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FINITE FRACTAL DIMENSIONALITY OF ATTRACTORS FOR NONLOCAL EVOLUTION EQUATIONS

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Abstract. In this work we consider the Dirichlet problem governed by a non local evolution equation. We prove the existence of exponential attractors for the flow generated by this problem, and as a consequence we obtain the finite dimensionality of the global attractor whose existence was proved in [1].

1. Introduction

Global attractors for dynamical systems generated by non local evolution equations in infinite dimensional Hilbert space have been considered in the literature within past few years, see [1, 6, 8] and references therein.

In the literature, there are some works on existence of exponential attractors for the flow governed by non local evolution equations and on the problem of determining upper bounds for the fractal dimension of these attractors (see for instance [2, 10, 9]).

In this paper, we consider the non linear Dirichlet problem with non local terms

$$\begin{gathered} \partial_t u(x,t) = -u(x,t) + g\big(\beta(Ku)(x,t)\big), \quad x \in \Omega, t>0 \\ u(x,0) = u_0(x), \quad x \in \Omega \\ u(x,t) = 0, \quad x \notin \Omega,\ t>0 \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded smooth domain, $\beta > 0$ and K is an integral operator with symmetric kernel

$$(Ku)(x) := \int_{\mathbb{R}^N} J(x,y)u(y)dy.$$

Here, $g : \mathbb{R} \to \mathbb{R}$ is a non linear real function of class C^1 with $g(0) = 0$, J is a non negative, symmetric bounded function with bounded derivative, satisfying $\int_{\mathbb{R}^N} J(x,y)dy = 1$ and

$$\sup_{x\in\mathbb{R}^N} \int_{\mathbb{R}^N} |\partial_x J(x,y)|dy \leq S, \quad \sup_{y\in\mathbb{R}^N} \int_{\mathbb{R}^N} |\partial_x J(x,y)|dx \leq S,$$

for some constant $0 < S < \infty$.

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For the sake of clarity and future reference, we list the hypotheses on g that were used in [1].

- (H1) The function $g : \mathbb{R} \to \mathbb{R}$, is globally Lipschitz continuous with constant k_1.
- (H2) The function $g \in C^1(\mathbb{R})$ and g' is Lipschitz continuous with constant k_2.
- (H3) There exists $a > 0$ such that $|g(x)| < a < \infty$, for all $x \in \mathbb{R}$.

Note that if (H1) and (H2) hold then

$$|g'(x)| \leq k_1, \quad \forall x \in \mathbb{R}.$$

Remark 1.1. To prevent the flow generated by (1.1) becomes a contraction (see Theorem 2.2) and, hence, the global attractor be reduced to single point, we assume $k_1\beta > 1$.

In this article, $\|\cdot\| = \langle \cdot, \cdot \rangle^{1/2}$ denotes the $L^2(\mathbb{R}^N)$ norm. We use $\|J\|_\infty$ to denote $\|J\|_{L^\infty(\mathbb{R}^N \times \mathbb{R}^N;\mathbb{R})}$ and $\|J'\|_\infty$ to denote $\|J'\|_{L^\infty(\mathbb{R}^N \times \mathbb{R}^N;L(\mathbb{R}^N \times \mathbb{R}^N;\mathbb{R}))}$.

Under hypothesis (H1)–(H3), it was proved in [1] that problem (1.1) has a global compact attractor $\mathcal{A}$ which is contained in ball centered at the origin of radius $a\sqrt{|\Omega|}$ in $L^2(\mathbb{R}^N)$. Also, under some additional hypotheses on g, the continuity of the global attractors and the existence of nonhomogeneous equilibria for (1.1) were proved in [1]. In [2] bi-space global and exponential attractors for the time continuous dynamical systems are considered and the bounds on their fractal dimension are discussed in the context of the smoothing properties of the system between appropriately chosen function spaces and applications to the sample problems are given, but no remark is made on problems governed by operators of the type Hilbert-Schmidt, where the symmetry of the problem is an extra difficulty inherent in the evolution equations with non local terms.

Our goal is to investigate, under the above conditions, the existence of an exponential attractor for the flow generated by (1.1) and consequently to obtain bounds on the fractal dimension of the global attractor associated to problem (1.1).

