true for $(P_{\lambda_0}(u_{\bar\lambda_0}) - \Delta\lambda G_p(u_{\bar\lambda_0}))^{\alpha/(pd)}$. Substituting these in (7.2) and using (3.2), (3.3) with $t_{\bar\lambda}, s_{\bar\lambda} = 1$ we obtain

$$\mathcal{J}_{\bar\lambda}(u_{\bar\lambda}, v_{\bar\lambda}) - \mathcal{J}_{\bar\lambda_0}(u_{\bar\lambda_0}, v_{\bar\lambda_0}) \le -\Delta\lambda\frac{1}{p}G_p(u_{\bar\lambda_0}) - \Delta\mu\frac{1}{q}G_q(v_{\bar\lambda_0}) + r_2(\Delta\lambda, \Delta\mu),$$

where $r_2(\Delta\lambda, \Delta\mu)$ is a sum of the reminder terms which orders with respect to $|\bar\lambda - \bar\lambda_0|$ are great or equal two. Thus we obtain the second inequality in (7.1). The first one is obtained by the same way.

To complete the proof of the proposition it remains to show that $r_i(\Delta\lambda, \Delta\mu) = o(|\bar\lambda - \bar\lambda_0|)$, $i = 1, 2$ uniformly on B. To this end (see (7.3) and (7.4)) it is sufficient to show that $\|u_{\bar\lambda}\|_p$, $\|v_{\bar\lambda}\|_q$ are uniformly bounded on B and

$$|F(u_{\bar\lambda}, v_{\bar\lambda})| \ge c_0 > 0, \quad |P_\lambda(u_{\bar\lambda})| \ge c_0 > 0, \quad |Q_\mu(v_{\bar\lambda})| \ge c_0 > 0, \tag{7.5}$$

with constant c_0 which does not depend on $\bar\lambda \in B$.

First consider the case $B \subset \Sigma_1$. Let us show the boundedness of $\|u_{\bar\lambda}\|_p$ and $\|v_{\bar\lambda}\|_q$. Suppose, contrary to our claim, that there is a sequence $\bar\lambda_m$ such that $\|u_{\bar\lambda_m}\|_p \to +\infty$ as $\bar\lambda_m \to \bar\lambda$ for some $\bar\lambda \in B$. Arguing as in the proof of Theorem 2.2 (see (5.2)) we obtain that in this case $P_{\lambda_m}(u_{\bar\lambda_m}) \to +\infty$ and $E_{\bar\lambda_m}(u_{\bar\lambda_m}) \to +\infty$ as $m \to +\infty$. But the last is impossible, since $E_{\bar\lambda}(u_{\bar\lambda}, v_{\bar\lambda})$ is uniformly bounded in B. Indeed, consider the first eigenfunctions $\varphi_1(U)$ and $\psi_1(U)$ of operators $-\Delta_p$ and $-\Delta_q$ with zero boundary conditions on a smooth subset $U \subset \Omega_+$. Then for all $\bar\lambda \in B$ we have

$$n_{\bar\lambda} \le \mathcal{J}_{\bar\lambda}(\varphi_1(U), \psi_1(U)) < c_3 < +\infty,$$

where c_3 does not depend on $\bar\lambda \in B$.

By (5.5) we have

$$F(u_{\bar\lambda}, v_{\bar\lambda}) \ge C_4(\lambda, \mu) > 0, \quad \bar\lambda \in B,$$

where by the construction $C_4(\lambda, \mu)$ is continuous nonzero function on the compact set B. This implies the first estimate in (7.5). At the same time we have the Nehari constraints $P_\lambda(u_{\bar\lambda}) = \alpha F(u_{\bar\lambda}, v_{\bar\lambda})$ and $Q_\mu(v_{\bar\lambda}) = \beta F(u_{\bar\lambda}, v_{\bar\lambda})$, which imply the last two estimates in (7.5).

Now consider the case $B \subset \Sigma^*$. Arguing as in (6.1) we obtain the first inequality in (7.5). Then again using Nehari constraints we obtain the last two estimates in (7.5). The boundedness of $\|u_{\bar\lambda}\|_p$ and $\|v_{\bar\lambda}\|_q$ is shown similar to the proof of Lemma 6.2 using Proposition 4.4. $\square$

