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EXISTENCE AND UNIQUENESS FOR A TWO-POINT INTERFACE BOUNDARY VALUE PROBLEM

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Abstract. We obtain sufficient conditions, easily verifiable, for the existence and uniqueness of piecewise smooth solutions of a linear two-point boundary-value problem with general interface conditions. The coefficients of the differential equation may have jump discontinuities at the interface point. As an example, the conditions obtained are applied to a problem with typical interface such as perfect contact, non-perfect contact, and flux jump conditions.

1. Introduction

In this article, we study existence and uniqueness of solutions of a two-point boundary-value problem with general interface conditions specified at an intermediate point. A linear differential equation of the problem has variable coefficients that may have jump discontinuities at the interface point. The problem may be viewed as a multi-point boundary value problem where solution and coefficient discontinuities are permitted at interface points. It may serve as a one-dimensional model problem for studying corresponding multi-dimensional, time dependent, or nonlinear interface problems.

Boundary-value problems with interface conditions are also known as BVPs with transmission (transmittal) conditions, or diffraction problems. BVPs with interface conditions arise in applications such as heat or mass transfer in composite materials or materials with thin porous barriers, elasticity problems for heterogeneous materials, and population genetics [2, 10]. For example, heat transfer in layered composite materials causes a finite temperature discontinuity while the heat flux is continuous across the interface; this phenomenon is described by the interfacial thermal resistance [14]. Similarly, a non-perfect contact of two materials causes the thermal contact resistance effect which also results in discontinuity of temperature across the interface [7]. In both cases, the temperature difference at the interface is proportional to the heat flux. Formal analytical solutions of conductive heat flow problems in a composite medium with perfect or contact interfaces is possible for differential equations with piecewise constant coefficients (see Ch. 9 and references on p. 378 in [7]).

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General linear two-point interface boundary value problems (IBVPs) for systems of ordinary differential equations were studied long ago in [11, 12, 13]. Note that existence and uniqueness of a solution is proved in [13, Theorem 2] assuming that $\det(H(Y)) \neq 0$, where $H(Y)$ is a matrix functional defining boundary conditions, and Y is a d-solution, a solution of the system that satisfies only the interface conditions. This condition is hard to verify in practice for general problems. Under a similar assumption, a d-solution of an interface problem with a more general boundary condition is obtained in [11, see (15)].

An interface BVP with a linear second-order differential equation that has a piecewise constant leading coefficient and a complex spectral parameter λ was studied in [8], where existence, uniqueness, and coerciveness in weighted Sobolev spaces are proved assuming that $|\lambda|$ is sufficiently large along with some other conditions. Similar results are obtained in [4] for an m-th order differential equation also with a piecewise constant leading coefficient but with a spectral parameter that may now appear in multi-point boundary-interface conditions. Sturm–Liouville problems for differential operators with interface conditions were studied in many works; for example, see [6, 15]. These studies address issues of existence of a sequence of real eigenvalues, zero counts of corresponding eigenfunctions, and their completeness, and do not focus on uniqueness of solutions of IBVPs.

The goal of this work is to formulate easily verifiable sufficient conditions for existence and uniqueness of the solution of the interface BVP with piecewise continuous coefficients and general interface conditions. The obtained conditions involve only the coefficients of the boundary and interface conditions and the coefficients of the differential operator. The results of this work may be useful for developing and analyzing numerical methods for solving IBVPs.

The presented simple analysis is based on the approach in [5, §1.2], which applies an alternative theorem and reduces the question of existence and uniqueness of solutions of two-point BVP to that of unique solvability of a scalar nonlinear equation using an auxiliary variational initial value problem (IVP). Similar results could be obtained in the case of multiple interface points or a nonlinear differential equation.

The outline of the article is as follows. In Section 2, we give the formulation of IBVP, introduce notation, and present auxiliary results. In Section 3, we show that there is a bijective map between the solution sets of IBVP and of a certain nonlinear system of two equations corresponding to the interface conditions, and then we prove an alternative theorem which associates unique solvability of IBVP with that of the corresponding homogeneous problem. In Section 4, we state the main result of this article which gives sufficient conditions for existence and uniqueness of solutions of IBVP, and we also present a counterpart statement obtained by change of variable. In Section 5, we find the Green's function of the problem, and list some of its properties.

