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NON-EXISTENCE OF GLOBAL SOLUTIONS FOR A DIFFERENTIAL EQUATION INVOLVING HILFER FRACTIONAL DERIVATIVE

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Abstract. We consider a basic fractional differential inequality with a fractional derivative named after Hilfer and a polynomial source. A non-existence of global solutions result is proved in an appropriate space and the critical exponent is shown to be optimal.

1. Introduction

We study the Cauchy problem of fractional order with a polynomial nonlinearity

$$\begin{gathered}(D_{0^+}^{\alpha,\beta}u)(t) \ge t^{\delta}|u(t)|^m, \quad t>0,\ m>1,\ \delta\in\mathbb{R}\\ (D_{0^+}^{\gamma-1}u)(0)=b>0,\end{gathered} \tag{1.1}$$

where

$$(D_{0^+}^{\alpha,\beta}y)(x) = \big(I_{0^+}^{\beta(1-\alpha)}\frac{d}{dx}I_{0^+}^{(1-\beta)(1-\alpha)}f\big)(x) \tag{1.2}$$

is the Hilfer fractional derivative (HFD) of order $0<\alpha<1$ and type $0\le\beta\le 1$, $\gamma=\alpha+\beta-\alpha\beta$ and I_{0+}^{σ}, $\sigma>0$, is the usual Riemann-Liouville fractional integral of order σ. This type of derivatives were introduced by Hilfer in [19, 20]. These references provide information about the applications of this derivative and how it arises. It is easy to see that this derivative interpolates the Riemann-Liouville fractional derivative ($\beta=0$) and the Caputo fractional derivative ($\beta=1$) (see [25, 33]). The special case $\beta=0$ has been discussed in [29].

In this article we find the range of values of m for which solutions do not exist globally and establish an optimal exponent (in some sense) by showing that solutions do exist beyond this bound in a certain space. The existence and uniqueness for the general problem

$$\begin{gathered}(D_{a^+}^{\alpha,\beta}u)(t)=f(t,u), \quad 0<\alpha<1,\ 0<\beta<1,\ t>a,\\ (D_{a^+}^{\gamma-1}u)(a+)=c>0,\end{gathered}$$

has been established in [11] in the space

$$C_{1-\gamma}^{\alpha,\beta}[a,b]=\{y\in C_{1-\gamma}[a,b],\ D_{a+}^{\alpha,\beta}y\in C_{1-\gamma}[a,b]\}$$

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where $C_{1-\gamma}[a,b]$ is the weighted space of continuous functions on $(a,b]$

$$C_{1-\gamma}[a,b] = \{g : (a,b] \to \mathbb{R} : (x-a)^{1-\gamma} g(x) \in C[a,b]\}.$$

The special cases $\beta = 0$ and $\beta = 1$ may be found in [21, 22, 23, 24, 25]. These cases correspond to the Riemann-Liouville derivative and the Caputo derivative cases, respectively. Problems with such derivatives have been treated in many papers, we cite a few of them [4, 5, 6, 7, 8, 9, 10, 13, 14, 15, 16, 27, 28, 29, 36], and refer the reader to the books [25, 33, 35] for many other properties of such derivatives. The applications of these types of derivatives are numerous. Some of them may be found in [1, 2, 3, 18, 26, 30, 31, 33, 34, 35]. However, we cannot find much on Hilfer type derivatives.

The next section contains some definitions, notation and some lemmas which will be useful later in our proof. In Section 3 we state and prove our non-existence result. Finally, in Section 4 we give an example showing the existence of solutions in case the exponent is higher than the critical one found in the previous section.

2. Preliminaries

In this section we present some definitions, lemmas, properties and notation which will be used in our results later.

Definition 2.1. Let $\Omega = [a,b]$ be a finite interval and $0 \le \gamma < 1$, we introduce the weighted space $C_\gamma[a,b]$ of continuous functions f on $(a,b]$

$$C_\gamma[a,b] = \{f : (a,b] \to \mathbb{R} : (x-a)^\gamma f(x) \in C[a,b]\}.$$

In the space $C_\gamma[a,b]$, we define the norm

$$\|f\|_{C_\gamma} = \|(x-a)^\gamma f(x)\|_C, \quad C_0[a,b] = C[a,b].$$

Definition 2.2. The Riemann-Liouville left-sided fractional integral $I_{a+}^\alpha f$ of order $\alpha > 0$ is defined by

$$(I_{a+}^\alpha f)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad (a < x \le b,\ \alpha > 0)$$

provided that the integral exists. Here $\Gamma(\alpha)$ is the Gamma function. When $\alpha = 0$, we define $I_{a+}^0 f = f$. In fact, one can prove that $I_{a+}^\alpha f$ converges to f when $\alpha \to 0$.