Corollary 7.3. *Assume* (1.2) *is satisfied and* $p, q \in (1, +\infty)$, $f \in L^\infty(\Omega)$. *Let* $(u_{\bar\lambda}, v_{\bar\lambda})$ *be a ground state in* Σ_1 *and* Σ^*. *Then the function* $\mathcal{E}(\bar\lambda) = E_{\bar\lambda}(u_{\bar\lambda}, v_{\bar\lambda})$ *is differentiable at any point* $\bar\lambda \in \Sigma_1$ *and* Σ^*. *Furthermore*

$$\frac{\partial}{\partial\lambda}E_{\bar\lambda}(u_{\bar\lambda}, v_{\bar\lambda}) = -\frac{1}{p}\int_\Omega |u_{\bar\lambda}|^p\,dx, \quad \frac{\partial}{\partial\mu}E_{\bar\lambda}(u_{\bar\lambda}, v_{\bar\lambda}) = -\frac{1}{q}\int_\Omega |v_{\bar\lambda}|^q\,dx.$$

Proof. Let $\bar\lambda \in \Sigma_1$ or Σ^*. Consider the sequence $\bar\lambda_i \to \bar\lambda$. From the continuity of $E_{\bar\lambda}(u_{\bar\lambda}, v_{\bar\lambda})$ it follows that $(E_{\bar\lambda_i}(u_{\bar\lambda_i}, v_{\bar\lambda_i}))$ is uniformly bounded. Using this and the fact that $(u_{\bar\lambda_i}, v_{\bar\lambda_i})$ are weak solutions of (1.1) it easily follows that there is a strong convergence in W subsequence of $(u_{\bar\lambda_i}, v_{\bar\lambda_i})$ with limit point $(u_{\bar\lambda}, v_{\bar\lambda})$. This and (7.1) yield the required. $\square$

8. Asymptotic behaviour of the energy level of ground states of (1.1)

In this section we prove statements (I): (b), (c), (II): (b), (c) and (III) of Theorem 2.3. The proofs will follow from three lemmas below.

Lemma 8.1. *Let $\Omega^+ \neq \emptyset$. Then*

- $\mathcal{E}(\bar\lambda) \to 0$ *as* $\lambda \uparrow \lambda_1$ *and* $\mu \uparrow \mu_1$*,*
- $\mathcal{E}(\bar\lambda) \to 0$ *as* $\bar\lambda \to (\lambda_1, \mu_0)$ *for any* $\mu_0 < \mu_1$*, ,*
- $\mathcal{E}(\bar\lambda) \to 0$ *as* $\bar\lambda \to (\lambda_0, \mu_1)$ *for any* $\lambda_0 < \lambda_1$*.*

Proof. All statements are proved in a similar way. We give the proof only for the first statement. Since $n_{\bar\lambda} = \mathcal{E}_{\bar\lambda}$, it is sufficient to show that $n_{\bar\lambda} \to 0$ as $\lambda \to \lambda_1$ and $\mu \to \mu_1$.

Consider the first eigenfunction φ_1. Using the assumption $\Omega^+ \neq \emptyset$ it is not hard to find $w \in W_0^{1,q}$ such that

$$Q_\mu(w) > 0, \quad F(\varphi_1, w) > 0,$$

for all $\mu \in (-\infty, \mu_1]$. This and the fact that $P_\lambda(\varphi_1) > 0$ for $\lambda < \lambda_1$ yield $(\varphi_1, w) \in \mathcal{A}$ and $(t\varphi_1, sw) \in \mathcal{N}_{\bar\lambda}$, where t, s are given by (3.2) and (3.3). At the same time

$$P_\lambda(\varphi_1) = (\lambda_1 - \lambda)\int_\Omega |\varphi_1|^p \to 0, \quad \text{as } \lambda \to \lambda_1.$$

From here it follows that

$$n_{\bar\lambda} \leq \mathcal{J}_{\bar\lambda}(\varphi_1, w) = C\frac{P_\lambda(\varphi_1)^{\alpha/(pd)} Q_\mu(w)^{\beta/(qd)}}{F(\varphi_1, w)^{1/d}} \to 0,$$

as $\lambda \to \lambda_1$ and $\mu \to \mu_1$. This completes the proof. □

The proof of the next lemma can be obtained in the standard way using statement II: (a) of Theorem 2.3.

