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WELL-POSEDNESS OF DISCONTINUOUS BOUNDARY-VALUE PROBLEMS FOR NONLINEAR ELLIPTIC COMPLEX EQUATIONS IN MULTIPLY CONNECTED DOMAINS

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Abstract. In the first part of this article, we study a discontinuous Riemann-Hilbert problem for nonlinear uniformly elliptic complex equations of first order in multiply connected domains. First we show its well-posedness. Then we give the representation of solutions for a modified Riemann-Hilbert problem for the complex equations. Then we obtain a priori estimates of the solutions and verify the solvability of the modified problem by using the Leray-Schauder theorem. Then the solvability of the original discontinuous Riemann-Hilbert boundary-value problem is obtained. In the second part, we study a discontinuous Poincaré boundary-value problem for nonlinear elliptic equations of second order in multiply connected domains. First we formulate the boundary-value problem and show its new well-posedness. Next we obtain the representation of solutions and obtain a priori estimates for the solutions of a modified Poincaré problem. Then with estimates and the method of parameter extension, we obtain the solvability of the discontinuous Poincaré problem.

1. Formulation of discontinuous Riemann-Hilbert problem

Lavrent′ev and Shabat [2] introduced the Keldych-Sedov formula for analytic functions in the upper half-plane, namely the representation of solutions of the mixed boundary-value problem for analytic functions, which is a special case of discontinuous boundary value problems with the integer index. The authors also pointed out that this formula has very important applications. However, for many problems in mechanics and physics, for instance some free boundary problems and the Tricomi problem for some mixed equations [1, 5, 6, 7, 8, 11, 12, 13, 14], one needs to apply more general discontinuous boundary-value problems of analytic functions and some elliptic equations in the simply and multiply connected domains. In [5] the author solved the general discontinuous Riemann-Hilbert problems for analytic functions in simply connected domains, but the general discontinuous boundary-value problems for elliptic equations in multiply connected domains have not been solved completely. In this article, we study the general discontinuous Riemann-Hilbert problem and discontinuous Poincaré problem and their new well-posedness for nonlinear elliptic equations in multiply connected domains.

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We study the nonlinear elliptic equations of first order

$$w_{\bar z} = F(z, w, w_z), \quad F = Q_1 w_z + Q_2 \overline{w}_{\bar z} + A_1 w + A_2 \overline{w} + A_3, \quad z \in D, \tag{1.1}$$

where $z = x + iy$, $w_{\bar z} = [w_x + iw_y]/2$, $Q_j = Q_j(z, w, w_z)$, $j = 1, 2$, $A_j = A_j(z, w)$, $j = 1, 2, 3$ and assume that equation (1.1) satisfy the following conditions:

(C1) $Q_j(z, w, U)$, $A_j(z, w)$ $(j = 1, 2, 3)$ are measurable in $z \in D$ for all continuous functions $w(z)$ in $D^* = \overline{D}\backslash Z$ and all measurable functions $U(z) \in L_{p_0}(D^*)$, and satisfy

$$L_p[A_j, \overline{D}] \le k_0, \quad j = 1, 2, \ L_p[A_3, \overline{D}] \le k_1, \tag{1.2}$$

where $Z = \{t_1, \dots, t_m\}$, $t_1, \dots, t_m$ are different points on the boundary $\partial D = \Gamma$ arranged according to the positive direction successively, and p, p_0, k_0, k_1 are non-negative constants, $2 < p_0 \le p$.

(C2) The above functions are continuous in $w \in \mathbb{C}$ for almost every point $z \in D$, $U \in \mathbb{C}$. and $Q_j = 0$ $(j = 1, 2)$, $A_j = 0$ $(j = 1, 2, 3)$ for $z \in \mathbb{C}\backslash D$.