2. Problem and auxiliary facts

Let (a, b) be a line interval, and let $c \in (a, b)$. For an integer $k \geq 0$, let Q^k be a vector space of functions v defined on $[a, c] \cup [c, b]$ such that

$$v|_{[a,c]} \in C^k[a, c], \quad v|_{[c,b]} \in C^k[c, b].$$

Note that function v and its derivatives can have only jump discontinuities at the interface point $x = c$, and v is double-valued at $x = c$. For every $v \in Q^0$, let

$$v^{\pm} = \lim_{x \to c\pm} v(x), \text{ and } [v] = v^+ - v^-.$$

Let functions p, q, and f belong to Q^0, and let α, β, a_j, b_j, $c_{ij}^{\pm}$, and γ_i for $i = 1, 2$, and $j = 0, 1$ be reals. Consider the following two-point IBVP:

$$Lu(x) \equiv -u''(x) + p(x)u'(x) + q(x)u(x) = f(x), \quad x \in (a, c) \cup (c, b), \tag{2.1a}$$

$$l_a(u) \equiv a_0 u(a) - a_1 u'(a) = \alpha, \quad l_b(u) \equiv b_0 u(b) + b_1 u'(b) = \beta \tag{2.1b}$$

with additional two interface conditions

$$l_i(u) \equiv c_{i,0}^- u^- + c_{i,1}^- (u')^- + c_{i,0}^+ u^+ - c_{i,1}^+ (u')^+ = \gamma_i, \ i = 1, 2. \tag{2.1c}$$

Letting

$$\mathbf{C} = (\mathbf{C}^- | \mathbf{C}^+), \ \mathbf{C}^{\pm} = \begin{pmatrix} c_{10}^{\pm} & \mp c_{11}^{\pm} \\ c_{20}^{\pm} & \mp c_{21}^{\pm} \end{pmatrix}, \tag{2.2}$$

the interface conditions (2.1c) can be written in the matrix-vector form

$$\mathbf{C}(u^-, (u')^-, u^+, (u')^+)^T = (\gamma_1, \gamma_2)^T.$$

Nonhomogeneous IBVP (2.1) can be reduced to a problem with homogeneous boundary and interface conditions by introducing a new dependent function $\tilde{u} = u - \phi$, where piecewise linear function ϕ is chosen to satisfy the nonhomogeneous conditions (assuming that the resulting 4×4 linear system is consistent).

To describe some typical interface conditions, let d_0, h, k_1 and k_2 be positive constants, and let

$$k(x) = \begin{cases} k_1, & x \in (a, c), \\ k_2, & x \in (c, b). \end{cases}$$

Consider the differential equation

$$-(ku')' + qu = f \quad \text{in } (a, c) \cup (c, b) \tag{2.3}$$

along with the boundary conditions (2.1b) and interface conditions presented in Table 1.

Table 1. Typical interface conditions.

Type	*Equations*	*Interface matrix* $\mathbf{C}$	*Matrix* $\hat{\mathbf{C}}$
perfect contact	$[u] = 0$, $[ku'] = 0$	$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & k_1 & 0 & -k_2 \end{pmatrix}$	$\frac{1}{k_2}\begin{pmatrix} k_2 & 0 \\ 0 & k_1 \end{pmatrix}$
flux jump	$[u] = 0$, $[ku'] = d_0 u(c)$	$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & k_1 & d_0 & -k_2 \end{pmatrix}$	$\frac{1}{k_2}\begin{pmatrix} k_2 & 0 \\ d_0 & k_1 \end{pmatrix}$
radiation / thermal resistance	$h[u] = ku'(c)$, $[ku'] = 0$	$\begin{pmatrix} h & 0 & -h & k_2 \\ 0 & k_1 & 0 & -k_2 \end{pmatrix}$	$\frac{1}{hk_2}\begin{pmatrix} hk_2 & k_1 k_2 \\ 0 & hk_1 \end{pmatrix}$

For each type of interface conditions in Table 1, submatrix $\mathbf{C}^-$ is nonnegative, and $-\mathbf{C}^+$ is an M-matrix; hence,

$$\hat{\mathbf{C}} \equiv -(\mathbf{C}^+)^{-1} \mathbf{C}^- \geq 0 \tag{2.4}$$

(see [1]). It will be shown in the sequel, that condition (2.4) is required for existence and uniqueness of a solution.

The following is a known existence and uniqueness statement for BVP (2.1a)–(2.1b) without interface conditions (see the corollary from Theorem 1.2.2 in [5]).

Theorem 2.1. *Let functions p, q, and f be continuous on $[a,b]$ with $q(x)>0$ for all $x \in [a,b]$. If reals a_0, a_1, b_0, and b_1 satisfy*

$$a_0a_1 \geq 0, \quad b_0b_1 \geq 0, \quad |a_0|+|a_1| \neq 0, \quad |b_0|+|b_1| \neq 0, \quad |a_0|+|b_0| \neq 0,$$

then BVP (2.1a), (2.1b) *has a unique solution for all reals α and β.*

The goal of this work is to obtain a similar result for IBVP (2.1); that is, a list of simple conditions involving only the coefficients of the differential equation, and the boundary and interface conditions that imply existence and uniqueness. The following statement is proved in [5] (see the proof of Theorem 1.2.2), and it plays a key role in the following analysis.