This article is organized as follows. In Section 2 we prove Lipschitz continuity of the dynamical system generated by (1.1), whose well posedness in $X = \{u \in L^2(\mathbb{R}^N) : u(x) = 0, \ if \ x \notin \Omega\}$ has been established in [1]. In Section 3 we prove that, in this phase space, the system has an exponential attractor and, as a consequence , we conclude that the global attractor has finite fractal dimension.

2. Dynamical system generated by (1.1)

It is known from previous work, see [1], that under hypotheses (H1) and (H2) the function

$$[F(u)](x) = \begin{cases} -u(x) + g(\beta(Ku)(x)), & x \in \Omega \\ 0, & x \notin \Omega, \end{cases}$$

is globally Lipschitz and continuously Fréchet differentiable in $X = \{u \in L^2(\mathbb{R}^N) : u(x) = 0, \text{ if } x \notin \Omega\}$. Therefore, the autonomous problem

$$\partial_t u = F(u) \tag{2.1}$$

with initial condition $u(x, 0) = u_0(x)$ generates a C^1 flow in X which is given by $T(t)u_0 = u(x,t)$ where $u(x,t)$ is given by the variation of constants formula by

$$u(x,t) = e^{-t}u(x,0) + \int_0^t e^{-(t-s)} g(\beta(Ku)(x,s))ds.$$

Furthermore, as consequence from Lemma 2.1 below, the problem (2.1) is also well posed in H^1.

Lemma 2.1. *Under hypotheses* (H1), (H2)*, the subset* H^1 *of* X *given by* $H^1 = \{u \in H^1(\mathbb{R}^N) : u(x) = 0, \text{ if } x \notin \Omega\}$ *is invariant under the map* F*.*

Proof. If $u \in H^1$, from hypothesis (H2) it follows that $F(u)$ is differentiable and

$$\begin{aligned}\partial_{x_i} F(u)(x) &= -\partial_{x_i} u(x) + g'(\beta(Ku)(x))\beta\partial_{x_i}(Ku)(x)\\ &= -\partial_{x_i} u(x) + g'(\beta(Ku)(x))\beta \int_{\mathbb{R}^N} \partial_{x_i} J(x,y)u(y)dy.\end{aligned}$$

Using hypotheses (H1) and (H2) and Generalized Young's Inequality (see [4]), we obtain

$$\begin{aligned}\|\partial_{x_i} F(u)\| &\le \|\partial_{x_i} u\| + \|g'(\beta(Ku))\beta \int_{\mathbb{R}^N} \partial_{x_i} J(x,y)u(y)dy\|\\ &\le \|\partial_{x_i} u\| + k_1\beta S\|u\|.\end{aligned}$$

It implies $F(u) \in H^1$, as claimed. □

In the next result we prove the Lipschitz continuity of the flow $T(t)$ generated by problem (2.1).

Theorem 2.2. *Assume hypothesis* (H1) *holds. Then, for* $u_1, u_2 \in X$ *and* $t \ge 0$*, we have*

$$\|T(t)u_1 - T(t)u_2\| \le e^{ct}\|u_1 - u_2\|, \tag{2.2}$$

for $c = k_1\beta - 1 > 0$*.*

Proof. Let $u_1, u_2 \in X$. Suppose that $T(t)u_1(x)$ and $T(t)u_2(x)$ are solutions of (2.1) with initial conditions u_1 and u_2, respectively. Then

$$\|T(t)u_1 - T(t)u_2\| \le e^{-t}\|u_1 - u_2\| + \int_0^t e^{-(t-s)}\|g(\beta(KT(s)u_1)) - g(\beta(KT(s)u_2))\|ds, \tag{2.3}$$

for $t \ge 0$. Using (H1) and Young's inequality, it follows that

$$\begin{aligned}\|g(\beta(KT(s)u_1)) - g(\beta(KT(s)u_2))\| &\le k_1\beta\|K(T(s)u_1 - T(s)u_2)\|\\ &\le k_1\beta\|T(s)u_1 - T(s)u_2\|.\end{aligned}$$

Then

$$\|T(t)u_1 - T(t)u_2\| \le e^{-t}\|u_1 - u_2\| + \int_0^t e^{-(t-s)}k_1\beta\|T(s)u_1 - T(s)u_2\|ds.$$

Thus, using Gronwall's inequality, we obtain

$$\|T(t)u_1 - T(t)u_2\| \le e^{(k_1\beta - 1)t}\|u_1 - u_2\|.$$

Hence (2.2) is satisfied. □

3. Existence of an exponential attractor

First we need to introduce some terminology. For a general introduction to theory of exponential attractors and fractal dimension see, for example [2, 3].

