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EXISTENCE OF SOLUTIONS FOR $(k, n-k-2)$ CONJUGATE BOUNDARY-VALUE PROBLEMS AT RESONANCE WITH $\dim\ker L = 2$

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Abstract. By constructing suitable project operators and using the coincidence degree theory due to Mawhin, the existence of solutions for $(k, n-k-2)$ conjugate boundary-value problems at resonance with $\dim ker L = 2$ is obtained.

1. Introduction

The existence of solutions for $(k, n-k)$ conjugate boundary-value problems at nonresonance has been studied in many papers (see [1, 2, 3, 6, 7, 9, 10, 11, 16, 14, 21, 25, 27, 29, 30, 31, 32]). The solvability of boundary-value problems at resonance has been investigated by many authors (see [4, 5, 8, 12, 13, 15, 17, 18, 19, 20, 26, 22, 24, 28, 33]). In [12], the existence of solutions for $(k, n-k)$ conjugate boundary-value problems at resonance with $\dim\ker L = 1$ has been studied. To the best of our knowledge, no paper discusses the existence of solutions for $(k, n-k-2)$ conjugate boundary-value problems at resonance with $\dim\ker L = 2$. We will fill this gap in the literature.

In this article, we investigate the existence of solutions for the $(k, n-k-2)$ conjugate boundary-value problem at resonance

$$(-1)^{n-k}y^{(n)}(t) = f\left(t, y(t), y'(t), \dots, y^{(n-1)}(t)\right) + \varepsilon(t), \quad \text{a.e. } t \in [0,1], \tag{1.1}$$

$$\begin{gathered} y^{(i)}(0) = y^{(j)}(1) = 0, \quad 0 \le i \le k-1,\ 0 \le j \le n-k-3, \\ y^{(n-2)}(1) = \sum_{j=1}^{l} \beta_j y^{(n-2)}(\eta_j), \quad y^{(n-1)}(1) = \sum_{i=1}^{m} \alpha_i y^{(n-1)}(\xi_i), \end{gathered} \tag{1.2}$$

where $1 \le k \le n-3$, $0 < \eta_1 < \eta_2 < \dots < \eta_l < 1$, $0 < \xi_1 < \xi_2 < \dots < \xi_m < 1$.

In this article, we assume that the following conditions hold.

(H1) $\sum_{i=1}^{m} \alpha_i = 1$, $\sum_{j=1}^{l} \beta_j = 1$, $\sum_{j=1}^{l} \beta_j \eta_j = 1$.

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(H2) $e = \begin{vmatrix} e_1 & e_2 \\ e_3 & e_4 \end{vmatrix} \neq 0$, where

$$e_1 = 1 - \sum_{i=1}^{m} a_i \xi_i, \quad e_2 = \frac{1}{2}\Big(1 - \sum_{j=1}^{l} \beta_j \eta_j^2\Big),$$
$$e_3 = \frac{1}{2}\Big(1 - \sum_{i=1}^{m} a_i \xi_i^2\Big), \quad e_4 = \frac{1}{6}\Big(1 - \sum_{j=1}^{l} \beta_j \eta_j^3\Big).$$

(H3) $\varepsilon(t) \in L^\infty[0,1]$, $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ satisfies Carathéodory conditions; i.e., $f(\cdot, x)$ is measurable for each fixed $x \in \mathbb{R}^n$, $f(t, \cdot)$ is continuous for a.e. $t \in [0,1]$, and for each $r > 0$, there exists $\Phi_r \in L^\infty[0,1]$ such that $|f(t, x_1, x_2, \dots, x_n)| \leq \Phi_r(t)$ for all $|x_i| \leq r$, $i = 1, 2, \dots, n$, a.e. $t \in [0,1]$.

2. Preliminaries

For convenience, we introduce some notation and a theorem. For more details see [23]. Let X and Y be real Banach spaces and $L : \operatorname{dom} L \subset X \to Y$ be a Fredholm operator with index zero, $P : X \to X$, $Q : Y \to Y$ be projectors such that

$$\operatorname{Im} P = \ker L, \quad \ker Q = \operatorname{Im} L, \quad X = \ker L \oplus \ker P, \quad Y = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

It follows that

$$L\big|_{\operatorname{dom} L \cap \ker P} : \operatorname{dom} L \cap \ker P \to \operatorname{Im} L$$

is invertible. We denote its inverse by K_P.