Lemma 2.2. *Let real values a_0 and a_1 satisfy $a_0a_1 \geq 0$, $|a_0|+|a_1| \neq 0$. Let functions $p(x)$ and $q(x)$ be continuous on the interval $[a,b]$, let $q(x)>0$ for $x \in [a,b]$, and let $M = \max_{x\in[a,b]} |p(x)|$. The solution of the initial value problem*

$$\xi'' = p\xi' + q\xi, \ \xi(a) = a_1, \ \xi'(a) = a_0,$$

satisfies the following inequalities:

$$\begin{gathered} \xi(b)\xi'(b) > 0, \\ |\xi(b)| > |a_1| + |a_0|(1-e^{-M(b-a)})/M > 0, \\ |\xi'(b)| > |a_0|e^{-M(b-a)} \geq 0. \end{gathered} \tag{2.5}$$

Note that, if $a_0 \neq 0$, then both $\xi(b)$ and $\xi'(b)$ are bounded away from zero. On the other hand, if $a_0 = 0$, then $\xi(b)$ is bounded away from zero while $\xi'(b)$ is not. These facts and the following one-dimensional form of the Hadamard theorem (see Theorem 5.3.10 in [9]) are used in the proof of Theorem 4.1, the main statement of this article.

Lemma 2.3. *Let $\phi : R \to R$ be a continuously differentiable function. If there is $\gamma > 0$ such that $|\phi'(s)| \geq \gamma$ for all $s \in R$, then ϕ is a homeomorphism.*

3. The alternative theorem

The alternative theorem for a boundary value problem is a statement that the nonhomogeneous BVP has a unique solution if and only if the corresponding reduced system is incompatible; that is, the corresponding homogeneous problem has only the trivial solution [3, Ch. IX]. In this section, we prove the alternative theorem for IBVP (2.1).

Let functionals $\tilde{l}_a$ and $\tilde{l}_b$ correspond to some linear initial conditions that are linearly independent with l_a and l_b, respectively, defined by (2.1b). For each $\mathbf{s} = (s_1, s_2) \in R^2$, the initial value problems,

$$\begin{gathered} Lu_1(x) = f(x), \ x \in (a,c), \quad l_a(u_1) = \alpha, \ \tilde{l}_a(u_1) = s_1, \\ Lu_2(x) = f(x), \ x \in (c,b), \quad l_b(u_2) = \beta, \ \tilde{l}_b(u_2) = s_2, \end{gathered} \tag{3.1}$$

have unique solutions $u_1(s_1, x)$ and $u_2(s_2, x)$, respectively. Let

$$u(\mathbf{s}; x) = \begin{cases} u_1(s_1, x), & x \in [a, c], \\ u_2(s_2, x), & x \in [c, b], \end{cases} \tag{3.2}$$

and note that $u(\mathbf{s}; x)$ belongs to Q^2. For functionals l_1 and l_2 defined in (2.1c), consider the following nonlinear system of equations for $\mathbf{s}$:

$$l_i(u(\mathbf{s}, \cdot)) = \gamma_i, \quad i = 1, 2. \tag{3.3}$$

Let $\mathcal{U}$ and $\mathcal{S}$ be the solution sets of IBVP (2.1) and the nonlinear system (3.3), respectively. If $\mathbf{s} \in \mathcal{S}$, then function $u(\mathbf{s}, x)$ defined by (3.2) is a solution of IBVP (2.1). Consider mapping $\hat{u} : \mathcal{S} \to \mathcal{U}$ defined by

$$\hat{u}(\mathbf{s}) = u(\mathbf{s}, \cdot), \quad \forall \mathbf{s} \in \mathcal{S}. \tag{3.4}$$

Lemma 3.1. *The mapping $\hat{u}$ is a bijection.*

Proof. Let us prove that $\hat{u}$ is onto. Let $U(x)$ be a solution of IBVP (2.1), let

$$s_1 = \tilde{l}_a(U), \; s_2 = \tilde{l}_b(U), \; \mathbf{s} = (s_1, s_2),$$

and let $u(\mathbf{s}; x)$ be given by (3.1), (3.2). Clearly, $u(\mathbf{s}, \cdot) = U$ since solutions of IVPs (3.1) are unique. Therefore, $u(\mathbf{s}, \cdot)$ satisfies equations (3.3). Thus, $\mathbf{s} = (s_1, s_2) \in \mathcal{S}$, and $\hat{u}(\mathbf{s}) = U(x)$; that is, mapping $\hat{u}$ is onto.