Recall that if $B \neq \emptyset$ is a precompact set in the Banach space $\mathcal{X}$ then its *fractal dimension* is given by

$$d_f^{\mathcal{X}}(B) = \limsup_{\epsilon \to 0} \log_{1/\epsilon} N_\epsilon(B),$$

where $N_\epsilon(B)$ denotes the smallest number of ϵ-balls in $\mathcal{X}$ needed to cover B.

We recall that a set $B \subset \mathcal{X}$ is an absorbing set for the flow $T(t)$ if, for any bounded set C in $\mathcal{X}$, there is a $t_1 = t_1(C)$ such that $T(t)C \subset B$ for any $t \geq t_1$ (see [9]).

Let $\mathcal{Y}$ and $\mathcal{X}$ be Banach spaces such that $\mathcal{Y}$ is compactly embedded in $\mathcal{X}$. Recalling the generalization of the notion of an exponential attractor, see [2], we will say that a nonvoid set $\mathcal{M} \subset \mathcal{Y}$ is called an *exponential* $(\mathcal{Y} - \mathcal{X})$ *attractor* for $T(t)$ if $\mathcal{M}$ is *positively invariant* under $T(t)$, closed in $\mathcal{Y}$, compact in $\mathcal{X}$, $d_f^{\mathcal{X}}(\mathcal{M}) < \infty$ and there exists $\omega > 0$ such that for all B bounded in $\mathcal{Y}$,

$$\lim_{t \to \infty} e^{\omega t} \operatorname{dist}_{\mathcal{X}}(T(t)B, \mathcal{M}) = 0.$$

Remark 3.1. If $u \in X$, proceeding as in [7], using Hölder's inequality, we obtain

$$\begin{aligned} |K(u)(x,s)| &\leq \int_{\mathbb{R}^N} J(x,y)|u(y,s)|dy \\ &\leq \int_{\mathbb{R}^N} \|J\|_\infty |u(y,s)|dy \\ &\leq \|J\|_\infty \sqrt{|\Omega|}\|u(\cdot,s)\|. \end{aligned} \tag{3.1}$$

Theorem 3.2. *Assume that* (H1)–(H3) *hold. Then , for any* $\varepsilon > 0$*, the ball centered at origin and radius* $\rho = (1 + k_1\beta\|J'\|_\infty|\Omega|)a\sqrt{|\Omega|} + \varepsilon$ *in* $H^1 = \{u \in H^1(\mathbb{R}^N) : u(x) = 0, \ \textit{if}\ x \neq \Omega\}$ *absorbs bounded subsets of* H^1.

Proof. Let $u(x,t)$ be the solution of (2.1) with initial condition $u(\cdot,0) \in B$, where B is a bounded subset of H^1. Then, if $x \notin \Omega$ we have $u(x,t) = 0$, and if $x \in \Omega$ we obtain, by the variation of constants formula,

$$u(x,t) = e^{-t}u(x,0) + \int_0^t e^{-(t-s)} g(\beta(Ku)(x,s))ds. \tag{3.2}$$

By (H3) we have

$$\|u(\cdot,t)\| \leq e^{-t}\|u(\cdot,0)\| + a\sqrt{|\Omega|}. \tag{3.3}$$

Hence, given $\varepsilon > 0$ there exists $t_1 = \ln(\frac{2\|u(\cdot,0)\|}{\varepsilon})$ such that, for $t > t_1$, we obtain

$$\|u(\cdot,t)\| < \frac{\varepsilon}{2} + a\sqrt{|\Omega|}. \tag{3.4}$$

Furthermore, using hypothesis (H2), from (3.2) we obtain

$$\partial_x u(x,t) = e^{-t}\partial_x u(x,0) + \int_0^t e^{-(t-s)} g'(\beta(Ku)(x,s))\beta\partial_x(Ku)(x,s)ds.$$

From (H1) and (H2) it follows that $|g'(x)| \leq k_1$, for all $x \in \mathbb{R}$. Then

$$|\partial_x u(x,t)| \leq e^{-t}|\partial_x u(x,0)| + \int_0^t e^{-(t-s)} k_1\beta|\partial_x(Ku)(x,s)|ds.$$