Let Ω be an open bounded subset of $X, \operatorname{dom} L \cap \overline{\Omega} \neq \emptyset$, the map $N : X \to Y$ will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \overline{\Omega} \to X$ is compact.

Theorem 2.1 ([23])**.** *Let $L : \operatorname{dom} L \subset X \to Y$ be a Fredholm operator of index zero and $N : X \to Y$ L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

1. *$Lx \neq \lambda N x$ for every $(x, \lambda) \in [(\operatorname{dom} L \setminus \ker L) \cap \partial\Omega] \times (0,1)$;*
2. *$Nx \notin \operatorname{Im} L$ for every $x \in \ker L \cap \partial\Omega$;*
3. *$\deg(QN|_{\ker L}, \Omega \cap \ker L, 0) \neq 0$, where $Q : Y \to Y$ is a projection such that $\operatorname{Im} L = \ker Q$.*

Then the equation $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$.

Take $X = C^{n-1}[0,1]$ with norm $\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty, \dots, \|u^{(n-1)}\|_\infty\}$, where $\|u\|_\infty = \max_{t \in [0,1]} |u(t)|$, $Y = L^1[0,1]$ with norm $\|x\|_1 = \int_0^1 |x(t)| dt$. Define operator $Ly(t) = (-1)^{n-k} y^{(n)}(t)$ with

$$\begin{aligned} \operatorname{dom} L = \Big\{ & y \in X : y^{(n)} \in Y,\ y^{(i)}(0) = y^{(j)}(1) = 0,\ 0 \leq i \leq k-1, \\ & 0 \leq j \leq n-k-3,\ y^{(n-2)}(1) = \sum_{j=1}^{l} \beta_j y^{(n-2)}(\eta_j), \\ & y^{(n-1)}(1) = \sum_{i=1}^{m} \alpha_i y^{(n-1)}(\xi_i) \Big\}. \end{aligned}$$

Let $N : X \to Y$ be defined as

$$Ny(t) = f\Big(t, y(t), y'(t), \dots, y^{(n-1)}(t)\Big) + \varepsilon(t), \quad t \in [0,1].$$

Then problem (1.1), (1.2) becomes $Ly = Ny$.

We use convention that $1/k! = 0$, for $k = -1, -2, \dots$. By simple calculation, we can get the following results.

$$\begin{vmatrix} \frac{1}{k!} & \frac{1}{(k+1)!} & \dots & \frac{1}{(n-3)!} \\ \frac{1}{(k-1)!} & \frac{1}{k!} & \dots & \frac{1}{(n-4)!} \\ & \dots & \dots & \\ \frac{1}{[k-(n-k-3)]!} & \frac{1}{[k+1-(n-k-3)]!} & \dots & \frac{1}{[n-3-(n-k-3)]!} \end{vmatrix}$$

$$= \frac{(n-k-3)!}{k!} \cdot \frac{(n-k-4)!}{(k+1)!} \dots \frac{1}{(n-3)!} \neq 0.$$

So, the following lemmas hold.

Lemma 2.2. *The system of linear equations*

$$\begin{gathered} \frac{x_k}{k!} + \frac{x_{k+1}}{(k+1)!} + \dots + \frac{x_{n-3}}{(n-3)!} + \frac{1}{(n-2)!} = 0, \\ \frac{x_k}{(k-1)!} + \frac{x_{k+1}}{k!} + \dots + \frac{x_{n-3}}{(n-4)!} + \frac{1}{(n-3)!} = 0, \\ \dots \\ \frac{x_k}{[k-(n-k-3)]!} + \frac{x_{k+1}}{[k+1-(n-k-3)]!} + \dots \\ + \frac{x_{n-3}}{[n-3-(n-k-3)]!} + \frac{1}{[n-2-(n-k-3)]!} = 0 \end{gathered}$$

has only one solution, its denoted by $(a_k, a_{k+1}, \dots, a_{n-3})$.