To prove that mapping $\hat{u}$ is one-to-one, suppose that $\mathbf{s}', \mathbf{s}'' \in \mathcal{S}$, and $\mathbf{s}' \neq \mathbf{s}''$. By uniqueness of solutions of IVPs (3.1) and definition (3.2), obtain $u(\mathbf{s}'; \cdot) \neq u(\mathbf{s}''; \cdot)$; that is, $\hat{u}(\mathbf{s}') \neq \hat{u}(\mathbf{s}'')$ by (3.4). $\square$

The following is the reduced system corresponding to IBVP (2.1):

$$\begin{gathered} Lw(x) = 0, \quad x \in (a, c) \cup (c, b), \\ l_a(w) = l_b(w) = l_1(w) = l_2(w) = 0. \end{gathered} \tag{3.5}$$

Theorem 3.2. *IBVP (2.1) has a unique solution $u \in Q^2$ if and only if the reduced system (3.5) has only the trivial solution $w = 0$.*

Proof. If IBVP (2.1) has a unique solution, then $w = 0$ is the unique solution of the reduced system (3.5). On the converse, assume that the reduced system has only the trivial solution.

Problem (3.1), (3.2) can be formulated in the following form: Let $v_1, w_1 \in C^2[a, c]$ and $v_2, w_2 \in C^2[c, b]$ be the unique solutions of the initial-value problems

$$\begin{gathered} Lv_1 = f \quad \text{on } (a, c), \quad l_a(v_1) = \alpha, \; \tilde{l}_a(v_1) = 0, \\ Lv_2 = f \quad \text{on } (c, b), \quad l_b(v_2) = \beta, \; \tilde{l}_b(v_2) = 0, \end{gathered}$$

and

$$\begin{gathered} Lw_1 = 0 \quad \text{on } (a, c), \quad l_a(w_1) = 0, \; \tilde{l}_a(w_1) = 1, \\ Lw_2 = 0 \quad \text{on } (c, b), \quad l_b(w_2) = 0, \; \tilde{l}_b(w_2) = 1. \end{gathered} \tag{3.6}$$

For $\mathbf{s} = (s_1, s_2) \in R^2$, let

$$u(\mathbf{s}; x) = \begin{cases} v_1(x) + s_1 w_1(x), & x \in [a, c], \\ v_2(x) + s_2 w_2(x), & x \in [c, b], \end{cases}$$

By requiring $u(\mathbf{s},x)$ to satisfy equations (3.3), for $i=1,2$, we obtain

$$\begin{aligned} &s_1(c_{i,0}^- w_1^- + c_{i,1}^-(w_1')^-) + s_2(c_{i,0}^+ w_2^+ - c_{i,1}^+(w_2')^+) \\ &= \gamma_i - c_{i,0}^-(v_1^- - c_{i,1}^-(v_1')^- - c_{i,0}^+ v_2^+ + c_{i,1}^+(v_2')^+. \end{aligned} \tag{3.7}$$

If (s_1,s_2) is a nontrivial solution of the corresponding homogeneous system

$$s_1(c_{i,0}^- w_1^- + c_{i,1}^-(w_1')^-) + s_2(c_{i,0}^+ w_2^+ - c_{i,1}^+(w_2')^+) = 0, \quad i=1,2,$$

then, by (3.6), the function

$$w(x) = \begin{cases} s_1 w_1(x), & x\in[a,c], \\ s_2 w_2(x), & x\in[c,b], \end{cases}$$

is a nontrivial solution of the reduced system (3.5); this contradicts the previous assumption. Therefore, the linear system (3.7) has a unique solution $\mathbf{s}^*\in R^2$, which is the unique solution of nonlinear system (3.3). By Lemma 3.1, $\hat{u}(\mathbf{s}^*) = u(\mathbf{s}^*;\cdot)$ is the unique solution of IBVP (2.1). □

4. Existence and uniqueness

Let $\mathbf{C}$ and $\hat{\mathbf{C}} = (\hat{c}_{ij})$ be the matrices defined in (2.2) and (2.4), respectively. The following is the main result of this article.

Theorem 4.1. *Assume that functions p and q belong to Q^0 and $q(x)>0$ for all $x\in[a,c]\cup[c,b]$. Assume that*

$$a_0a_1 \ge 0, \quad a_0+a_1\ne 0, \tag{4.1a}$$

$$b_0b_1 \ge 0, \quad b_0+b_1\ne 0, \tag{4.1b}$$

and that

$$\det(\mathbf{C}^+) \ne 0, \tag{4.2a}$$

$$\hat{\mathbf{C}} \le 0 \quad \text{or} \quad \hat{\mathbf{C}} \ge 0, \tag{4.2b}$$

$$\hat{\mathbf{C}} \ne 0. \tag{4.2c}$$

Also assume that at least one of the following assumptions holds:

(1) $b_0\ne 0$ *and* $\hat{c}_{11}+\hat{c}_{21}\ne 0$;
(2) $b_0\ne 0$, $\hat{c}_{11}+\hat{c}_{21}=0$, *and* $a_0\ne 0$;
(3) $b_1\ne 0$ *and* $\hat{c}_{21}\ne 0$;
(4) $b_1\ne 0$, $\hat{c}_{22}\ne 0$, *and* $a_0\ne 0$.