Definition 2.3. The Riemann-Liouville right-sided fractional integral $I_{b-}^\alpha f$ of order $\alpha > 0$ is defined by

$$(I_{b-}^\alpha f)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)}{(t-x)^{1-\alpha}} dt, \quad (a \le x < b,\ \alpha > 0)$$

provided that the integral exists. When $\alpha = 0$, we define $I_{b-}^0 f = f$.

Definition 2.4. The Riemann-Liouville left-sided fractional derivative $D_{a+}^\alpha f$ of order α $(0 \le \alpha < 1)$ is defined by

$$(D_{a+}^\alpha f)(x) = \frac{d}{dx} (I_{a+}^{1-\alpha} f)(x);$$

that is,

$$(D_{a+}^\alpha f) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(t)}{(x-t)^\alpha} dt \quad (x > a,\ 0 < \alpha < 1),$$

when $\alpha = 1$ we have $D_{a+}^\alpha f = Df$. In particular, when $\alpha = 0$, $D_{a+}^0 f = f$.

Definition 2.5. The Riemann-Liouville right-sided fractional derivative $D^\alpha_{b-}f$ of order α $(0 \le \alpha < 1)$ is defined by

$$(D^\alpha_{b-}f)(x) = -\frac{d}{dx}(I^{1-\alpha}_{a+}f)(x);$$

that is,

$$(D^\alpha_{b-}f) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^b \frac{f(t)}{(t-x)^\alpha}dt \quad (a \le x < b,\ 0 < \alpha < 1).$$

In particular, when $\alpha = 0$, $D^0_{b-}f = f$.

Definition 2.6. We define the space

$$C^\gamma_{1-\gamma}[a,b] = \{y \in C_{1-\gamma}[a,b], D^\gamma_{a+}y \in C_{1-\gamma}[a,b]\}.$$

Lemma 2.7 ([25, 35])**.** *Let $0 < \alpha < 1$ and $0 \le \gamma < 1$. If $f \in C^1_\gamma$, the space of continuous functions on $[a,b]$ such that their derivatives are in C_γ, then the fractional derivatives D^α_{a+} and D^α_{b-} exist on $(a,b]$ and $[a,b)$ respectively, and can be represented in the forms*

$$(D^\alpha_{a+}f)(x) = \frac{1}{\Gamma(1-\alpha)}\Big[\frac{f(a)}{(x-a)^\alpha} + \int_a^x \frac{f'(t)dt}{(x-t)^\alpha}\Big]$$

and

$$(D^\alpha_{b-}f)(x) = \frac{1}{\Gamma(1-\alpha)}\Big[\frac{f(b)}{(b-x)^\alpha} - \int_x^b \frac{f'(t)dt}{(t-x)^\alpha}\Big].$$

Next, we have the Semigroup property of the fractional integration operator I^α_{a+}.

Lemma 2.8 ([25, 35])**.** *Let $\alpha > 0$, $\beta > 0$ and $0 \le \gamma < 1$. If $f \in L_p(a,b)$, $1 \le p \le \infty$ then the equation*

$$I^\alpha_{a+}I^\beta_{a+}f = I^{\alpha+\beta}_{a+}f$$

holds at almost every point $x \in [a,b)]$. When $\alpha + \beta > 1$, this relation is valid at any point $x \in [a,b]$.

Next is the fractional integration by parts.

Lemma 2.9 ([25, 35])**.** *Let $\alpha > 0$, $p \ge 1$, $q \ge 1$ and $\frac{1}{p} + \frac{1}{q} \le 1 + \alpha$ ($p \ne 1$ and $q \ne 1$ in the case when $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$). If $\varphi \in L_p(a,b)$ and $\psi \in L_q(a,b)$, then*

$$\int_a^b \varphi(x)(I^\alpha_{a+}\psi)(x)dx = \int_a^b \psi(x)(I^\alpha_{b-}\varphi)(x)dx.$$

Definition 2.10. The fractional derivative ${}^cD^\alpha_{a+}f$ of order $\alpha \in \mathbb{R}$ $(0 < \alpha < 1)$ on $[a,b]$ defined by

$${}^cD^\alpha_{a+}f = I^{1-\alpha}_{a+}Df,$$

where $D = \frac{d}{dx}$, is called the Caputo fractional derivative of f of order $\alpha \in \mathbb{R}$.