Lemma 2.3. *The system of linear equations*

$$\begin{gathered} \frac{x_k}{k!} + \frac{x_{k+1}}{(k+1)!} + \dots + \frac{x_{n-3}}{(n-3)!} + \frac{1}{(n-1)!} = 0, \\ \frac{x_k}{(k-1)!} + \frac{x_{k+1}}{k!} + \dots + \frac{x_{n-3}}{(n-4)!} + \frac{1}{(n-2)!} = 0, \\ \dots \\ \frac{x_k}{[k-(n-k-3)]!} + \frac{x_{k+1}}{[k+1-(n-k-3)]!} + \dots \\ + \frac{x_{n-3}}{[n-3-(n-k-3)]!} + \frac{1}{[n-1-(n-k-3)]!} = 0 \end{gathered}$$

has only one solution, it is denoted by $(b_k, b_{k+1}, \dots, b_{n-3})$.

Lemma 2.4. *For given* $u \in Y$, *the system of linear equations*

$$\begin{gathered} \frac{x_k}{k!} + \frac{x_{k+1}}{(k+1)!} + \dots + \frac{x_{n-3}}{(n-3)!} + \frac{(-1)^{n-k}}{(n-1)!} \int_0^1 (1-s)^{n-1} u(s) ds = 0, \\ \frac{x_k}{(k-1)!} + \frac{x_{k+1}}{k!} + \dots + \frac{x_{n-3}}{(n-4)!} + \frac{(-1)^{n-k}}{(n-2)!} \int_0^1 (1-s)^{n-2} u(s) ds = 0, \\ \dots \end{gathered}$$

$$\frac{x_k}{[k-(n-k-3)]!}+\frac{x_{k+1}}{[k+1-(n-k-3)]!}+\dots+\frac{x_{n-3}}{[n-3-(n-k-3)]!}$$
$$+\frac{(-1)^{n-k}}{[n-1-(n-k-3)]!}\int_0^1(1-s)^{n-1-(n-k-3)}u(s)ds=0$$

has only one solution, its denoted by $(B_k(u),B_{k+1}(u),\dots,B_{n-3}(u))$.

Define the operators $T_1,T_2,Q_1,Q_2:Y\to R$ as follows:

$$T_1u(t)=\sum_{i=1}^m\alpha_i\int_{\xi_i}^1u(s)ds,$$
$$T_2u(t)=\sum_{j=1}^l\beta_j\Big[\int_{\eta_j}^1(1-s)u(s)ds+(1-\eta_j)\int_0^{\eta_j}u(s)ds\Big],$$
$$Q_1u=\frac{1}{e}(e_4T_1u-e_3T_2u),\quad Q_2u=\frac{1}{e}(-e_2T_1u+e_1T_2u).$$

Obviously, $e_1=T_1(1)$, $e_2=T_2(1)$, $e_3=T_1(t)$, $e_4=T_2(t)$.

Lemma 2.5. *Assume that* (H1) *holds, then* $L:\operatorname{dom}L\subset X\to Y$ *is a Fredholm operator of index zero and the linear continuous projector* $Q:Y\to Y$ *can be defined as*

$$Qu=Q_1u+t\cdot Q_2u,$$

and the linear operator $K_P:\operatorname{Im}L\to\operatorname{dom}L\cap\ker P$ *can be written as*

$$K_Pu=\sum_{i=k}^{n-3}\frac{B_i(u)}{i!}t^i+\frac{(-1)^{n-k}}{(n-1)!}\int_0^t(t-s)^{n-1}u(s)ds.$$

Proof. Take $y\in\ker L$. We obtain $y=\sum_{i=k}^{n-1}\frac{x_i}{i!}t^i$ satisfying

$$\frac{x_k}{k!}+\frac{x_{k+1}}{(k+1)!}+\dots+\frac{x_{n-2}}{(n-2)!}+\frac{x_{n-1}}{(n-1)!}=0,$$
$$\frac{x_k}{(k-1)!}+\frac{x_{k+1}}{k!}+\dots+\frac{x_{n-2}}{(n-3)!}+\frac{x_{n-1}}{(n-2)!}=0,$$
$$\dots$$
$$\frac{x_k}{[k-(n-k-3)]!}+\frac{x_{k+1}}{[k+1-(n-k-3)]!}+\dots$$
$$+\frac{x_{n-2}}{[n-2-(n-k-3)]!}+\frac{x_{n-1}}{[n-1-(n-k-3)]!}=0.$$

Setting $x_{n-2}=1$, $x_{n-1}=0$, and $x_{n-2}=0$, $x_{n-1}=1$, respectively, by Lemmas 2.2, 2.3, we have

$$y=\sum_{i=k}^{n-3}\frac{ca_i+db_i}{i!}t^i+\frac{c}{(n-2)!}t^{n-2}+\frac{d}{(n-1)!}t^{n-1},c,d\in\mathbb{R}.$$