Lemma 8.2. *Let $F(\varphi_1, \psi_1) < 0$. Then*

- $\mathcal{E}(\bar\lambda) \to 0$ *as* $\lambda \downarrow \lambda_1$ *and* $\mu \downarrow \mu_1$*,*
- $\mathcal{E}(\bar\lambda) \to 0$ *as* $\bar\lambda \to (\lambda_1, \mu_0)$ *for any* $\mu_0 \in (\mu_1, \mu^*)$ *and* $\mathcal{E}(\bar\lambda) \to 0$ *as* $\bar\lambda \to (\lambda_0, \mu_1)$ *for any* $\lambda_0 \in (\lambda_1, \lambda^*)$*.*

Lemma 8.3. *If $f(x) \leq 0$, $p, q \geq 2$ and $\max\{p, q\} > 2$, then $\mathcal{E}(\bar\lambda) \to -\infty$ as $\bar\lambda \to (\lambda^*, \mu^*)$.*

Proof. Let (u^*, v^*) be a minimizer of (2.2) such that $u^*, v^* \geq 0$. Observe that $F(u^*, v^*) = 0$, since by the assumption $f(x) \leq 0$. This implies that

$$\Omega \neq \operatorname{supp} u^* \cap \operatorname{supp} v^* \subset \Omega_0 = \{x \in \Omega : f(x) = 0\}. \tag{8.1}$$

Assume first that for every minimizer (u^*, v^*) of (2.2) it holds

$$\sigma^* = \frac{1}{\lambda_1}\frac{\int_\Omega |\nabla u^*|^p}{\int_\Omega |u^*|^p} = \frac{1}{\mu_1}\frac{\int_\Omega |\nabla v^*|^q}{\int_\Omega |v^*|^q}.$$

Since $f(x) \leq 0$, by Proposition 4.3 we know that $\sigma^* > 1$ and therefore, $\lambda^* > \lambda_1$ and $\mu^* > \mu_1$. This fact, as well as $u^*, v^* \geq 0$ and (8.1) imply that the equations $D_u P_{\lambda^*}(u^*) = 0$ and $D_v Q_{\mu^*}(v^*) = 0$ cannot be satisfied on Ω. Hence there exist $\theta_1, \theta_2 \in C_0^\infty(\Omega)$ such that

$$\langle D_u P_{\lambda^*}(u^*), \theta_1\rangle < 0, \quad \langle D_v Q_{\mu^*}(v^*), \theta_2\rangle < 0.$$

Consider the functions

$$u_r := u^* + r\theta_1, \quad v_r := v^* + r^\delta \theta_2,$$

for $r > 0$ and some constant $\delta > 0$ which will be defined below. Then by Taylor's theorem we obtain

$$P_{\lambda^*}(u_r) = P_{\lambda^*}(u^*) + r\langle D_u P_{\lambda^*}(u^*), \theta_1\rangle + R_1(r), \tag{8.2}$$

$$Q_{\mu^*}(v_r) = Q_{\mu^*}(v^*) + r^\delta\langle D_v Q_{\mu^*}(v^*), \theta_2\rangle + R_2(r), \tag{8.3}$$

for sufficiently small $r > 0$, where $R_1(r)$ and $R_2(r)$ are reminders, for instance, in the Lagrange form; i.e.,

$$\begin{aligned} R_1(r) = r^2\frac{p(p-1)}{2!}\Big(&\int_\Omega |\nabla(u^* + r\kappa\theta_1)|^{p-2}|\nabla\theta_1|^2 dx \\ &- \lambda\int_\Omega |(u^* + r\kappa\theta_1)|^{p-2}|\theta_1|^2 dx\Big), \end{aligned}$$

for some $\kappa \in (0,1)$, and similar equality for $R_2(r)$. Since $p, q \geq 2$ by assumption, we guarantee that $R_1(r) = o(r)$ and $R_2(r) = o(r^\delta)$. Using this fact and the observation that $P_{\lambda^*}(u^*) = 0$, $Q_{\mu^*}(v^*) = 0$, from (8.2) and (8.3) we obtain

$$P_{\lambda^*}(u_r) = r\langle D_u P_{\lambda^*}(u^*), \theta_1\rangle + o(r) < 0, \tag{8.4}$$

$$Q_{\mu^*}(v_r) = r^\delta\langle D_v Q_{\mu^*}(v^*), \theta_2\rangle + o(r^\delta) < 0, \tag{8.5}$$

for sufficiently small $r > 0$. On the other hand, we have

$$\begin{aligned} F(u_r, v_r) &= r^{\delta\beta}F_1 + r^\alpha F_2 \\ &:= r^{\delta\beta}\int_{\Omega_{u^*}\setminus\Omega_0} f|u^* + r\theta_1|^\alpha|\theta_2|^\beta\,dx + r^\alpha\int_{\Omega_{v^*}\setminus\Omega_0} f|\theta_1|^\alpha|v^* + r^\delta\theta_2|^\beta\,dx \\ &< 0. \end{aligned}$$

Here it holds a strong inequality. Indeed, if one suppose $F_1 = 0$ and $F_2 = 0$, then (u_r, v_r) would be an admissible point for (2.2) but by (8.4), (8.5) $P_{\lambda^*}(u_r) < 0$, $Q_{\mu^*}(v_r) < 0$ that contradicts the definition of σ^*.