Theorem 2.11 (Young's inequality)**.** *If a and b are nonnegative real numbers and p and q are positive real numbers such that $1/p + 1/q = 1$ then we have*

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}.$$

3. Non-existence result

In this section we establish sufficient conditions ensuring non-existence of global solutions. In particular we find a range of values for the exponent m for which solutions cannot be continued for all time. The proof is based mainly on the test function method developed by Mitidieri and Pohozaev [32] and some adequate manipulations of the fractional derivatives and integrals. In addition to the results stated in the Preliminaries Section we need the following lemma.

Lemma 3.1. *If $\alpha > 0$ and $f \in C[a,b]$, then*

$$(I_{a+}^{\alpha} f)(a) = \lim_{t\to a} (I_{a+}^{\alpha} f)(t) = 0$$

and

$$(I_{b-}^{\alpha} f)(b) = \lim_{t\to b} (I_{b-}^{\alpha} f)(t) = 0.$$

Proof. Since $f \in C[a,b]$, on $[a,b]$, we have $|f(t)| < M$ for some positive constant M. Therefore

$$\begin{aligned} |(I_{a+}^{\alpha} f)(t)| &\le \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} |f(s)| ds \\ &\le \frac{M}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} ds \\ &\le \frac{M}{\alpha\Gamma(\alpha)} [-(t-s)^{\alpha}]_{s=a}^{t} = \frac{M}{\Gamma(\alpha+1)} (t-a)^{\alpha}. \end{aligned}$$

As $\alpha > 0$ we see that

$$(I_{a+}^{\alpha} f)(a) = \lim_{t\to a} (I_{a+}^{\alpha} f)(t) = 0.$$

The second part is proved similarly. □

Theorem 3.2. *Assume that $\delta > -\alpha$ and $1 < m \le \frac{\delta+1}{1-\alpha}$. Then, Problem (1.1) does not admit global nontrivial solutions in $C_{1-\gamma}^{\gamma}$, when $b > 0$.*

Proof. Assume, on the contrary, that a nontrivial solution u exists for all time $t > 0$. Let $\varphi \in C^1([0,\infty))$ be a test function satisfying: $\varphi(t) \ge 0$ and φ is non-increasing such that

$$\varphi(t) := \begin{cases} 1, & t \in [0, T/2], \\ 0, & t \in [T, \infty), \end{cases}$$

for some $T > 0$. Multiplying the inequality in (1.1) by $\varphi(t)$ and integrating we obtain

$$\int_0^T (D_{0^+}^{\alpha,\beta} u)(t)\varphi(t) dt \ge \int_0^T t^{\delta} |u(t)|^m \varphi(t) dt \tag{3.1}$$

and from the definition of $(D_{0^+}^{\alpha,\beta} u)(t)$ (see (1.2)) we can write

$$\int_0^T I_{0^+}^{\beta(1-\alpha)} \frac{d}{dt} (I_{0^+}^{1-\gamma} u)(t) \varphi(t) dt \ge \int_0^T t^{\delta} |u(t)|^m \varphi(t) dt. \tag{3.2}$$

By Lemma 2.9, we may deduce from (3.2) that

$$\int_0^T \frac{d}{dt} \big(I_{0^+}^{1-\gamma} u\big)(t) \big(I_{T-}^{\beta(1-\alpha)} \varphi\big)(t) dt \ge \int_0^T t^{\delta} |u(t)|^m \varphi(t) dt. \tag{3.3}$$

An integration by parts yields

$$
\begin{aligned}
&[(I_{0^+}^{1-\gamma}u)(t)(I_{T-}^{\beta(1-\alpha)}\varphi)(t)]_{t=0}^{T} - \int_0^T (I_{0^+}^{1-\gamma}u)(t)\frac{d}{dt}(I_{T-}^{\beta(1-\alpha)}\varphi)(t)dt \\
&\geq \int_0^T t^\delta |u(t)|^m \varphi(t)dt.
\end{aligned}
$$