Therefore,

$$\ker L=\big\{y:y=\sum_{i=k}^{n-3}\frac{ca_i+db_i}{i!}t^i+\frac{c}{(n-2)!}t^{n-2}+\frac{d}{(n-1)!}t^{n-1},c,d\in R\big\}.$$

Define the linear operator $P:X\to X$ as follows

$$Py(t)=\sum_{i=k}^{n-3}\frac{y^{(n-2)}(0)a_i+y^{(n-1)}(0)b_i}{i!}t^i+\frac{y^{(n-2)}(0)}{(n-2)!}t^{n-2}+\frac{y^{(n-1)}(0)}{(n-1)!}t^{n-1}.$$

Obviously, $\operatorname{Im} P = \ker L$ and $P^2 y = Py$. For any $y \in X$, it follows from $y = (y - Py) + Py$ that $X = \ker P + \ker L$. By simple calculation, we can get that $\ker L \cap \ker P = \{0\}$. So, we have

$$X = \ker L \oplus \ker P. \tag{2.1}$$

We will show that

$$\begin{aligned}\operatorname{Im} L = \Big\{ u \in Y : &\sum_{i=1}^{m} \alpha_i \int_{\xi_i}^{1} u(s)ds = 0, \\ &\sum_{j=1}^{l} \beta_j \Big[\int_{\eta_j}^{1} (1-s)u(s)ds + (1-\eta_j) \int_0^{\eta_j} u(s)ds \Big] = 0 \Big\}.\end{aligned}$$

In fact, if $u \in \operatorname{Im} L$, there exists $y \in \operatorname{dom} L$ such that $u = Ly \in Y$. This, together with $y^i(0) = 0$, $0 \le i \le k-1$, implies that

$$y(t) = \sum_{i=k}^{n-1} \frac{c_i}{i!} t^i + \frac{(-1)^{n-k}}{(n-1)!} \int_0^t (t-s)^{n-1} u(s)ds.$$

Since $\sum_{i=1}^{m} \alpha_i = 1$ and $y^{(n-1)}(1) = \sum_{i=1}^{m} \alpha_i y^{(n-1)}(\xi_i)$, we obtain

$$\sum_{i=1}^{m} \alpha_i \int_{\xi_i}^{1} u(s)ds = 0. \tag{2.2}$$

Since $\sum_{j=1}^{l} \beta_j = 1$, $\sum_{j=1}^{l} \beta_j \eta_j = 1$ and $y^{(n-2)}(1) = \sum_{j=1}^{l} \beta_j y^{(n-2)}(\eta_j)$, we obtain

$$\sum_{j=1}^{l} \beta_j \Big[\int_{\eta_j}^{1} (1-s)u(s)ds + (1-\eta_j) \int_0^{\eta_j} u(s)ds \Big] = 0. \tag{2.3}$$

On the other hand, if $u \in Y$ satisfies (2.2) and (2.3), take

$$y = \sum_{i=k}^{n-3} \frac{B_i(u)}{i!} t^i + \frac{(-1)^{n-k}}{(n-1)!} \int_0^t (t-s)^{n-1} u(s)ds.$$

It follows from (2.2), (2.3) and Lemma 2.4 that $y \in \operatorname{dom} L$. Obviously, $Ly = u$. So, we get $u \in \operatorname{Im} L$.

Now we will prove that $Q : Y \to Y$ is a projector such that $\ker Q = \operatorname{Im} L$, $Y = \operatorname{Im} L \oplus \operatorname{Im} Q$. For $u \in Y$, since

$$\begin{gathered} Q_1(1) = \frac{1}{e}[e_4 T_1(1) - e_3 T_2(1)] = 1, \quad Q_1(t) = \frac{1}{e}[e_4 T_1(t) - e_3 T_2(t)] = 0, \\ Q_2(1) = \frac{1}{e}[-e_2 T_1(1) + e_1 T_2(1)] = 0, \quad Q_2(t) = \frac{1}{e}[-e_2 T_1(t) + e_1 T_2(t)] = 1, \end{gathered}$$

we have

$$\begin{gathered} Q_1(Qu) = Q_1(Q_1 u + t \cdot Q_2 u) = Q_1 u \cdot Q_1(1) + Q_2 u \cdot Q_1(t) = Q_1 u, \\ Q_2(Qu) = Q_2(Q_1 u + t \cdot Q_2 u) = Q_1 u \cdot Q_2(1) + Q_2 u \cdot Q_2(t) = Q_2 u. \end{gathered}$$