Then the homogeneous interface problem (3.5) has only the trivial solution.

Proof. Since a_0 and a_1 are not both zeros by assumption (4.1a), let reals d_0 and d_1 satisfy the identity

$$a_1d_0 - a_0d_1 = 1. \tag{4.3}$$

For each $s\in R$, the interface IVP

$$l_a(w)=0,\ d_0w(a)-d_1w'(a)=s, \tag{4.4a}$$

$$Lw(x)=0,\ x\in(a,c), \tag{4.4b}$$

$$\mathbf{C}(w^-,(w')^-,w^+,(w')^+)^T = \mathbf{0}, \tag{4.4c}$$

$$Lw(x)=0,\ x\in(c,b), \tag{4.4d}$$

has a unique solution $w(s;\cdot) \in Q^2$. Indeed, by (4.3), IVP (4.4a), (4.4b) has a unique solution $w_1(s;\cdot) \in C^2[a,c]$. Using assumption (4.2a), $\det(\mathbf{C}^+) \neq 0$, let

$$\begin{pmatrix} c_1(s) \\ c_0(s) \end{pmatrix} \equiv \hat{\mathbf{C}} \begin{pmatrix} w_1(s;c) \\ w_1'(s;c) \end{pmatrix}.$$

Function $w_2(s;\cdot) \in C^2[c,b]$ is the unique solution of IVP with the differential equation (4.4d) subject to the initial conditions

$$w_2(s;c) = c_1(s), \quad w_2'(s;c) = c_0(s).$$

Then then function

$$w(s;x) = \begin{cases} w_1(s;x), & x \in [a,c], \\ w_2(s;x), & x \in [c,b], \end{cases}$$

belongs to Q_2, and it is the solution of problem (4.4). Note that $d^i w/dx^i(s;x)$, $i \leq 2$, continuously depends on parameter s at $x = b$.

Differentiating the equations in (4.4) with respect to s and using condition (4.3), obtain the following variational interface IVP for the unknown function $\xi(s;x) = \partial w(s;x)/\partial s$:

$$\xi(a) = a_1, \ \xi'(a) = a_0, \tag{4.5a}$$

$$L\xi(x) = 0, \ x \in (a,c), \tag{4.5b}$$

$$\mathbf{C}(\xi^-, (\xi')^-, \xi^+, (\xi')^+)^T = \mathbf{0},$$

$$L\xi(x) = 0, \ x \in (c,b).$$

Functions $d^i\xi/dx^i$, $i \leq 2$ continuously depend on parameter s at $x = b$.

For the functional l_b defined in (2.1b), let $\phi : R \to R$ be given by

$$\phi(s) = l_b(w(s;\cdot)), \ s \in R.$$

Since

$$\phi'(s) = b_0\xi(s;b) + b_1\frac{d\xi}{dx}(s;b), \tag{4.6}$$

function ϕ is continuously differentiable. Since $w_1(0;\cdot) = 0$ and $w_2(0;\cdot) = 0$, it follows that $\phi(0) = 0$. Let us prove that $s = 0$ is the only solution of the equation $\phi(s) = 0$, $s \in R$, which then implies that IBVP (3.5) also has only the trivial solution. To this end, we need to prove that the derivative $\phi'(s)$ is bounded away from zero; that is, there is $\gamma > 0$ such that $|\phi'(s)| \geq \gamma > 0$ for all $s \in R$.

Let $M = \max_{x\in[a,b]} |p(x)|$ and

$$\delta = \min\{1, (1 - e^{-M\min\{c-a,b-c\}})/M\} > 0. \tag{4.7}$$

Let $s \in R$. By Lemma 2.2 applied on IVP (4.5a), (4.5b), and assumption (4.1a), it follows that

$$\xi^-(\xi')^- > 0, \tag{4.8a}$$

$$|\xi^-| > |a_1| + |a_0|(1 - e^{-M(c-a)})/M \geq \delta|a_0 + a_1| > 0, \tag{4.8b}$$

$$|(\xi')^-| > |a_0|e^{-M(c-a)} \geq 0. \tag{4.8c}$$

Since $\det(\mathbf{C}^+) \neq 0$, matrix $\hat{\mathbf{C}} = -(\mathbf{C}^+)^{-1}\mathbf{C}^-$ exists. Let

$$\begin{pmatrix} c_1' \\ c_0' \end{pmatrix} \equiv \hat{\mathbf{C}} \begin{pmatrix} \xi^- \\ (\xi')^- \end{pmatrix}. \tag{4.9}$$