But, as in (3.1), using Hölder's inequality, we obtain

$$\begin{aligned} |\partial_x K(u)(x,s)| &\le \int_{\mathbb{R}^N} |\partial_x J(x,y)||u(y,s)|dy \\ &\le \int_{\mathbb{R}^N} \|J'\|_\infty |u(y,s)|dy \\ &\le \|J'\|_\infty \sqrt{|\Omega|}\|u(\cdot,s)\|. \end{aligned} \tag{3.5}$$

Thus, using (3.3) and (3.5), we obtain

$$\begin{aligned} |\partial_x u(x,t)| &\le e^{-t}|\partial_x u(x,0)| + k_1\beta\|J'\|_\infty \sqrt{|\Omega|}\int_0^t e^{-(t-s)}(e^{-s}\|u(\cdot,0)\| + a\sqrt{|\Omega|})ds \\ &\le e^{-t}|\partial_x u(x,0)| + k_1\beta\|J'\|_\infty \sqrt{|\Omega|}\|u(\cdot,0)\|e^{-t}t + k_1\beta\|J'\|_\infty a|\Omega|. \end{aligned}$$

Hence

$$\|\partial_x u(\cdot,t)\| \le e^{-t}\|\partial_x u(\cdot,0)\| + k_1\beta\|J'\|_\infty|\Omega|\|u(\cdot,0)\|e^{-t}t + k_1\beta\|J'\|_\infty|\Omega|a\sqrt{|\Omega|}. \tag{3.6}$$

But, there exists $t_2 = \ln(4\|\partial_x u(\cdot,0)\|/\varepsilon)$ such that for $t > t_2$ we have

$$e^{-t}\|\partial_x u(\cdot,0)\| < \frac{\varepsilon}{4}, \tag{3.7}$$

and, since $\lim_{t\to\infty} e^{-t}t = 0$, there exists $t_3 > 0$ such that, for $t > t_3$, we have

$$k_1\beta\|J'\|_\infty|\Omega|\|u(\cdot,0)\|e^{-t}t < \frac{\varepsilon}{4}. \tag{3.8}$$

Then, using (3.6), (3.7) and (3.8), we obtain

$$\|\partial_x u(\cdot,t)\| < \frac{\varepsilon}{2} + k_1\beta\|J'\|_\infty|\Omega|a\sqrt{|\Omega|}.$$

for all $t > t^* := \max\{t_2, t_3\}$. It follows that for $t > \max\{t_1, t^*\}$,

$$\|u(\cdot,t)\| + \|\partial_x u(\cdot,t)\| \le (1 + k_1\beta\|J'\|_\infty|\Omega|)a\sqrt{|\Omega|} + \varepsilon.$$

From this, the result follows immediately. □

For the rest of this article, we denote by B_0 the ball in H^1 centered at origin and radius $\rho = (1 + k_1\beta\|J'\|_\infty|\Omega|)a\sqrt{|\Omega|} + \varepsilon$, (with $\varepsilon > 0$ fixed arbitrarily).

Theorem 3.3. *Assume* (H1)–(H3)*, and let B_0 be the set that absorbs bounded subsets of H^1, given in Theorem 3.2. Then, there exists $t_0 \ge t_1(B_0)$ such that the following conditions hold: $T(t_0)$ admits a decomposition*

$$T(t_0) = P(t_0) + M(t_0)$$

where $P(t_0) : B_0 \to X \subset L^2(\mathbb{R}^N)$ is a contraction on B_0; that is,

$$\|P(t_0)u_1 - P(t_0)u_2\| \le \delta\|u_1 - u_2\|, \quad \forall u_1, u_2 \in B_0, \tag{3.9}$$

for some $0 \le \delta < 1/2$, and $M(t_0) : B_0 \to H^1$ satisfies

$$\|M(t_0)u_1 - M(t_0)u_2\|_{H^1} \le k\|u_1 - u_2\|, \quad \forall u_1, u_2 \in B_0, \tag{3.10}$$

for some $k > 0$.