Thus,

$$Q^2 u = Q_1(Qu) + t \cdot Q_2(Qu) = Q_1 u + t \cdot Q_2 u = Qu.$$

Since $u \in \ker Q$, we have

$$\begin{gathered} e_4 T_1 u - e_3 T_2 u = 0, \\ -e_2 T_1 u + e_1 T_2 u = 0. \end{gathered}$$

It follows from (H2) that $T_1u = T_2u = 0$. So, $u \in \operatorname{Im} L$; i.e., $\ker Q \subset \operatorname{Im} L$. Clearly, $\operatorname{Im} L \subset \ker Q$. So, $\operatorname{Im} L = \ker Q$. This, together with $Q^2y = Qy$, means that $\operatorname{Im} L \cap \operatorname{Im} Q = \{0\}$. Thus, we have $Y = \operatorname{Im} L \oplus \operatorname{Im} Q$. Considering (2.1), we know that L is a Frdholm operator of index zero.

Define the operator $K_P : Y \to X$ as follows

$$K_P u = \sum_{i=k}^{n-3} \frac{B_i(u)}{i!} t^i + \frac{(-1)^{n-k}}{(n-1)!} \int_0^t (t-s)^{n-1} u(s) ds.$$

For $u \in \operatorname{Im} L$, by Lemma 2.4, we have $K_P u \in \operatorname{dom} L$. Clearly, $K_P u \in \ker P$. So, we get that $K_P(\operatorname{Im} L) \subset \operatorname{dom} L \cap \ker P$. Now we will prove that K_P is the inverse of $L|_{\operatorname{dom} L \cap \ker P}$.

Obviously, $LK_P u = u$, for $u \in \operatorname{Im} L$. On the other hand, for $y \in \operatorname{dom} L \cap \ker P$, we have

$$\begin{aligned} K_P Ly(t) &= \sum_{i=k}^{n-3} \frac{B_i(Ly)}{i!} t^i + \frac{(-1)^{n-k}}{(n-1)!} \int_0^t (t-s)^{n-1} (-1)^{n-k} y^{(n)}(s) ds \\ &= \sum_{i=k}^{n-3} \Big(\frac{B_i(Ly) - y^{(i)}(0)}{i!}\Big) t^i + y(t). \end{aligned}$$

Since $K_P(Ly) \in \operatorname{dom} L$ and $y \in \operatorname{dom} L$, we obtain $(K_P Ly)^{(j)}(1) = y^{(j)}(1) = 0$, $0 \le j \le n-k-3$. Thus $(B_k(Ly) - y^{(k)}(0), B_{k+1}(Ly) - y^{(k+1)}(0), \dots, B_{n-3}(Ly) - y^{(n-3)}(0))$ is the only zero solution of the system of linear equations

$$\begin{gathered} \frac{x_k}{k!} + \frac{x_{k+1}}{(k+1)!} + \dots + \frac{x_{n-3}}{(n-3)!} = 0, \\ \frac{x_k}{(k-1)!} + \frac{x_{k+1}}{k!} + \dots + \frac{x_{n-3}}{(n-4)!} = 0, \\ \dots \\ \frac{x_k}{[k-(n-k-2)]!} + \frac{x_{k+1}}{[k+1-(n-k-2)]!} + \dots \\ + \frac{x_{n-3}}{[n-3-(n-k-3)]!} = 0. \end{gathered}$$

So, we have $K_P Ly = y$, for $y \in \operatorname{dom} L \cap \ker P$. Thus, $K_P = (L|_{\operatorname{dom} L \cap \ker P})^{-1}$. The proof is complete. $\square$

3. Main results

Lemma 3.1. *Assume $\Omega \subset X$ is an open bounded subset and $\operatorname{dom} L \cap \overline{\Omega} \neq \emptyset$, then N is L-compact on $\overline{\Omega}$.*

Proof. By (H3), we have that $QN(\overline{\Omega})$ is bounded. Now we will show that $K_P(I-Q)N : \overline{\Omega} \to X$ is compact.