Let t_r and s_r are given by (3.2) and (3.3). Then $(t_r u_r, s_r v_r) \in \mathcal{N}_{(\lambda^*,\mu^*)}$ and

$$\begin{aligned} &\mathcal{J}_{(\lambda^*,\mu^*)}(u_r, v_r) \\ &= -C\frac{|P_{\lambda^*}(u_r)|^{\alpha/(pd)}|Q_{\mu^*}(v_r)|^{\beta/(qd)}}{|F(u_r, v_r)|^{1/d}} \\ &= -C\frac{r^{\frac{\alpha}{pd}+\frac{\delta\beta}{qd}}|\langle D_u P_{\lambda^*}(u^*), \theta_1\rangle + o(r)/r|^{\alpha/(pd)}|\langle D_v Q_{\mu^*}(v^*), \theta_2\rangle + o(r^\delta)/r^\delta|^{\beta/(qd)}}{|r^{\delta\beta}F_1 + r^\alpha F_2|^{1/d}}. \end{aligned} \tag{8.6}$$

We will prove the theorem, if we find $\delta > 0$, for which the system

$$\begin{gathered} \delta\beta - \frac{\alpha}{p} - \frac{\delta\beta}{q} > 0, \\ \alpha - \frac{\alpha}{p} - \frac{\delta\beta}{q} > 0, \end{gathered}$$

will be consistent. Expressing δ from the first and second inequalities, we obtain

$$\frac{\alpha q}{\beta p}\frac{1}{(q-1)} < \delta < \frac{\alpha q}{\beta p}(p-1).$$

From the assumptions $p,q \geq 2$ and $\max\{p,q\} > 2$ we conclude that such δ exists and therefore,

$$E_{(\lambda^*,\mu^*)}(t_r u_r, s_r v_r) = \mathcal{J}_{(\lambda^*,\mu^*)}(u_r, v_r) \to -\infty \quad \text{as } r \to 0, \tag{8.7}$$

Hence $n_{(\lambda^*,\mu^*)} = -\infty$.

Assume now that there exists a minimizer (u^*, v^*) of (2.2) such that

$$\sigma^* = \frac{1}{\lambda_1} \frac{\int_\Omega |\nabla u^*|^p}{\int_\Omega |u^*|^p} > \frac{1}{\mu_1} \frac{\int_\Omega |\nabla v^*|^q}{\int_\Omega |v^*|^q}.$$

In this case the proof is actually the same as in the previous case except that now we have to take into account that $Q_{\mu^*}(v^*) < 0$ and therefore (8.6) will be changed.

Let us indicate the proof briefly. Since $\lambda^* > \lambda_1$, we can find θ such that $\langle D_u P_{\lambda^*}(u^*), \theta \rangle < 0$. Consider the function $u_r := u^* + r\theta$, for $r > 0$. Then, in view of assumption $p \geq 2$, we have

$$P_{\lambda^*}(u_r) = P_{\lambda^*}(u^*) + r\langle D_u P_{\lambda^*}(u^*), \theta\rangle + o(r) = r\langle D_u P_{\lambda^*}(u^*), \theta\rangle + o(r) < 0,$$

for sufficiently small $r > 0$. Furthermore, we have

$$F(u_r, v^*) = r^\alpha F_3 := r^\alpha \int_{\Omega\setminus\Omega_0} f|\theta|^\alpha |v^*|^\beta \,dx < 0.$$

Thus, we obtain

$$\begin{aligned}
\mathcal{J}_{(\lambda^*,\mu^*)}(u_r, v^*) &= \frac{|Q_{\lambda^*}(v^*)|^{\beta/(qd)} |P_{\lambda^*}(u_r)|^{\alpha/(pd)}}{|F(u_r, v^*)|^{1/d}} \\
&= \frac{r^{\alpha/(pd)} |\langle D_u P_{\lambda^*}(u^*), \theta\rangle + o(r)/r|^{\alpha/(pd)} |Q_{\lambda^*}(v^*)|^{\beta/(qd)}}{r^{\alpha/d} |F_3|^{1/d}}.
\end{aligned}$$

Hence we again get (8.7). Note that in this case we have not used the assumption $\max\{\alpha, \beta\} > 2$.