By (4.2b), (4.2c), and (4.8a), it follows that

$$c_0'c_1' \geq 0, \; |c_0'| + |c_1'| \neq 0. \tag{4.10}$$

Function $\xi|_{[c,b]}$ is the unique solution of the IVP

$$\xi'' = p\xi' + q\xi \text{ on } (c,b), \; \xi(c) = c_1', \; \xi'(c) = c_0'.$$

Applying Lemma 2.2 on the interval (c,b) with a_i replaced by c_i' and using (4.7) and (4.10), conclude that

$$\xi(b)\xi'(b) > 0, \tag{4.11a}$$

$$|\xi(b)| > |c_1'| + |c_0'|(1 - e^{-M(b-c)})/M \geq \delta|c_0' + c_1'| > 0, \tag{4.11b}$$

$$|\xi'(b)| > |c_0'|e^{-M(b-c)} \geq 0. \tag{4.11c}$$

By (4.6), (4.1b), and (4.11a), we obtain

$$|\phi'(s)| = |b_0\xi(b) + b_1\xi'(b)| = |b_0\xi(b)| + |b_1\xi'(b)|. \tag{4.12}$$

The last identity and (4.11b) imply

$$|\phi'(s)| \geq |b_0\xi(b)| \geq \delta|b_0(c_0' + c_1')|. \tag{4.13}$$

First, let $b_0 \neq 0$ and $\hat{c}_{11} + \hat{c}_{21} \neq 0$ (Assumption 1). Using (4.9), (4.8a), (4.2b), and (4.8b), we obtain

$$|c_0' + c_1'| \geq |(\hat{c}_{11} + \hat{c}_{21})\xi^-| > \delta|(\hat{c}_{11} + \hat{c}_{21})(a_0 + a_1)|,$$

which, along with (4.13), (4.7), and (4.1a), gives

$$|\phi'(s)| > \delta^2|b_0(\hat{c}_{11} + \hat{c}_{21})(a_0 + a_1)| > 0.$$

Now suppose that $b_0 \neq 0$, $\hat{c}_{11} + \hat{c}_{21} = 0$ and $a_0 \neq 0$ (Assumption 2). Using (4.2b), and (4.2c), we obtain

$$|\hat{c}_{12} + \hat{c}_{22}| > 0. \tag{4.14}$$

By (4.9) and (4.8c), we obtain

$$|c_0' + c_1'| = |(\hat{c}_{12} + \hat{c}_{22})(\xi')^-| > |(\hat{c}_{12} + \hat{c}_{22})a_0|e^{-M(c-a)}.$$

Therefore, by (4.13), the last bound, (4.7), and (4.14), get

$$|\phi'(s)| \geq \delta|b_0||c_0' + c_1'| > \delta|(\hat{c}_{12} + \hat{c}_{22})a_0b_0|e^{-M(c-a)} > 0.$$

Using (4.12), (4.11c), (4.9), (4.2b), and (4.8), we obtain

$$\begin{aligned} |\phi'(s)| &\geq |b_1\xi'(b)| \geq |b_1|e^{-M(b-c)}|c_0'| \\ &= |b_1|e^{-M(b-c)}|\hat{c}_{21}\xi^- + \hat{c}_{22}(\xi')^-| \\ &\geq |b_1|e^{-M(b-c)}\Big(\delta|\hat{c}_{21}(a_0 + a_1)| + |\hat{c}_{22}a_0|e^{-M(c-a)}\Big). \end{aligned} \tag{4.15}$$

If $b_1 \neq 0$ and $\hat{c}_{21} \neq 0$ (Assumption 3), then, from (4.15), by (4.7) and (4.1a), we get

$$|\phi'(s)| \geq \delta e^{-M(b-c)}\,|\hat{c}_{21}b_1(a_0 + a_1)| > 0.$$

If $b_1 \neq 0$, $\hat{c}_{22} \neq 0$, and $a_0 \neq 0$ (Assumption 4), then, from (4.15), we obtain

$$|\phi'(s)| \geq e^{-M(b-a)}\,|\hat{c}_{22}a_0b_1| > 0.$$

Thus, under each of Assumptions 1–4 in the statement of the theorem, function ϕ' is bounded away from zero. By Lemma 2.3, function $\phi(s)$ is a homeomorphism on R. In particular, $s = 0$ is the only solution of the equation $\phi(s) = 0$, which implies that problem (3.5) has only the trivial solution. $\square$

Applying Theorem 3.2, obtain the following statement.