It follows from (H3) that there exists constant $M_0 > 0$ such that $|(I-Q)Ny| \le M_0$, a.e. $t \in [0,1], y \in \overline{\Omega}$. Thus, $K_P(I-Q)N(\overline{\Omega})$ is bounded. By (H3) and Lebesgue Dominated Convergence theorem, we get that $K_P(I-Q)N : \overline{\Omega} \to X$ is continuous. Since $\{\int_0^t (t-s)^j (I-Q)Ny(s) ds, y \in \overline{\Omega}\}$, $j = 0, 1 \dots, n-1$ are equi-continuous, and t^j, $j = 0, 1 \dots, n-1$ are uniformly continuous on [0,1], using Ascoli-Arzela theorem, we obtain that $K_P(I-Q)N : \overline{\Omega} \to X$ is compact. The proof is complete. $\square$

To obtain our main results, we need the following assumptions.

(H4) There exist constants $M_1 > 0, M_2 > 0$ such that if $|y^{(n-1)}(t)| > M_1$, $t \in [\xi_m, 1]$ then
$$\sum_{i=1}^{m} \alpha_i \int_{\xi_i}^{1} Ny(s)ds \neq 0,$$
and if $|y^{(n-2)}(t)| > M_2$, $t \in [0, \eta_1]$ then
$$\sum_{j=1}^{l} \beta_j \Big[\int_{\eta_j}^{1} (1-s)Ny(s)ds + (1-\eta_j) \int_{0}^{\eta_j} Ny(s)ds \Big] \neq 0.$$

(H5) There exist functions $g, h, \psi_i \in L^1[0,1]$, $i = 1, 2, \dots, n$, with $\|\psi_n\|_1 := r_1 < 1/2$, $\sum_{i=1}^{n-1} \|\psi_i\|_1 := r_2 < \frac{1-2r_1}{4}$, $\theta \in [0,1)$, and some $1 \le j \le n-1$ such that
$$|f(t, x_1, x_2, \dots, x_n)| \le g(t) + \sum_{i=1}^{n} \psi_i(t)|x_i| + h(t)|x_j|^{\theta}.$$

(H6) There exist constants $c_0 > 0, d_0 > 0$ such that, for
$$y = \sum_{i=k}^{n-3} \frac{ca_i + db_i}{i!} t^i + \frac{c}{(n-2)!} t^{n-2} + \frac{d}{(n-1)!} t^{n-1} \in \ker L,$$
one of the following two conditions holds
(1) $c \cdot T_1 Ny < 0$, if $|c| > c_0$, $d \cdot T_2 Ny < 0$, if $|d| > d_0$,
(2) $c \cdot T_1 Ny > 0$, if $|c| > c_0$, $d \cdot T_2 Ny > 0$, if $|d| > d_0$,

Lemma 3.2. *Suppose* (H1)–(H5) *hold, then the set*
$$\Omega_1 = \{y \in \operatorname{dom} L \setminus \ker L : Ly = \lambda Ny, \lambda \in (0,1)\}$$
is bounded.

Proof. Take $y \in \Omega_1$. By $Ny \in \operatorname{Im} L$, we have
$$\sum_{i=1}^{m} \alpha_i \int_{\xi_i}^{1} Ny(s)ds = 0, \tag{3.1}$$
$$\sum_{j=1}^{l} \beta_j \Big[\int_{\eta_j}^{1} (1-s)Ny(s)ds + (1-\eta_j) \int_{0}^{\eta_j} Ny(s)ds \Big] = 0. \tag{3.2}$$
Since $Ly = \lambda Ny$ and $y \in \operatorname{dom} L$, we obtain
$$y(t) = \sum_{i=k}^{n-1} \frac{c_i}{i!} t^i + \frac{(-1)^{n-k}}{(n-1)!} \lambda \int_0^t (t-s)^{n-1} Ny(s)ds, \tag{3.3}$$
where $c_k, c_{k+1}, \dots, c_{n-1}$ satisfy
$$\sum_{i=k}^{n-1} \frac{c_i}{i!} = -\frac{(-1)^{n-k}}{(n-1)!} \lambda \int_0^1 (1-s)^{n-1} Ny(s)ds,$$
$$\sum_{i=k}^{n-1} \frac{c_i}{(i-1)!} = -\frac{(-1)^{n-k}}{(n-2)!} \lambda \int_0^1 (1-s)^{n-2} Ny(s)ds,$$
$$\dots$$

$$\sum_{i=k}^{n-1} \frac{c_i}{[i-(n-k-3)]!} = -\frac{(-1)^{n-k}}{[i-(n-k-3)]!} \lambda \int_0^1 (1-s)^{i-(n-k-3)} Ny(s) ds.$$