(C3) The complex equation (1.1) satisfies the uniform ellipticity condition

$$|F(z, w, U_1) - F(z, w, U_2)| \le q_0 |U_1 - U_2|, \tag{1.3}$$

for almost every point $z \in D$, in which $w, U_1, U_2 \in \mathbb{C}$ and q_0 is a non-negative constant, $q_0 < 1$.

Let $N \ge 1$ and let D be an $N+1$-connected bounded domain in $\mathbb{C}$ with the boundary $\partial D = \Gamma = \cup_{j=0}^N \Gamma_j \in C^1_\mu$ $(0 < \mu < 1)$. Without loss of generality, we assume that D is a circular domain in $|z| < 1$, bounded by the $(N+1)$-circles $\Gamma_j : |z - z_j| = r_j$, $j = 0, 1, \dots, N$ and $\Gamma_0 = \Gamma_{N+1} : |z| = 1$, $z = 0 \in D$. In this article, we use the same notation as in references [1, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. Now we formulate the general discontinuous Riemann-Hilbert problem for equation (1.1) as follows.

Problem A. The general discontinuous Riemann-Hilbert problem for (1.1) is to find a continuous solution $w(z)$ in D^* satisfying the boundary condition:

$$\mathrm{Re}[\overline{\lambda(z)} w(z)] = c(z), \ z \in \mathit{\Gamma}^* = \mathit{\Gamma}\backslash Z, \tag{1.4}$$

where $\lambda(z), c(z)$ satisfy the conditions

$$C_\alpha[\lambda(z), \hat{\mathit{\Gamma}}_j] \le k_0, \quad C_\alpha[|z - t_{j-1}|^{\beta_{j-1}} |z - t_j|^{\beta_j} c(z), \hat{\mathit{\Gamma}}_j] \le k_2, \quad j = 1, \dots, m, \tag{1.5}$$

in which $\lambda(z) = a(z) + ib(z)$, $|\lambda(z)| = 1$ on Γ, and $Z = \{t_1, \dots, t_m\}$ are the first kind of discontinuous points of $\lambda(z)$ on Γ, $\hat{\Gamma}_j$ is an arc from the point t_{j-1} to t_j on Γ, and does not include the end point t_j $(j = 1, 2, \dots, m)$, we can assume that $t_j \in \Gamma_0$ $(j = 1, \dots, m_0)$, $t_j \in \Gamma_1$ $(j = m_0 + 1, \dots, m_1)$, $\dots$, $t_j \in \Gamma_N$ $(j = m_{N-1} + 1 \dots, m)$ are all discontinuous points of $\lambda(z)$ on Γ; If $\lambda(z)$ on Γ_l $(0 \le l \le N)$ has no discontinuous point, then we can choose a point $t_j \in \Gamma_l$ $(0 \le l \le N)$ as a discontinuous point of $\lambda(z)$ on Γ_l $(0 \le l \le N)$, in this case $t_j = t_{j+1}$; $\alpha (1/2 < \alpha < 1)$, k_0, k_2, $\beta_j (0 < \beta_j < 1)$ are positive constants and satisfy the conditions

$$\beta_j + |\gamma_j| < 1, \quad j = 1, \dots, m,$$

where γ_j $(j = 1, \dots, m)$ are as stated in (1.6) below.

Denote by $\lambda(t_j-0)$ and $\lambda(t_j+0)$ the left limit and right limit of $\lambda(t)$ as $t\to t_j$ $(j=1,2,\dots,m)$ on Γ, and

$$\begin{gathered} e^{i\phi_j}=\frac{\lambda(t_j-0)}{\lambda(t_j+0)},\quad \gamma_j=\frac{1}{\pi i}\ln\frac{\lambda(t_j-0)}{\lambda(t_j+0)}=\frac{\phi_j}{\pi}-K_j,\\ K_j=\big[\frac{\phi_j}{\pi}\big]+J_j,\quad J_j=0 \text{ or } 1,\quad j=1,\dots,m, \end{gathered}\tag{1.6}$$

in which $0\le\gamma_j<1$ when $J_j=0$, and $-1<\gamma_j<0$ when $J_j=1$, $j=1,\dots,m$. The index K of Problem A is defined as

$$K=\frac{1}{2}(K_1+\dots+K_m)=\sum_{j=1}^{m}\big[\frac{\phi_j}{2\pi}-\frac{\gamma_j}{2}\big].$$