Proof. Let $u(x,t)$ be the solution of (2.1) with initial condition u, then
$$T(t)u = e^{-t}u + \int_0^t e^{-(t-s)} g(\beta(KT(s))u)ds.$$
Write $P(t)u = e^{-t}u$ and $M(t)u = \int_0^t e^{-(t-s)} g(\beta(KT(s))u)ds$. Note that, choosing $t_0 > \ln 2$, it follows that
$$\|P(t_0)u_1 - P(t_0)u_2\| \le e^{-t_0}\|u_1 - u_2\|, \quad \forall u_1, u_2 \in B_0,$$
then (3.9) is satisfied with $\delta = e^{-t_0}$. Now, from (H3), it follows that
$$\|M(t)u\| \le \int_0^t e^{-(t-s)}\|g(\beta(KT(s)u)\|ds \le \int_0^t e^{-(t-s)} a\sqrt{|\Omega|}ds \le a\sqrt{|\Omega|}.$$
Using (H1),(H2), and (3.5), we have
$$\begin{aligned}
|\partial_x M(t)u(x)| &\le \int_0^t e^{-(t-s)}\beta|g'(\beta(KT(s)u)(x))||\partial_x(KT(s)u)(x)|ds \\
&\le \int_0^t e^{-(t-s)} k_1\beta\|J'\|_\infty\sqrt{|\Omega|}\|T(s)u\|ds.
\end{aligned}$$
Since $u \in B(0,\rho)$, it follows that $\|T(s)u\| \le \rho + a\sqrt{|\Omega|}$. Hence
$$|\partial_x M(t)u(x)| \le k_1\beta\|J'\|_\infty\sqrt{|\Omega|}(\rho + a\sqrt{|\Omega|}).$$
Therefore, $M(t): B_0 \to H^1$ for all $t \ge 0$.

Also, using (H1) and Theorem 2.2, we obtain
$$\begin{aligned}
\|M(t)u_1 - M(t)u_2\| &\le \int_0^t e^{-(t-s)}\|g(\beta KT(s)u_1) - g(\beta KT(s)u_2)\|ds \\
&\le \int_0^t e^{-(t-s)}\beta k_1\|T(s)u_1 - T(s)u_2\|ds \\
&\le \int_0^t e^{-(t-s)}\beta k_1 e^{(k_1\beta-1)s}\|u_1 - u_2\|ds \\
&= k_1\beta\|u_1 - u_2\|\int_0^t e^{-(t-s)}e^{(k_1\beta-1)s}ds \\
&\le \|u_1 - u_2\|e^{(k_1\beta-1)t}.
\end{aligned}$$
Using hypothesis (H1), (H2), (3.1) and (3.5), it follows that
$$\begin{aligned}
&|\partial_x M(t)u_1(x) - \partial_x M(t)u_2(x)| \\
&\le \int_0^t e^{-(t-s)}\beta|g'(\beta(KT(s)u_1)(x)) - g'(\beta(KT(s)u_2)(x))||\partial_x(KT(s)u_1)(x)|ds \\
&\quad + \int_0^t e^{-(t-s)}\beta|g'(\beta(KT(s)u_2)(x))||\partial_x[K(T(s)u_1 - T(s)u_2)](x)|ds \\
&\le \int_0^t e^{-(t-s)}k_2\beta^2\|J\|_\infty\sqrt{|\Omega|}\|T(s)u_1 - T(s)u_2\|\|J'\|_\infty\sqrt{|\Omega|}\|T(s)u_1\|ds \\
&\quad + \int_0^t e^{-(t-s)}k_1\beta\|J'\|_\infty\sqrt{|\Omega|}\|T(s)u_1 - T(s)u_2\|ds.
\end{aligned}$$
Thus, using Theorem 2.2, we have
$$|\partial_x M(t)u_1(x) - \partial_x M(t)u_2(x)|$$

$$
\begin{aligned}
&\le \int_0^t e^{-(t-s)} k_2\beta^2 \|J\|_\infty |\Omega| \|J'\|_\infty (\rho + a\sqrt{|\Omega|}) e^{(k_1\beta-1)s} \|u_1 - u_2\| ds \\
&\quad + \int_0^t e^{-(t-s)} k_1\beta \|J'\|_\infty \sqrt{|\Omega|} e^{(k_1\beta-1)s} \|u_1 - u_2\| ds \\
&\le \Big[\frac{k_2\beta^2 \|J\|_\infty |\Omega| \|J'\|_\infty (\rho + a\sqrt{|\Omega|}) + k_1\beta \|J'\|_\infty \sqrt{|\Omega|}}{k_1\beta - 1}\Big] e^{(k_1\beta-1)t} \|u_1 - u_2\|.
\end{aligned}
$$