To conclude the proof, we observe that for any fixed $(u,v) \in W$ the function $\mathcal{J}_{\bar\lambda}(u,v)$ is a continuous map with respect to $\bar\lambda$. Furthermore $\mathcal{E}_{\bar\lambda} \leq \mathcal{J}_{\bar\lambda}(u_r, v_r)$ for any $\bar\lambda \in \Sigma^*$. These and (8.7) imply that $\mathcal{E}_{\bar\lambda} \to -\infty$ as $\lambda \to \lambda^*$, $\mu \to \mu^*$. $\square$

9. Blow-up results

In this Section we prove Theorems 2.4 and 2.5.

Proof of Theorem 2.4. The proofs of statements (1) and (2) are similar; so we give the proofs of (1) and (3) only.

Proof of statement (1). Let $(u_{\bar\lambda}, v_{\bar\lambda})$ be a solution of (1.1) and $\bar\lambda \to (\lambda_1, \mu_0)$, where $\mu_0 < \mu_1$. Let

$$u_{\bar\lambda} = t_{\bar\lambda} \hat u_{\bar\lambda}, \quad v_{\bar\lambda} = s_{\bar\lambda} \hat v_{\bar\lambda},$$

where $\|\hat u_{\bar\lambda}\|_q = 1$, $\|\hat v_{\bar\lambda}\|_q = 1$ and $t_{\bar\lambda} = \|u_{\bar\lambda}\|_q$, $s_{\bar\lambda} = \|v_{\bar\lambda}\|_q$. From (3.4) and Lemma 8.1 we know that

$$\mathcal{J}_{\bar\lambda}(u_{\bar\lambda}, v_{\bar\lambda}) = \mathcal{J}_{\bar\lambda}(\hat u_{\bar\lambda}, \hat v_{\bar\lambda}) = C \frac{P_\lambda(\hat u_{\bar\lambda})^{\alpha/(pd)} Q_\mu(\hat v_{\bar\lambda})^{\beta/(qd)}}{F(\hat u_{\bar\lambda}, \hat v_{\bar\lambda})^{1/d}} \to 0, \tag{9.1}$$

as $\bar\lambda \to (\lambda_1, \mu_0)$. At the same time, since $\mu_0 < \mu_1$, we have

$$Q_\mu(\hat v_{\bar\lambda}) = 1 - \mu \int_\Omega |\hat v_{\bar\lambda}|^q \,dx > 1 - \frac{\mu}{\mu_1} > 0$$

uniformly by $\bar\lambda$. Therefore from (9.1) it follows that

$$\frac{P_\lambda(\hat u_{\bar\lambda})^{\alpha/(pd)}}{F(\hat u_{\bar\lambda},\hat v_{\bar\lambda})^{1/d}}=\Big(\frac{P_\lambda(\hat u_{\bar\lambda})^{\alpha}}{F(\hat u_{\bar\lambda},\hat v_{\bar\lambda})^{p}}\Big)^{1/(pd)}\to 0. \tag{9.2}$$

Hence, from (3.3) we obtain that $s_{\bar\lambda}\to 0$ and consequently $\|v_{\bar\lambda}\|_q\to 0$ as $\bar\lambda\to(\lambda_1,\mu_0)$.

From (9.2) it follows that $P_\lambda(\hat u_{\bar\lambda})\to 0$, since $F(\hat u_{\bar\lambda},\hat v_{\bar\lambda})$ is bounded. This, (3.2) and the assumption $q<\beta$ imply that $t_{\bar\lambda}\to+\infty$ and consequently $\|u_{\bar\lambda}\|_p\to+\infty$ as $\bar\lambda\to(\lambda_1,\mu_0)$.

Proof of statement (3). Suppose $(\bar\lambda_m)$, $m\in\mathbb{N}$ is a sequence in Σ_1 such that $\lambda_m\to\lambda_1$ and $\mu_m\to\mu_1$ as $m\to\infty$. As above, consider $u_{\bar\lambda_m}=t_{\bar\lambda_m}\hat u_{\bar\lambda_m}$ and $v_{\bar\lambda_m}=s_{\bar\lambda_m}\hat v_{\bar\lambda_m}$, $m=1,2,\dots$.

Since $\|\hat u_{\bar\lambda_m}\|_p=1$ and $\|\hat v_{\bar\lambda_m}\|_q=1$ for $m\in\mathbb{N}$, we can apply the Eberlein-Shmulyan theorem and the Sobolev embedding theorem. Thus, there exist $(\hat u_0,\hat v_0)\in W$ and a subsequence, which we denote again $(\bar\lambda_m)$, such that $(\hat u_{\bar\lambda_m},\hat v_{\bar\lambda_m})\to(\hat u_0,\hat v_0)$ as $m\to\infty$ weakly in W and strongly in $L_r(\Omega)\times L_s(\Omega)$ as $r\in(1,p^*)$, $s\in(1,q^*)$.