If $\lambda(t)$ on Γ is continuous, then $K=\Delta_\Gamma\arg\lambda(t)/2\pi$ is a unique integer. Now the function $\lambda(t)$ on Γ is not continuous, we can choose $J_j=0$ or $1\,(j=1,\dots,m)$, hence the index K is not unique. Later on there is no harm in assuming that the partial indexes K_l of $\lambda(z)$ on Γ_l $(l=1,\dots,N_0\le N)$ are not integers, and the partial indexes K_l of $\lambda(z)$ on Γ_l $(j=0,N_0+1,\dots,N)$ are integers; (if K_0 of $\lambda(z)$ on Γ_0 is not integer, then we can similarly discuss). We can require that the solution $w(z)$ possesses the property

$$\begin{gathered} R(z)w(z)\in C_\delta(\overline{D}),\quad R(z)=\prod_{j=1}^{m}|z-t_j|^{\eta_j/\tau^2},\\ \eta_j=\begin{cases}\beta_j+\tau, & \text{if } \gamma_j\ge 0,\ \gamma_j<0,\ \beta_j>|\gamma_j|,\\ |\gamma_j|+\tau, & \text{if } \gamma_j<0,\ \beta_j\le|\gamma_j|,\end{cases} \end{gathered}\tag{1.7}$$

in which γ_j $(j=1,\dots,m)$ are real constants as stated in (1.6), $\tau\le\min(\alpha,1-2/p_0)$ and $\delta<\min(\beta_1,\dots,\beta_m,\tau)$ are small positive constants.

When the index $K<0$, Problem A may not be solvable, when $K\ge0$, the solution of Problem A is not necessarily unique. Hence we put forward a new concept of well-posedness of Problem A with modified boundary conditions as follows.

Problem B. Find a continuous solution $w(z)$ of the complex equation (1.1) in D^* satisfying the boundary condition

$$\operatorname{Re}[\overline{\lambda(z)}w(z)]=r(z)+h(z)\overline{\lambda(z)}X(z),\quad z\in\Gamma^*,\tag{1.8}$$

where $X(z)$ is as stated in (1.9) below, and

$$h(z)=\begin{cases}0, & z\in\Gamma_0,\ K\ge0\\ h_j, & z\in\Gamma_j,\ j=1,\dots,N,\ K\ge0\\ h_j, & z\in\Gamma_j,\ j=1,\dots,N,\ K<0\\ [1+(-1)^{2K}]h_0 & \\ +\operatorname{Re}\sum_{m=1}^{[|K|+1/2]-1}(h_m^+ + ih_m^-)z^m, & z\in\Gamma_0,\ K<0\end{cases}$$

in which h_j $(j=[1-(-1)^{2K}]/2,\dots,N)$, h_m^+,h_m^-, $(m=1,\dots,[|K|+1/2]-1)$ are unknown real constants to be determined appropriately, and $h_{N+1}(=h_0)=0$, if $2|K|$ is an odd integer; and

$$Y(z)=\prod_{j=1}^{m_0}(z-t_j)^{\gamma_j}\prod_{l=l}^{N}(z-z_l)^{-[\tilde K_l]}\prod_{j=m_0+1}^{m_1}\Big(\frac{z-t_j}{z-z_1}\Big)^{\gamma_j}\Big(\frac{z-t_1'}{z-z_1}\Big)$$