Therefore, (3.10) is satisfied with $t_0 > \ln 2$ and

$$
\begin{aligned}
k = \max\Big\{ e^{(k_1\beta-1)t_0}, \big[\frac{k_2\beta^2 \|J\|_\infty |\Omega| \|J'\|_\infty (\rho + a\sqrt{|\Omega|}) + k_1\beta \|J'\|_\infty \sqrt{|\Omega|}}{k_1\beta - 1}\big] \\
\times \sqrt{|\Omega|} e^{(k_1\beta-1)t_0} \Big\}.
\end{aligned} \tag{3.11}
$$

It completes the proof. $\square$

The proof of the following lemma is very easy and it will be omitted.

Lemma 3.4. *For any bounded interval I, there exist $0 < \theta < 1$ and $c > 0$ such that for $t, s \in I$ we have*

$$
|e^{-t} - e^{-s}| \le c|t - s|^\theta. \tag{3.12}
$$

In particular, when $I = [t_0, 2t_0]$ the inequality above is obtained with $c = 1$.

Theorem 3.5. *Assume* (H1) *and* (H3)*. Then, for $t_1, t_2 \in [t_0, 2t_0]$ and $u_1, u_2 \in B_0$, $T(t)$ satisfies*

$$
\|T(t_1)u_1 - T(t_2)u_2\| \le \mu(|t_1 - t_2|^\theta + \|u_1 - u_2\|) \tag{3.13}
$$

with some $\mu > 0$ and $0 < \theta < 1$.

Proof. Note that

$$
\begin{aligned}
T(t_1)u_1 - T(t_2)u_2 = (e^{-t_1}u_1 - e^{-t_2}u_2) + \Big(\int_0^{t_1} e^{-(t_1-s)} g(\beta K T(s)u_1) ds \\
- \int_0^{t_2} e^{-(t_2-s)} g(\beta K T(s)u_2) ds \Big).
\end{aligned}
$$

Using Lemma 3.4, we obtain

$$
\begin{aligned}
\|e^{-t_1}u_1 - e^{-t_2}u_2\| &\le |e^{-t_1} - e^{-t_2}| \|u_1\| + e^{-t_2}\|u_1 - u_2\| \\
&\le \|u_1\| |t_1 - t_2|^\theta + \|u_1 - u_2\| \\
&= \mu_1(|t_1 - t_2|^\theta + \|u_1 - u_2\|),
\end{aligned}
$$

where $\mu_1 = \max\{\rho, 1\}$. Now,

$$
\begin{aligned}
&\| \int_0^{t_1} e^{-(t_1-s)} g(\beta K T(s)u_1) ds - \int_0^{t_2} e^{-(t_2-s)} g(\beta K T(s)u_2) ds \| \\
&\le \| \int_0^{t_1} e^{-(t_1-s)} g(\beta K T(s)u_1) ds - \int_0^{t_1} e^{-(t_2-s)} g(\beta K T(s)u_2) ds \| \\
&\quad + \| \int_0^{t_1} e^{-(t_2-s)} g(\beta K T(s)u_2) ds - \int_0^{t_2} e^{-(t_2-s)} g(\beta K T(s)u_2) ds \|.
\end{aligned}
$$

Using (H1) and (H3), Young's inequality, Lemma 3.12 and Theorem 3.4, we have

$$
\| \int_0^{t_1} e^{-(t_1-s)} g(\beta K T(s)u_1) ds - \int_0^{t_1} e^{-(t_2-s)} g(\beta K T(s)u_2) ds \|
$$

$$
\begin{aligned}
&\le \int_0^{t_1} \|e^{-(t_1-s)}g(\beta KT(s)u_1) - e^{-(t_2-s)}g(\beta KT(s)u_2)\|ds \\
&\le \int_0^{t_1} |e^{-(t_1-s)} - e^{-(t_2-s)}|\|g(\beta KT(s)u_1)\|ds \\
&\quad + \int_0^{t_1} e^{-(t_2-s)}\|g(\beta KT(s)u_1) - g(\beta KT(s)u_2)\|ds \\
&\le \int_0^{t_1} a\sqrt{|\Omega|}c|t_1-t_2|^\theta ds + \int_0^{t_1} e^{-(t_2-s)}k_1\beta e^{(k_1\beta-1)s}\|u_1-u_2\|ds \\
&\le 2t_0 a\sqrt{|\Omega|}c|t_1-t_2|^\theta + \|u_1-u_2\|e^{k_1\beta 2t_0} \\
&= \mu_2(|t_1-t_2|^\theta + \|u_1-u_2\|),
\end{aligned}
$$

where $\mu_2 = \max\{2t_0 ac\sqrt{|\Omega|},\, e^{k_1\beta 2t_0}\}$.