Using Lemma 3.1 we see that $(I_{T-}^{\beta(1-\alpha)}\varphi)(T) = 0$ and $(I_{0^+}^{1-\gamma}u)(0) = (D_{0^+}^{\gamma-1}u)(0) = b$, so

$$
-b(I_{T-}^{\beta(1-\alpha)}\varphi)(0) - \int_0^T (I_{0^+}^{1-\gamma}u)(t)\frac{d}{dt}(I_{T-}^{\beta(1-\alpha)}\varphi)(t)dt \geq \int_0^T t^\delta |u(t)|^m \varphi(t)dt.
$$

From Definition 2.5, it follows that

$$
-b(I_{T-}^{\beta(1-\alpha)}\varphi)(0) + \int_0^T (I_{0^+}^{1-\gamma}u)(t)(D_{T-}^{1-\beta(1-\alpha)}\varphi)(t)dt \geq \int_0^T t^\delta |u(t)|^m \varphi(t)dt
$$

and from Lemma 2.7 we see that

$$
\begin{aligned}
&- b(I_{T-}^{\beta(1-\alpha)}\varphi)(0) \\
&+ \int_0^T (I_{0^+}^{1-\gamma}u)(t)\Big[\frac{1}{\Gamma[\beta(1-\alpha)]}\Big(\frac{\varphi(T)}{(T-t)^{1-\beta(1-\alpha)}} - \int_t^T \frac{\varphi'(s)ds}{(s-t)^{1-\beta(1-\alpha)}}\Big)\Big] \\
&\geq \int_0^T t^\delta |u(t)|^m \varphi(t)dt.
\end{aligned} \tag{3.4}
$$

Since $\varphi(T) = 0$, relation (6) becomes

$$
-b(I_{T-}^{\beta(1-\alpha)}\varphi)(0) - \int_0^T (I_{0^+}^{1-\gamma}u)(t)(I_{T^-}^{\beta(1-\alpha)}\varphi')(t)dt \geq \int_0^T t^\delta |u(t)|^m \varphi(t)dt.
$$

Lemma 2.9 allows us to write

$$
-b(I_{T-}^{\beta(1-\alpha)}\varphi)(0) - \int_0^T \varphi'(t)(I_{0^+}^{\beta(1-\alpha)}I_{0^+}^{1-\gamma}u)(t)dt \geq \int_0^T t^\delta |u(t)|^m \varphi(t)dt,
$$

and by Lemma 2.8

$$
- b(I_{T-}^{\beta(1-\alpha)}\varphi)(0) - \int_0^T \varphi'(t)(I_{0^+}^{1-\alpha}u)(t)dt \geq \int_0^T t^\delta |u(t)|^m \varphi(t)dt. \tag{3.5}
$$

Notice that

$$
\begin{aligned}
- \int_0^T \varphi'(t)(I_{0^+}^{1-\alpha}u)(t)dt &= \frac{-1}{\Gamma(1-\alpha)}\int_0^T \varphi'(t)\int_0^t \frac{u(s)}{(t-s)^\alpha}ds\,dt \\
&\leq \frac{1}{\Gamma(1-\alpha)}\int_0^T |\varphi'(t)|\int_0^t \frac{|u(s)|}{(t-s)^\alpha}dsdt.
\end{aligned}
$$

Since $\varphi(t)$ is nonincreasing, $\varphi(s) \geq \varphi(t)$ for all $t \geq s$, and

$$
\frac{1}{\varphi(s)^{1/m}} \leq \frac{1}{\varphi(t)^{1/m}}, \quad 0 \leq s \leq t < T,\ m > 1.
$$

Also we have

$$
\varphi'(t) = 0, \quad t \in [0, T/2].
$$

Therefore,

$$
- \int_0^T \varphi'(t)(I_{0^+}^{1-\alpha}u)(t)dt \leq \frac{1}{\Gamma(1-\alpha)}\int_0^T |\varphi'(t)|\int_0^t \frac{|u(s)|}{(t-s)^\alpha}\frac{\varphi(s)^{1/m}}{\varphi(s)^{1/m}}dsdt
$$

$$\begin{aligned}&\le \frac{1}{\Gamma(1-\alpha)}\int_0^T \frac{|\varphi'(t)|}{\varphi(t)^{1/m}}\int_0^t \frac{|u(s)|}{(t-s)^\alpha}\varphi(s)^{1/m}dsdt\\ &\le \frac{1}{\Gamma(1-\alpha)}\int_{T/2}^T \frac{|\varphi'(t)|}{\varphi(t)^{1/m}}\int_0^t \frac{|u(s)|}{(t-s)^\alpha}\varphi(s)^{1/m}dsdt.\end{aligned}$$