$$\times \prod_{j=m_{N_0-1}+1}^{m_{N_0}} \Big(\frac{z-t_j}{z-z_{N_0}}\Big)^{\gamma_j} \Big(\frac{z-t'_{N_0}}{z-z_{N_0}}\Big) \prod_{j=m_{N_0}+1}^{m_{N_0+1}} \Big(\frac{z-t_j}{z-z_{N_0+1}}\Big)^{\gamma_j} \dots$$
$$\times \prod_{j=m_{N-1}+1}^{m} \Big(\frac{z-t_j}{z-z_N}\Big)^{\gamma_j},$$

where $\tilde{K}_l = \sum_{j=m_{l-1}+1}^{m_l} K_j$ denote the partial index on Γ_l ($l = 1,\dots,N$), t'_l ($\in \Gamma_l$, $l =,\dots,N_0$) are fixed points, which are not the discontinuous points at Z; we must give the attention that the boundary circles Γ_j ($j = 0,1,\dots,N$) of the domain D are moved round the positive direct. Similarly to [5, (1.7)–(1.12) Chapter V], we see that

$$\frac{\lambda(t_j-0)}{\lambda(t_j+0)}\overline{\Big[\frac{Y(t_j-0)}{Y(t_j+0)}\Big]} = \frac{\lambda(t_j-0)}{\lambda(t_j+0)} e^{-i\pi\gamma_j} = \pm 1, \quad j = 1,\dots,m,$$

it only needs to charge the symbol on some arcs on Γ, then $\lambda(z)\overline{Y(z)}/|Y(z)|$ on Γ is continuous. In this case, its index

$$\kappa = \frac{1}{2\pi}\Delta_\Gamma[\lambda(z)\overline{Y(z)}] = K - \frac{N_0}{2}$$

is an integer; and

$$\begin{gathered}
X(z) = \begin{cases} iz^{[\kappa]}e^{iS(z)}Y(z), & z\in\Gamma_0,\\ ie^{i\theta_j}e^{iS(z)}Y(z), & z\in\Gamma_j, j=1,\dots,N,\end{cases}\\
\operatorname{Im}[\overline{\lambda(z)}X(z)] = 0, \quad z\in\Gamma,\\
\operatorname{Re} S(z) = S_1(z) - \theta(t),\\
S_1(z) = \begin{cases} \arg\lambda(z) - [\kappa]\arg z - \arg Y(z), & z\in\Gamma_0,\\ \arg\lambda(z) - \arg Y(z), & z\in\Gamma_j,\ j=1,\dots,N,\end{cases}\\
\theta(z) = \begin{cases} 0, & z\in\Gamma_0,\\ \theta_j, & z\in\Gamma_j,\ j=1,\dots,N,\end{cases}\\
\operatorname{Im}[S(1)] = 0,
\end{gathered} \tag{1.9}$$

in which $S(z)$ is a solution of the modified Dirichlet problem with the above boundary condition for analytic functions, θ_j ($j = 1,\dots,N$) are real constants, and $\kappa = K - N_0/2$.

In addition, we may assume that the solution $w(z)$ satisfies the following point conditions

$$\operatorname{Im}[\overline{\lambda(a_j)}w(a_j)] = b_j, \quad j\in J = \{1,\dots,2K+1\}, \quad \text{if } K\ge 0, \tag{1.10}$$

where $a_j \in \Gamma_0\,(j\in J)$ are distinct points; and $b_j (j \in J)$ are all real constants satisfying the conditions

$$|b_j| \le k_3, \quad j\in J$$

with the positive constant k_3. Problem B with $A_3(z,w) = 0$ in D, $c(z) = 0$ on Γ and $b_j = 0$ ($j \in J$) is called Problem B_0.

We mention that the undetermined real constants $h_j, h_m^{\pm}$ in (1.8) are for ensuring the existence of continuous solutions, and the point conditions in (1.10) are for ensuring the uniqueness of continuous solutions in D. The condition $0 < K < N$

is called the singular case, which only occurs in the case of multiply connected domains, and is not easy handled.