Corollary 4.2. *If the assumptions of Theorem* 4.1 *hold, then IBVP* (2.1) *has a unique solution.*

Let us apply Theorem 4.1 to the model problem (2.3) with interface conditions given in Table 1. For all three types of interface conditions, assumptions in (4.2), $\hat{c}_{11} + \hat{c}_{21} \neq 0$, and $\hat{c}_{22} \neq 0$ are satisfied. For both the perfect contact and the radiation conditions, $\hat{c}_{21} = 0$. Therefore, using the assumptions 1 and 4 in Theorem 4.1, obtain that the corresponding interface BVPs have unique solutions for all a_0, a_1, b_0, and b_1 that satisfy (4.1a), (4.1b), and the condition $|a_0| + |b_0| \neq 0$. This result is similar to that given in Theorem 2.1 for the two-point BVP. For the flux jump interface condition, we have $\hat{c}_{21} \neq 0$. Assumptions 1 and 3 imply that the corresponding IBVP has a unique solution for all a_0, a_1, b_0, and b_1 that satisfy (4.1a) and (4.1b), even if the condition $|a_0| + |b_0| \neq 0$ is not satisfied. This differs from the conclusion in Theorem 2.1. For example, the IBVP with the flux jump interface condition has a unique solution for the Neumann boundary conditions.

By changing variable x to $-x$ in the homogeneous interface BVP (3.5), we obtain an equivalent problem:

$$\begin{gathered}
-w'' - p(-x)w' + q(-x)w = 0, \; x \in (-b, -c) \cup (-c, -a), \\
b_0 w(-b) - b_1 w'(-b) = \beta, \; a_0 w(-a) + a_1 w'(-a) = \alpha, \\
c^+_{i0} w(-c-) + c^+_{i1} w'(-c-) + c^-_{i0} w(-c+) - c^-_{i1} w'(-c+) = 0, \; i = 1, 2,
\end{gathered}$$

with the matrix

$$\tilde{\mathbf{C}} = \begin{pmatrix} c^-_{10} & -c^-_{11} \\ c^-_{20} & -c^-_{21} \end{pmatrix}^{-1} \begin{pmatrix} c^+_{10} & c^+_{11} \\ c^+_{20} & c^+_{21} \end{pmatrix}$$

playing the role of matrix $\hat{\mathbf{C}}$ for IBVP (3.5). Thus we also have the following counterpart uniqueness and existence result.

Theorem 4.3. *Assume that functions p and q belong to Q^0 and $q(x) > 0$ for $x \in [a, c) \cup (c, b]$. Assume that*

$$a_0 a_1 \geq 0, \quad a_0 + a_1 \neq 0, \quad b_0 b_1 \geq 0, \quad b_0 + b_1 \neq 0.$$

Assume that matrix $\tilde{\mathbf{C}}$ exists, and

$$\tilde{\mathbf{C}} \leq 0 \quad \text{or} \quad \tilde{\mathbf{C}} \geq 0, \quad \text{and} \quad \tilde{\mathbf{C}} \neq 0.$$

Assume that at least one of the following assumptions hold:

(1) *$a_0 \neq 0$ and $\tilde{c}_{11} + \tilde{c}_{21} \neq 0$;*
(2) *$a_0 \neq 0$, $\tilde{c}_{11} + \tilde{c}_{21} = 0$, and $b_0 \neq 0$;*
(3) *$a_1 \neq 0$ and $\tilde{c}_{21} \neq 0$;*
(4) *$a_1 \neq 0$, $\tilde{c}_{22} \neq 0$, and $b_0 \neq 0$.*

Then the interface problem (3.5) *has only the trivial solution.*

Applying Theorem 4.3 to each interface condition in Table 1, obtain matrix $\tilde{\mathbf{C}}$ equal to the corresponding matrix $\hat{\mathbf{C}}$ with k_1 and k_2 interchanged, which yields identical conditions for existence and uniqueness.

5. Green's function

To define the Green's function of IBVP (2.1), consider first the homogeneous differential equation subject to the homogeneous interface conditions only:

$$Lw = 0 \text{ on } (a,c) \cup (c,b), \quad l_1(w) = l_2(w) = 0. \tag{5.1}$$

Since $\operatorname{rank}(\mathbf{C}) = 2$ and matrix $\mathbf{C}$ has four columns, $\operatorname{nullity}(\mathbf{C}) = 2$. Let $\{\mathbf{b}_1,\ \mathbf{b}_2\} \subset R^4$ be a basis for the nullspace of $\mathbf{C}$, and define subvectors $\mathbf{b}_i^\pm \in R^2$ by

$$\mathbf{b}_i = \begin{pmatrix} \mathbf{b}_i^- \\ \mathbf{b}_i^+ \end{pmatrix}, \quad i = 1, 2.$$