Hence,

$$-\int_0^T \varphi'(t)(I_{0^+}^{1-\alpha}u)(t)dt \le \int_{T/2}^T \frac{|\varphi'(t)|}{\varphi(t)^{1/m}}(I_{0^+}^{1-\alpha}\varphi^{1/m}|u|)(t)dt\,.$$

By Lemma 2.9,

$$-\int_0^T \varphi'(t)(I_{0^+}^{1-\alpha}u)(t)dt \le \int_{T/2}^T \Big(I_{T-}^{1-\alpha}\frac{|\varphi'|}{\varphi^{1/m}}\Big)(t)\varphi(t)^{1/m}|u(t)|dt. \tag{3.6}$$

(Note that we may assume that $|\varphi'(t)|\varphi(t)^{-1/m}$ is summable even though $\varphi(t)\to 0$ as $t\to T$, for otherwise we consider $\varphi^\lambda(t)$ with sufficiently large exponent λ). Next, we multiply by $t^{\delta/m}.t^{-\delta/m}$ inside the integral in the right hand side of (8)

$$-\int_0^T \varphi'(t)(I_{0^+}^{1-\alpha}u)(t)dt \le \int_{T/2}^T \Big(I_{T-}^{1-\alpha}\frac{|\varphi'|}{\varphi^{1/m}}\Big)(t)\varphi(t)^{1/m}\frac{t^{\delta/m}}{t^{\delta/m}}|u(t)|dt.$$

For $-\alpha<\delta<0$ we have $t^{-\delta/m}<T^{-\delta/m}$ (because $t<T$) and for $\delta>0$ we obtain $t^{-\delta/m}<2^{\delta/m}T^{-\delta/m}$ (because $T/2<t$), that is

$$t^{-\delta/m}<\max\{1,2^{\delta/m}\}T^{-\delta/m}.$$

Therefore,

$$\begin{aligned}&-\int_0^T \varphi'(t)(I_{0^+}^{1-\alpha}u)(t)dt\\ &\le \max\{1,2^{\delta/m}\}T^{-\delta/m}\int_{T/2}^T (I_{T-}^{1-\alpha}\frac{|\varphi'|}{\varphi^{1/m}})(t)t^{\delta/m}\varphi(t)^{1/m}|u(t)|dt.\end{aligned} \tag{3.7}$$

A simple application of the Young inequality (Theorem 2.11) with m and m' such that $\frac{1}{m}+\frac{1}{m'}=1$ gives

$$\begin{aligned}&-\int_0^T \varphi'(t)(I_{0^+}^{1-\alpha}u)(t)dt\\ &\le \frac{1}{m}\int_{T/2}^T t^\delta\varphi(t)|u(t)|^m dt+\frac{(\max\{1,2^{\delta/m}\})^{m'}}{m'}T^{-\frac{\delta m'}{m}}\int_{T/2}^T\Big(I_{T-}^{1-\alpha}\frac{|\varphi'|}{\varphi^{1/m}}\Big)^{m'}(t)dt\\ &\le \frac{1}{m}\int_0^T t^\delta\varphi(t)|u(t)|^m dt+\frac{(\max\{1,2^{\delta/m}\})^{m'}}{m'}T^{-\frac{\delta m'}{m}}\int_{T/2}^T\Big(I_{T-}^{1-\alpha}\frac{|\varphi'|}{\varphi^{1/m}}\Big)^{m'}(t)dt.\end{aligned}$$

or

$$\begin{aligned}&\int_0^T \varphi'(t)(I_{0^+}^{1-\alpha}u)(t)dt\\ &\ge -\frac{1}{m}\int_0^T t^\delta\varphi(t)|u(t)|^m dt-\frac{(\max\{1,2^{\delta/m}\})^{m'}}{m'}T^{-\frac{\delta m'}{m}}\int_{T/2}^T(I_{T-}^{1-\alpha}\frac{|\varphi'|}{\varphi^{1/m}})^{m'}(t)dt.\end{aligned} \tag{3.8}$$