Now we introduce the previous well-posedness of the discontinuous Riemann-Hilbert problem of elliptic complex equations, which are we always use here.

Problem C. Find a continuous solution $w(z)$ in D of (1.1) with the modified boundary condition (1.8), where

$$h(z)=\begin{cases}0, & z\in\Gamma,\ K>N-1,\\ h_j, & z\in\Gamma_j,\ j=1,\dots,N-K',\ 0\le K\le N-1\\ 0, & z\in\Gamma_j,\ j=N-K'+1,\dots,N-K'+[K]+1,\\ & \quad 0\le K\le N-1,\\ h_j, & z\in\Gamma_j,\ j=1,\dots,N,\ K<0,\\ [1+(-1)^{2K}]h_0 & \\ +\operatorname{Re}\sum_{m=1}^{[|K|+1/2]-1}(h_m^+ & \\ +ih_m^-)z^m, & z\in\Gamma_0,\ K<0\,.\end{cases} \tag{1.11}$$

in which $K'=[K+1/2]$, $[K]$ denotes the integer part of K, $h_0,h_m^+,h_m^-\,(m=1,\dots,[|K|+1/2]-1)$ are unknown real constants to be determined appropriately, and $h_{N+1}(=h_0)=0$, if $2|K|$ is an odd integer; and the solution $w(z)$ satisfies the point conditions

$$\operatorname{Im}[\overline{\lambda(a_j)}w(a_j)]=b_j,\quad j\in J=\begin{cases}1,\dots,2K-N+1, & \text{if } K>N-1,\\ 1,\dots,[K]+1, & \text{if } 0\le K\le N-1,\end{cases} \tag{1.12}$$

in which $a_j\in\Gamma_{j+N_0}$ $(j=1,\dots,N-N_0)$, $a_j\in\Gamma_0$ $(j=N-N_0+1,\dots,2K-N+1$, if $K\ge N)$ are distinct points; and when $[K]+1\le N-N_0$, a_j $(\in\Gamma_{j+N-[K]-1}$, $j=1,\dots,[K]+1)$, otherwise a_j $(\in\Gamma_{j+N-N_0}$, $j=1,\dots,N_0)$, and a_j $(\in\Gamma_0$, $j=N_0+1,\dots,[K]+1)$ are distinct points, and

$$|b_j|\le k_3,\quad j\in J$$

with a non-negative constant k_3.

We can prove the equivalence of Problem B and Problem C for for equation (1.1). From this, we see that the advantages of the new well-posedness are as follows:

(1) The statement of the new well-posedness is simpler than others (see [5, 6, 13]).
(2) The point conditions in $\Gamma_0=\{|z|=1\}$ are similar to those for the simple connected domain $D=\{|z|<1\}$.
(3) The new well-posedness statement does not distinguished the singular case $0<K<N$ and non-singular case $K\ge N$.

We mention the equivalence of these well-posedness statements; i.e. if there exists the unique solvability under one well-posedness statement, then we can derive the unique solvability under under the other well-posedness. Hence it is best to choose the simplest well-posedness statement.

To prove the solvability of Problem B for the complex equation (1.1), we need to give a representation theorem.

Theorem 1.1. *Suppose that the complex equation* (1.1) *satisfies conditions* (C1)–(C3)*, and $w(z)$ is a solution of Problem* B *for* (1.1)*. Then $w(z)$ is representable* *as*

$$w(z) = [\Phi(\zeta(z)) + \psi(z)]e^{\phi(z)}, \tag{1.13}$$

where $\zeta(z)$ *is a homeomorphism in* $\overline{D}$, *which maps quasi-conformally* D *onto the* $N+1$-*connected circular domain* G *with boundary* $L = \zeta(\Gamma)$ *in* $\{|\zeta| < 1\}$, *such that: three points on* Γ *are mapped into three points on* L *respectively;* $\Phi(\zeta)$ *is an analytic function in* G; $\psi(z), \phi(z), \zeta(z)$ *and its inverse function* $z(\zeta)$ *satisfy the estimates*