Without loss of generality assuming that $t_1 < t_2$, using hypothesis (H3), we obtain

$$
\begin{aligned}
&\|\int_0^{t_1} e^{-(t_2-s)}g(\beta KT(s)u_2)ds - \int_0^{t_2} e^{-(t_2-s)}g(\beta KT(s)u_2)ds\| \\
&\le \int_{t_1}^{t_2} e^{-(t_2-s)}\|g(\beta KT(s)u_2)\|ds \\
&\le \int_{t_1}^{t_2} e^{-(t_2-s)}a\sqrt{|\Omega|}ds \\
&\le (t_2-t_1)a\sqrt{|\Omega|} \\
&\le t_0^{1-\theta}a\sqrt{|\Omega|}(t_2-t_1)^\theta \\
&= \mu_3(t_2-t_1)^\theta,
\end{aligned}
$$

where $\mu_3 = t_0^{1-\theta}a\sqrt{|\Omega|}$. Therefore,

$$
\|T(t_1)u_1 - T(t_2)u_2\| \le \mu(|t_2-t_1|^\theta + \|u_1-u_2\|)
$$

for some $\mu > 0$ and $0 < \theta < 1$. $\square$

Since $W^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$ it follows that $H^1 \hookrightarrow X$. Then, from Theorems 3.2, 3.3 and 3.5, it follows that the assumptions in the Proposition 2.7 of [2] are satisfied. Therefore, we have the following result.

Theorem 3.6. *Under the hypotheses* (H1)-(H3), *for any* $\nu \in (0, \frac{1}{2}-\delta)$, *there exists a nonvoid set* $\mathcal{M} = \mathcal{M}_\nu \subset B(0,\rho)$, *positively invariant under* $T(t)$ *and precompact in* X, *with the following properties:*

(1) *there exists* $\omega > 0$ *such that for any bounded set* $B \subset H^1$ *we have*

$$
\lim_{t\to\infty} e^{\omega t}\operatorname{dist}(T(t)B, \mathcal{M}) = 0
$$

where $\operatorname{dist}(\cdot,\cdot)$ *denotes the Hausdorff semi-distance in* X *(see* [5]*).*

(2) $\mathcal{M}$ *possesses finite fractal dimension in* X*; more precisely, we have for any* $\nu \in (0, \frac{1}{2}-\delta)$

$$
d_f^X(\mathcal{M}) \le \frac{1}{\theta}(1 + \log_{\frac{1}{2(\delta+\nu)}} N_{\nu/k}(B(0,1)),
$$

where $d_f(\mathcal{M})$ *denotes the fractal dimension of* $\mathcal{M}$, $B(0,1)$ *denotes the open ball centered at* 0 *and radius* 1 *in* $H^1 = \{u \in H^1(\mathbb{R}^N) : u(x) = 0,\ \text{if } x \notin \Omega\}$,

$N_{\nu/k}(B)$ denotes the smallest number of $\frac{\nu}{k}-$balls in $L^2(\mathbb{R}^N)$ needed to cover $B(0,1)$ and k is the constant given in (3.11).

Denoting by $\overline{\mathcal{M}_\nu}$ the closure in X of the set $\mathcal{M}_\nu$, from [2, Corollary 2.8] we have the following result.

Corollary 3.7. *Under the hypotheses of Theorem 3.6, for any $\nu \in (0, \frac{1}{2} - \delta)$:*

(1) *$\overline{\mathcal{M}_\nu}$ is an exponential $(H^1 - X)$ attractor bounded in H^1;*
(2) *there exists a finite dimensional global $(H^1 - X)$ attractor $\mathcal{A} \subset \overline{\mathcal{M}_\nu}$.*

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