It follows from $y^{(i)}(0) = y^{(j)}(1) = 0$, $0 \le i \le k-1$, $0 \le j \le n-k-3$ that there exist points $\delta_i \in [0,1]$ such that $y^{(i)}(\delta_i) = 0$, $i = 0, 1, \dots, n-3$. So, we have

$$y^{(i)}(t) = \int_{\delta_i}^t y^{(i+1)}(s) ds, \quad i = 0, 1, \dots, n-3.$$

Therefore,

$$\|y^{(i)}\|_\infty \le \|y^{(i+1)}\|_1 \le \|y^{(i+1)}\|_\infty, i = 0, 1, \dots, n-3. \tag{3.4}$$

By (3.1) and (H4), there exists $t_0 \in [\xi_m, 1]$ such that $|y^{(n-1)}(t_0)| \le M_1$. This, together with (3.3), implies that

$$|c_{n-1}| \le M_1 + \int_0^1 \big| f(s, y(s), y'(s), \dots, y^{(n-1)}(s)) \big| ds + \|\varepsilon\|_1.$$

By (3.2) and (H4), we get that there exists $t_1 \in [0, \eta_1]$ such that $|y^{(n-2)}(t_1)| \le M_2$. It follows from (3.3) that

$$\begin{aligned} |c_{n-2}| &\le M_2 + |c_{n-1}| + \int_0^1 \big| f(s, y(s), y'(s), \dots, y^{(n-1)}(s)) \big| ds + \|\varepsilon\|_1 \\ &\le M_1 + M_2 + 2 \int_0^1 \big| f(s, y(s), y'(s), \dots, y^{(n-1)}(s)) \big| ds + 2\|\varepsilon\|_1. \end{aligned}$$

Thus,

$$\begin{gathered} \|y^{(n-1)}\|_\infty \le M_1 + 2 \int_0^1 \big| f(s, y(s), y'(s), \dots, y^{(n-1)}(s)) \big| ds + 2\|\varepsilon\|_1, \\ \|y^{(n-2)}\|_\infty \le 2M_1 + M_2 + 4 \int_0^1 \big| f(s, y(s), y'(s), \dots, y^{(n-1)}(s)) \big| ds + 4\|\varepsilon\|_1. \end{gathered}$$

By (H5) and (3.4) we have

$$\|y^{(n-1)}\|_\infty \le r_3 + 2r_2 \|y^{(n-2)}\|_\infty + 2r_1 \|y^{(n-1)}\|_\infty + 2\|h\|_1 \|y^{(n-2)}\|_\infty^\theta$$

and

$$\|y^{(n-2)}\|_\infty \le 2r_3 + M_2 + 4r_2 \|y^{(n-2)}\|_\infty + 4r_1 \|y^{(n-1)}\|_\infty + 4\|h\|_1 \|y^{(n-2)}\|_\infty^\theta, \tag{3.5}$$

where $r_3 = M_1 + 2\|g\|_1 + 2\|\varepsilon\|_1$. So, we obtain

$$\|y^{(n-1)}\|_\infty \le \frac{1}{1-2r_1} [r_3 + 2r_2 \|y^{(n-2)}\|_\infty + 2\|h\|_1 \|y^{(n-2)}\|_\infty^\theta]. \tag{3.6}$$

By (3.5) and (3.6), we have

$$\|y^{(n-2)}\|_\infty \le \frac{2r_3}{1-2r_1} + M_2 + \frac{4r_2}{1-2r_1} \|y^{(n-2)}\|_\infty + \frac{4\|h\|_1}{1-2r_1} \|y^{(n-2)}\|_\infty^\theta.$$

Therefore,

$$\|y^{(n-2)}\|_\infty \le \frac{1}{1-2r_1-4r_2} [2r_3 + (1-2r_1)M_2 + 4\|h\|_1 \|y^{(n-2)}\|_\infty^\theta].$$

It follows from $\theta \in [0,1)$ that $\{\|y^{(n-2)}\|_\infty : y \in \Omega_1\}$ is bounded. By (3.4) and (3.6), we get that Ω_1 is bounded. □

Lemma 3.3. *Suppose* (H1)–(H3), (H6) *hold. Then the set*

$$\Omega_2 = \{y \in \ker L : Ny \in \operatorname{Im} L\}$$

is bounded.