Let $w_i^-(x)$ and $w_i^+(x)$ be the solutions of the corresponding initial value problems with the initial data $\mathbf{b}_i^-$ and $\mathbf{b}_i^+$, respectively; that is,

$$Lw_i^\pm = 0, \quad (w_i^\pm(c), (w_i^\pm)'(c))^T = \mathbf{b}_i^\pm, \quad i = 1, 2,$$

where the domains of the problems are the intervals (a,c) and (c,b) for the superscripts $-$ and $+$, respectively. Assume that both sets $\{\mathbf{b}_1^-, \mathbf{b}_2^-\}$ and $\{\mathbf{b}_1^+, \mathbf{b}_2^+\}$ are linearly independent. Then $\{w_1^-, w_2^-\}$ and $\{w_1^+, w_2^+\}$ are fundamental solution sets of the differential operator L on the intervals (a,c) and (c,b), respectively. For $i = 1, 2$, let

$$w_i = \begin{cases} w_i^-, & x \in [a,c], \\ w_i^+, & x \in [c,b]. \end{cases}$$

Functions w_1 and w_2 are double valued at $x = c$. For every $C_1,\ C_2 \in R$, let

$$w_C = C_1 w_1 + C_2 w_2. \tag{5.2}$$

Obviously, function w_C satisfies the homogeneous interface conditions. Let us impose on w_C the homogeneous boundary conditions

$$l_a(w_C) = 0, \quad l_b(w_C) = 0.$$

The system can be written in the form

$$C_1 l_a(w_1^-) + C_2 l_a(w_2^-) = 0, \tag{5.3a}$$

$$C_1 l_b(w_1^+) + C_2 l_b(w_2^+) = 0, \tag{5.3b}$$

A necessary and sufficient condition for a unique trivial solution of this system is

$$\det \begin{pmatrix} l_a(w_1^-) & l_a(w_2^-) \\ l_b(w_1^-) & l_b(w_2^-) \end{pmatrix} \neq 0, \tag{5.4}$$

(this is an analog of the condition $\det(H(X)) \neq 0$ in [11, p. 6]). According to Theorem 3.2, inequality (5.4) is also a necessary and sufficient condition for solution existence and uniqueness of the non-homogeneous IBVP (2.1).

Let us define the Green's function of IBVP (2.1). Let w_a be function w_C in (5.2) that satisfies condition (5.3a). Similarly, let w_b satisfy (5.3b). Then $l_a(w_a) = l_b(w_b) = 0$ and both w_a and w_b satisfy the homogeneous interface conditions $l_i(w_a) = l_i(w_b) = 0$, $i = 1, 2$. Let W be the Wronskian of w_a and w_b. Then, for $(x,s) \in [a,b]^2$,

$$G(x,s) = \frac{1}{W(s)} \begin{cases} w_a(s) w_b(x), & s \le x, \\ w_a(x) w_b(s), & x \le s, \end{cases} \tag{5.5}$$

is the Green's function of problem (2.1).

By definition (5.5), it follows that function G is continuous in $[a,b]^2$ everywhere except the lines $x = c$ and $s = c$, where it may have finite discontinuities. For a fixed $s \in [a,b]$, $G(x,s)$ satisfies the homogeneous equations in (3.5) almost everywhere on (a,b). The derivative $\partial G/\partial x$ has, in addition, the jump discontinuity across the line $x = s$. The domain of the Green's function is shown in Figure 1.

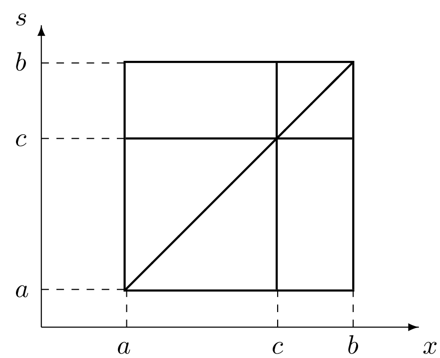

FIGURE 1. The domain of the Green's function (one interface point)

As shown in Figure 1, the lines $x = c$, $s = c$, and $x = s$ divide the square $[a,b]^2$ into 4 triangular and 2 rectangular closed regions. Let $\mathcal{D}$ be the set consisting of these 6 regions. For each integer $k \geq 0$, function G is in C^{k+2} on every region in $\mathcal{D}$ provided that the coefficients p and q of the differential operator L are in Q^k.

Conclusions. Uniqueness and existence of piecewise smooth solutions of the linear two-point BVP with general interface conditions can be established by verifying simple sufficient conditions that only involve coefficients of the boundary and interface conditions and the differential equation. IBVPs with perfect contact or radiation type interface conditions have unique solutions under assumptions on the coefficients of boundary conditions identical to those for the corresponding two-point BVP. IBVP with the flux jump interface condition has a unique solution under weaker assumptions. The interface problem has the Green's function with regularity properties similar to those of the standard two-point BVP.

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