Clearly from (3.5) and (3.8), we see that

$$-b(I_{T-}^{\beta(1-\alpha)}\varphi)(0)+\frac{(\max\{1,2^{\delta/m}\})^{m'}}{m'}T^{-\frac{\delta m'}{m}}\int_{T/2}^T(I_{T-}^{1-\alpha}\frac{|\varphi'|}{\varphi^{1/m}})^{m'}(t)dt$$

$$\geq (1-\frac{1}{m}) \int_0^T t^\delta |u(t)|^m \varphi(t)dt,$$

or since $b > 0$,

$$\frac{1}{m'} \int_0^T t^\delta |u(t)|^m \varphi(t)dt \leq \frac{(\max\{1, 2^{\delta/m}\})^{m'}}{m'} T^{-\frac{\delta m'}{m}} \int_{T/2}^T \Big(I_{T-}^{1-\alpha} \frac{|\varphi'|}{\varphi^{1/m}} \Big)^{m'} (t)dt.$$

Therefore, by Definition 2.3 we have

$$\begin{aligned}
&\int_0^T t^\delta |u(t)|^m \varphi(t)dt \\
&\leq (\max\{1, 2^{\delta/m}\})^{m'} T^{-\frac{\delta m'}{m}} \int_{T/2}^T \Big(\frac{1}{\Gamma(1-\alpha)} \int_t^T (s-t)^{-\alpha} \frac{|\varphi'(s)|}{\varphi(s)^{1/m}} ds \Big)^{m'} dt.
\end{aligned}$$

The change of variable $\sigma T = t$ yields

$$\begin{aligned}
&\int_0^T t^\delta |u(t)|^m \varphi(t)dt \\
&\leq (\max\{1, 2^{\delta/m}\})^{m'} T^{-\frac{\delta m'}{m}} \int_{1/2}^1 (\frac{1}{\Gamma(1-\alpha)} \int_{\sigma T}^T \Big(s - \sigma T \Big)^{-\alpha} \frac{|\varphi'(s)|}{\varphi(s)^{1/m}} ds)^{m'} T d\sigma.
\end{aligned}$$

Another change of variable $s = rT$ gives

$$\begin{aligned}
&\int_0^T t^\delta |u(t)|^m \varphi(t)dt \\
&\leq (\max\{1, 2^{\delta/m}\})^{m'} T^{-\frac{\delta m'}{m}} \int_{1/2}^1 \Big(\frac{1}{\Gamma(1-\alpha)} \int_\sigma^1 (rT - \sigma T)^{-\alpha} \frac{|\varphi'(r)|}{\varphi(r)^{1/m}} dr \Big)^{m'} T d\sigma,
\end{aligned}$$

or

$$\begin{aligned}
&\int_0^T t^\delta |u(t)|^m \varphi(t)dt \\
&\leq \frac{(\max\{1, 2^{\delta/m}\})^{m'}}{\Gamma^{m'}(1-\alpha)} T^{1-\alpha m' - \delta m'/m} \int_{1/2}^1 \Big(\int_\sigma^1 (r-\sigma)^{-\alpha} \frac{|\varphi'(r)|}{\varphi(r)^{1/m}} dr \Big)^{m'} d\sigma.
\end{aligned} \tag{3.9}$$

It is clear that we may assume that the integral term in the right-hand side of (3.9) is bounded; that is,

$$\int_{1/2}^1 \Big(\int_\sigma^1 (r-\sigma)^{-\alpha} \frac{|\varphi'(r)|}{\varphi(r)^{1/m}} dr \Big)^{m'} d\sigma \leq K_1,$$

for some positive constant K_1, otherwise we consider $\varphi^\lambda(r)$ with some sufficiently large λ. Therefore,

$$\int_0^T t^\delta |u(t)|^m \varphi(t)dt \leq K_2 T^{1-\alpha m' - \delta m'/m}, \tag{3.10}$$

with

$$K_2 := \frac{(\max\{1, 2^{\delta/m}\})^{m'}}{\Gamma^{m'}(1-\alpha)} K_1.$$

If $m < \frac{\delta+1}{1-\alpha}$ we see that $1 - \alpha m' - \delta m'/m < 0$ and consequently $T^{1-\alpha m' - \delta m'/m} \to 0$ as $T \to \infty$. Then from (3.10) we obtain

$$\lim_{T\to\infty} \int_0^T t^\delta |u(t)|^m \varphi(t)dt = 0.$$

This is a contradiction since the solution is supposed to be nontrivial.