For $(u_{\bar\lambda_m},v_{\bar\lambda_m})\in\mathcal{N}_{\bar\lambda_m}$ we have

$$E_{\bar\lambda_m}(u_{\bar\lambda_m},v_{\bar\lambda_m})=t^p_{\bar\lambda_m}\frac{d}{\alpha}P_{\lambda_m}(\hat u_{\bar\lambda_m})=s^q_{\bar\lambda_m}\frac{d}{\beta}Q_{\lambda_m}(\hat v_{\bar\lambda_m})\to 0\quad\text{as } m\to\infty. \tag{9.3}$$

Suppose, for instance, that $\hat v_0=0$. Then $Q_\mu(\hat v_{\bar\lambda_m})\to 1$. This allows us to apply the arguments from the proof of statement (1). Indeed, (9.3) implies that $s_{\bar\lambda_m}\to 0$ as $m\to\infty$. Furthermore, similar to (9.2) it is deduced that

$$\frac{P_{\lambda_m}(\hat u_{\bar\lambda_m})^{\alpha}}{F(\hat u_{\bar\lambda_m},\hat v_{\bar\lambda_m})^{p}}\to 0.$$

Then $P_{\lambda_m}(\hat u_{\bar\lambda_m})\to 0$ and due to (3.2) and the assumption $q<\beta$ we have $t_{\bar\lambda}\to+\infty$. Thus in this case statement (3) is true.

Consider now the case $\hat u_0\neq 0$ and $\hat v_0\neq 0$. Suppose that simultaneously $P_{\lambda_m}(\hat u_{\bar\lambda_m})\to 0$ and $Q_{\lambda_m}(\hat v_{\bar\lambda_m})\to 0$ as $m\to\infty$. Then by the weak lower semi-continuity of $P_\lambda(u)$, $Q_\lambda(v)$ it follows that $P_{\lambda_1}(\hat u_0)\le 0$ and $Q_{\mu_1}(\hat v_0)\le 0$. This is possible only if $\hat u_0=\varphi_1$ and $\hat v_0=\psi_1$ up to nonzero multiplier. Hence we have $0\le F(\hat u_0,\hat v_0)=F(\varphi_1,\psi_1)$. However this contradicts the assumption $F(\varphi_1,\psi_1)<0$. Thus we can assume without loss of generality that $Q_{\lambda_m}(\hat v_{\bar\lambda_m})\to c_1>0$ as $m\to\infty$. However we are again in the position to apply the arguments from the proof of statement (1) and therefore we obtain anew the desired $s_{\bar\lambda_m}\to 0$, $t_{\bar\lambda_m}\to+\infty$ as $m\to\infty$. □

Proof of Theorem 2.5. Note that for $(u_{\bar\lambda},v_{\bar\lambda})\in\mathcal{N}_{\bar\lambda}$ we have

$$E_{\bar\lambda}(u_{\bar\lambda},v_{\bar\lambda})=\big(\frac{\alpha}{p}+\frac{\beta}{q}-1\big)\frac{1}{\alpha}P_\lambda(u_{\bar\lambda}).$$

From here it follows that

$$E_{\bar\lambda}(u_{\bar\lambda},v_{\bar\lambda})\to-\infty\Longleftrightarrow P_\lambda(u_{\bar\lambda})\to-\infty, \tag{9.4}$$

as $\bar\lambda\to(\lambda^*,\mu^*)$. On the other hand

$$P_\lambda(u_{\bar\lambda})=\|u_{\bar\lambda}\|_p^p-\lambda\int_\Omega|u_{\bar\lambda}|^p\,dx\ge-\frac{\lambda}{\lambda_1}\|u_{\bar\lambda}\|_p^p.$$

From here and (9.4) we obtain $\|u_{\bar\lambda}\|_p\to+\infty$. By the same arguments it also follows that $\|v_{\bar\lambda}\|_q\to+\infty$. □

10. Final remarks

Let us compare the obtained results for (1.1) in the special case $p = q$, $\lambda = \mu$ with known results for (1.3) when $\gamma = \alpha + \beta$.