Proof. Take $y \in \Omega_2$, then

$$y(t) = \sum_{i=k}^{n-3} \frac{ca_i + db_i}{i!} t^i + \frac{c}{(n-2)!} t^{n-2} + \frac{d}{(n-1)!} t^{n-1}.$$

By $Ny \in \operatorname{Im} L$, we have $T_1 Ny = 0, T_2 Ny = 0$. By (H6), we get that $|c| \le c_0, |d| \le d_0$. This means that Ω_2 is bounded. $\square$

Lemma 3.4. *Suppose* (H1)–(H3), (H6) *hold. Then the set*

$$\Omega_3 = \{y \in \ker L : \lambda Jy + (1-\lambda)\omega QNy = 0, \lambda \in [0,1]\}$$

is bounded, where $J : \ker L \to \operatorname{Im} Q$ *is a linear isomorphism given by*

$$J\Big(\sum_{i=k}^{n-3} \frac{ca_i + db_i}{i!} t^i + \frac{c}{(n-2)!} t^{n-2} + \frac{d}{(n-1)!} t^{n-1}\Big) = \frac{1}{e}(e_4 c - e_3 d) + \frac{1}{e}(-e_2 c + e_1 d)t,$$

where $c, d \in \mathbb{R}$ *and*

$$\omega = \begin{cases} -1, & \text{if (H6)(1) holds,} \\ 1, & \text{if (H6)(2) holds.} \end{cases}$$

Proof. Take $y \in \Omega_3$. $y \in \ker L$ implies that

$$y = \sum_{i=k}^{n-3} \frac{ca_i + db_i}{i!} t^i + \frac{c}{(n-2)!} t^{n-2} + \frac{d}{(n-1)!} t^{n-1}, c, d \in \mathbb{R}.$$

Since $\lambda Jy + (1-\lambda)\omega QNy = 0$, we obtain

$$\lambda c = -(1-\lambda)\omega T_1 Ny, \quad \lambda d = -(1-\lambda)\omega T_2 Ny.$$

If $\lambda = 0$, by (H6), we get $|c| \le c_0, |d| \le d_0$. If $\lambda = 1$, then $c = d = 0$. For $\lambda \in (0,1)$, if $|c| > c_0$ or $|d| > d_0$, then

$$\lambda c^2 = -(1-\lambda)\omega c \cdot T_1 Ny < 0$$

or

$$\lambda d^2 = -(1-\lambda)\omega d \cdot T_2 Ny < 0.$$

A contradiction. So, Ω_3 is bounded. $\square$

Theorem 3.5. *Suppose* (H1)–(H6) *hold. Then* (1.1)–(1.2) *has at least one solution in* X.

Proof. Let $\Omega \supset \cup_{i=1}^{3} \overline{\Omega_i} \cup \{0\}$ be a bounded open subset of X. It follows from Lemma 3.1 that N is L-compact on $\overline{\Omega}$. By Lemmas 3.2 and 3.3, we obtain

(1) $Ly \ne \lambda Ny$ for every $(y, \lambda) \in [(\operatorname{dom} L \setminus \ker L) \cap \partial\Omega] \times (0,1)$;
(2) $Ny \notin \operatorname{Im} L$ for every $y \in \ker L \cap \partial\Omega$.

We need to prove only that:

$$\deg(QN|_{\ker L}, \Omega \cap \ker L, 0) \neq 0.$$

Take

$$H(y, \lambda) = \lambda Jy + \omega(1-\lambda)QNy.$$

According to Lemma 3.4, we know that $H(y, \lambda) \neq 0$ for $y \in \partial\Omega \cap \ker L$, $\lambda \in [0, 1]$. By the homotopy of degree, we obtain

$$\begin{aligned} \deg(QN|_{\ker L}, \Omega \cap \ker L, 0) &= \deg(\omega H(\cdot, 0), \Omega \cap \ker L, 0) \\ &= \deg(\omega H(\cdot, 1), \Omega \cap \ker L, 0) \\ &= \deg(\omega J, \Omega \cap \ker L, 0) \neq 0. \end{aligned}$$

By Theorem 2.1, we can obtain that $Ly = Ny$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$; i.e., (1.1)–(1.2) has at least one solution in X. The prove is complete. □

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