In the case $m = \frac{\delta+1}{1-\alpha}$ we have $1 - \alpha m' - \delta m'/m = 0$ and the relation (3.10) ensures that

$$\lim_{T\to\infty} \int_0^T t^\delta |u(t)|^m \varphi(t) dt \le K_2. \tag{3.11}$$

Moreover, it is clear that

$$\begin{aligned}
&\int_{T/2}^T (I_{T-}^{1-\alpha} \frac{|\varphi'|}{\varphi^{1/m}})(t) t^{\delta/m} \varphi(t)^{1/m} |u(t)| dt \\
&\le \Big[\int_{T/2}^T \Big(I_{T-}^{1-\alpha} \frac{|\varphi'|}{\varphi^{1/m}} \Big)^{m'} (t) dt \Big]^{1/m'} \Big[\int_{T/2}^T t^\delta \varphi(t) |u(t)|^m dt \Big]^{1/m}.
\end{aligned}$$

This relation, together with (3.5) and (3.7), implies that

$$\int_0^T t^\delta \varphi(t) |u(t)|^m dt \le K_3 \Big[\int_{T/2}^T t^\delta \varphi(t) |u(t)|^m dt \Big]^{1/m}$$

for some positive constant K_3, with

$$\lim_{T\to\infty} \int_{T/2}^T t^\delta \varphi(t) |u(t)|^m dt = 0$$

due to the convergence of the integral in (3.11). This leads again to a contradiction. The proof is complete. □

4. Sharpness of the bound

In this section we want to prove that the exponent $\frac{\delta+1}{1-\alpha}$ is sharp in some sense. We will show that solutions exist for exponents strictly bigger than $\frac{\delta+1}{1-\alpha}$. For that we need the following lemma

Lemma 4.1. *The following identity holds*

$$(D_{a^+}^{\alpha,\beta}[(s-a)^{\sigma-1}])(t) = \frac{\Gamma(\sigma)}{\Gamma(\sigma-\alpha)} (t-a)^{\sigma-\alpha-1}, \quad t > a, \ \sigma > 0,$$

where $0 < \alpha < 1$ *and* $0 \le \beta \le 1$.

Example 4.2. Consider the following differential equation with Hilfer fractional derivative of order $0 < \alpha < 1$ and $0 \le \beta \le 1$,

$$(D_{a^+}^{\alpha,\beta} y)(t) = \lambda (t-a)^\delta [y(t)]^m, \quad t > a, \ m > 1 \tag{4.1}$$

with λ, $\delta \in \mathbb{R}$ $(\lambda \ne 0)$.

Look for a solution of the form $y(t) = c(t-a)^\nu$ for some $\nu \in \mathbb{R}$. Let us find the values of c and ν. By using Lemma 4.1 we have

$$(D_{a^+}^{\alpha,\beta}[c(s-a)^\nu])(t) = \frac{c\Gamma(\nu+1)}{\Gamma(\nu-\alpha+1)} (t-a)^{\nu-\alpha}, \quad \nu > -1, \ t > a.$$

Plugging this expression in (4.1) yields

$$\frac{c\Gamma(\nu+1)}{\Gamma(\nu-\alpha+1)} (t-a)^{\nu-\alpha} = \lambda (t-a)^\delta [c(t-a)^\nu]^m.$$

We obtain $\nu = \frac{\alpha+\delta}{1-m}$ and $c = [\frac{\Gamma(\frac{\alpha+\delta}{1-m}+1)}{\lambda\Gamma(\frac{m\alpha+\delta}{1-m}+1)}]^{1/(m-1)}$. That is,

$$y(t) = \Big[\frac{\Gamma(\frac{\alpha+\delta}{1-m}+1)}{\lambda\Gamma(\frac{m\alpha+\delta}{1-m}+1)}\Big]^{1/(m-1)}(t-a)^{(\alpha+\delta)/(1-m)}$$

is a solution of (4.1). One can easily check that $y \in C_{1-\gamma}$ with $m = 1 + \frac{\alpha+\delta}{1-\gamma}$ which is clearly bigger than the critical exponent $\frac{\delta+1}{1-\alpha}$ if $\delta > -\alpha$. Moreover, the condition $(D_{a^+}^{\gamma-1}u)(0) = b$ is satisfied with

$$b = \Big[\frac{\Gamma(\frac{\alpha+\delta}{1-m}+1)}{\lambda\Gamma(\frac{m\alpha+\delta}{1-m}+1)}\Big]^{1/(m-1)}.$$

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