In [23, 24] to study (1.3) the following critical value is introduced

$$\lambda^*_{\rm sing} = \inf_u \Big\{ \frac{\int |\nabla u|^p \, dx}{\int |u|^p \, dx} : F(u) := \int f|u|^\gamma \, dx \geq 0, \ u \in W^1_p(\Omega) \Big\},$$

and under the same assumptions of Corollary 2.6 the following is proven:

(I) $\lambda^*_{sing} < +\infty$ if and only if $\Omega^0 \cup \Omega^+ \neq \emptyset$;

(II) $\lambda_1 < \lambda^*_{sing}$ if and only if $F(\phi_1) < 0$;

(III) problem (1.3) has two sets of positive solutions:

  (1) w^1_λ for $\lambda \in (-\infty, \lambda^*_{sing})$ in the case $\Omega^+ \neq \emptyset$, such that $E_\lambda(w^1_\lambda) > 0$;

  (2) w^2_λ for $\lambda \in (\lambda_1, \lambda^*_{sing})$ in the case $F(\phi_1) < 0$, such that $E_\lambda(w^2_\lambda) < 0$;

(IV) the solutions w^1_λ on $(-\infty, \lambda_1)$ and w^2_λ on $(\lambda_1, \lambda^*_{sing})$ are the ground states of (1.3) and the functions $\mathcal{E}^1(\lambda) := E_\lambda(w^1_\lambda)$, $\mathcal{E}^2(\lambda) := E_\lambda(w^2_\lambda)$ are continuous in the intervals of their determination.

To compare (1.3) and (1.1) it is necessary also to take into consideration that the critical values λ^* and λ^*_{sing} are essentially the same objects but for different problems. For instance, both of them define a threshold of the applicability of Nehari manifold method (see Section 4, and [21, 23, 24]). Moreover, they can be obtained as a consequence of a general approach (see [23, 24]).

First, we see that in contrast to (1.3) we always have $\lambda^* < +\infty$ for (1.1). The next distinction is that in the case $F(\phi_1) < 0$, $\Omega^+ \neq \emptyset$ the ground state level of (1.3) has a discontinuity at the point λ_1. Furthermore, in the interval $(\lambda_1, \lambda^*_{sing})$ one has a multiplicity of solutions to (1.3); i.e., there are two branches positive solutions w^1_λ and w^2_λ, whereas this property is not observed for (1.1) (see Figure 2).

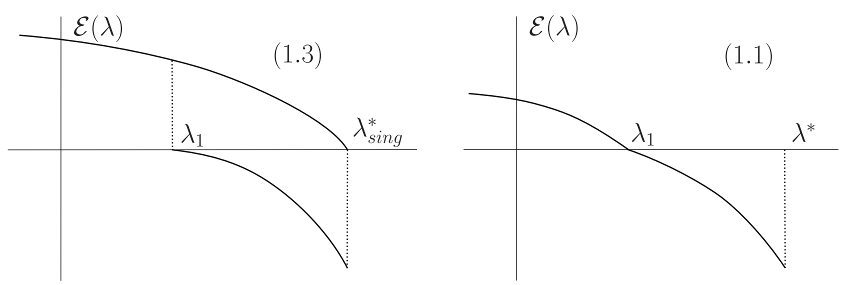

Figure 2. $p = q$, $\lambda = \mu$, $F(\varphi_1, \varphi_1) < 0$, $\Omega_+ \neq \emptyset$

For (1.3) similar results like in Theorem 2.3 are known [24]. In particular, $\mathcal{E}^1(\lambda) \to 0$ as $\lambda \uparrow \lambda^*_{sing}$ and $\mathcal{E}^2(\lambda) \to 0$ as $\lambda \downarrow \lambda_1$. Additionally, $\mathcal{E}^2(\lambda) \to -\infty$ as $\lambda \to \lambda^*_{sing}$ if and only if $\Omega^+ = \emptyset$, where $\lambda^*_{sing} < +\infty$ if and only if $\Omega^0 \neq \emptyset$.

Thus, we see that in case $f(x) < 0$ and $p > 2$ ground states of the system (1.1) blow up at the finite value λ^*, whereas this phenomenon for the scalar equation (1.3) is impossible (see Figure 3).

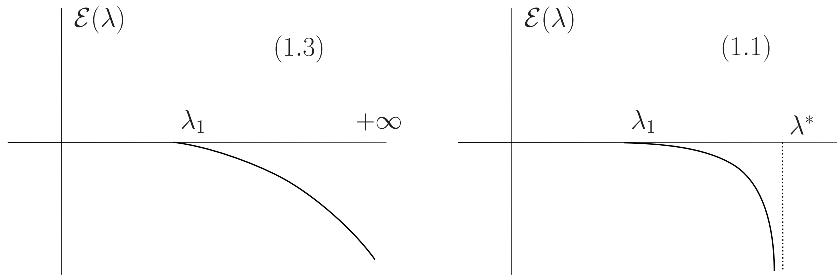

FIGURE 3. $p = q$, $p > 2$, $\lambda = \mu$, $f(x) < 0$

Remark 10.1. Consider the positive solutions w^1_λ and w^2_λ of (1.3) for $\lambda \in (-\infty, \lambda^*)$ and $\lambda \in (\lambda_1, \lambda^*)$, respectively. Then it easy to verify that the following pairs of functions $u^1_\lambda = c_1 w^1_\lambda$, $v^1_\lambda = c_2 w^1_\lambda$ and $u^2_\lambda = c_1 w^2_\lambda$, $v^2_\lambda = c_2 w^2_\lambda$, where

$$c_1 = \frac{\alpha^{(\beta-p)/(p^2 d)}}{\beta^{\beta/(p^2 d)}}, \quad c_2 = \frac{\beta^{(\alpha-p)/(p^2 d)}}{\alpha^{\alpha/(p^2 d)}},$$

satisfy the system (1.1) with $p = q$, $\lambda = \mu$, $\alpha + \beta = \gamma$ in the intervals $(-\infty, \lambda^*)$ and (λ_1, λ^*), respectively. It is interesting to compare these solutions with those obtained in Corollary 2.6. The properties of the corresponding energy functionals $\mathcal{E}_\lambda$ (see Corollary 2.7 and Figures 2 and 3) show that these solutions are different. The differences can be seen also from the blow-up behaviour of the ground state branches of (1.1) obtained in Theorem 2.4,2.5, which, as it is easy to see, is impossible for the ground state branches of (1.3) under the same assumptions.

Remark 10.2. Observe that $n_{\bar\lambda} = 0$ when $\bar\lambda$ belongs to quadrant II or III (see Figure 1). This can be seen from the following. First of all, note that the set $\mathcal{B}$ is empty when $\bar\lambda$ lies in quadrants II and III. Further, if we consider, for example, quadrant III; i.e., $\lambda > \lambda_1$, $\mu < \mu_1$, then one can find a sequence of functions $(\xi_k, \zeta_k) \in W$ such that $P_\lambda(\xi_k) > 0$, $Q_\mu(\zeta_k) > 0$, $F(\xi_k, \zeta_k) > c_0 > 0$ uniformly by $k \in \mathbb{N}$ and $P_\lambda(\xi_k) \to 0$.

Remark 10.3. As it was noted above, the scalar problem (1.3) in case $F(\phi_1) < 0$ has multiple positive solutions w^1_λ, w^2_λ for $\lambda \in (\lambda_1, \lambda^*_{sing})$. On the other hand it is known (see [27, 32]) that this problem possesses a turning point (λ_0, w_0), where $\lambda_0 > \lambda^*_{sing}$ and w_0 is a positive solution of (1.3). Thus we can assume up to the uniqueness of positive branches (w^1_λ), (w^2_λ) that they are linked at the point (λ_0, w_0). However, we did not find for (1.1) the second branch of solutions which could be considered as a contender to link with the branch of ground states (u_λ, v_λ) found in Theorems 2.2, 2.3. In view of this, we conjecture that there are only two scenarios of the behaviour of the ground state branch of (1.1): it continues on the whole quadrant IV or there is a threshold in IV where it blows up.

Remark 10.4. It is known that for the scalar problem (1.3), the assumption $F(\phi_1) < 0$ is necessary and sufficient for the existence of positive solutions for $\lambda > \lambda_1$. We cannot prove the similar statement for (1.1). In particular it is an open question whether (1.1) has positive solutions outside of the quadrant I, if $F(\varphi_1, \psi_1) \geq 0$.

Remark 10.5. In this paper we consider only the case $d = \frac{\alpha}{p} + \frac{\beta}{q} - 1 > 0$ which corresponds to the case when Hessian $\Gamma_{\bar{\lambda}}(u, v)$ is indefinite. The cases $d = 0$ and $d < 0$ for (1.1) have been investigated in the literature (see e.g. [9, 18, 19, 20]). However, in these cases, to our knowledge, there are no investigations similar to those obtained in the present paper.

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Yavdat Il'yasov
Institute of Mathematics of RAS, Ufa, Russia
E-mail address: ilyasov02@gmail.com

Vladimir Bobkov
Institute of Mathematics of RAS, Ufa, Russia
Ufa State Aviation Technical University, Ufa, Russia
E-mail address: bobkovve@gmail.com