$$C_\beta[\psi, \overline{D}] \le k_4, \quad C_\beta[\phi, \overline{D}] \le k_4, \quad C_\beta[\zeta(z), \overline{D}] \le k_4, \tag{1.14}$$

$$L_{p_0}[|\psi_{\bar z}| + |\psi_z|, \quad \overline{D}] \le k_4,\ L_{p_0}[|\phi_{\overline{z}}| + |\phi_z|, \overline{D}] \le k_4, \tag{1.15}$$

$$C_\beta[z(\zeta), \overline{G}] \le k_4, \quad L_{p_0}[|\chi_{\bar z}| + |\chi_z|, \overline{D}] \le k_5, \tag{1.16}$$

in which $\chi(z)$ *is as stated in* (1.20) *below,* $\beta = \min(\alpha, 1 - 2/p_0)$, p_0 $(2 < p_0 \le p)$, $k_j = k_j(q_0, p_0, \beta, k_0,\ k_1, D)$ $(j = 4, 5)$ *are non-negative constants depending on* $q_0, p_0, \beta, k_0, k_1, D$. *Moreover, the function* $\Phi[\zeta(z)]$ *satisfies the estimate*

$$C_\delta[R(z)\Phi[\zeta(z)], \overline{D}] \le M_1 = M_1(q_0, p_0, \beta, k, D) < \infty, \tag{1.17}$$

in which $R(z)$, γ_j $(j = 1, \dots, m)$ *are as stated in* (1.7) *and* $\tau \le \min(\alpha, 1 - 2/p_0)$, $\delta < \min(\beta_1, \dots, \beta_m, \tau)$ *are small positive constants,* $k = k(k_0, k_1, k_2, k_3)$, *and* M_1 *is a non-negative constant dependent on* q_0, p_0, β, k, D.

Proof. We substitute the solution $w(z)$ of Problem B into the coefficients of equation (1.1) and consider the system

$$\begin{gathered}
\phi_{\bar z} = Q\phi_z + A, \quad A = \begin{cases} A_1 + A_2\overline{w}/w & \text{for } w(z) \ne 0, \\ 0 & \text{for } w(z) = 0 \text{ or } z \notin D, \end{cases} \\
\psi_{\bar z} = Q\psi_z + A_3 e^{-\phi(z)}, \quad Q = \begin{cases} Q_1 + Q_2\overline{w_z}/w_z & \text{for } w_z \ne 0, \\ 0 & \text{for } w_z = 0 \text{ or } z \notin D, \end{cases} \\
W_{\bar z} = QW_z, \quad W(z) = \Phi[\zeta(z)] \quad \text{in } D.
\end{gathered} \tag{1.18}$$

By using the continuity method and the principle of contracting mapping, we can find the solution

$$\begin{gathered}
\psi(z) = T_0 f = -\frac{1}{\pi}\int\int_D \frac{f(\zeta)}{\zeta - z} d\sigma_\zeta, \\
\phi(z) = T_0 g,\ \zeta(z) = \Psi[\chi(z)], \quad \chi(z) = z + T_0 h
\end{gathered} \tag{1.19}$$

of (1.18), in which $f(z), g(z), h(z) \in L_{p_0}(\overline{D})$, $2 < p_0 \le p$, $\chi(z)$ is a homeomorphic solution of the third equation in (1.18), $\Psi(\chi)$ is a univalent analytic function, which conformally maps $E = \chi(D)$ onto the domain G (see [3, 6]), and $\Phi(\zeta)$ is an analytic function in G. We can verify that $\psi(z), \phi(z), \zeta(z)$ satisfy the estimates (1.14) and (1.15). It remains to prove that $z = z(\zeta)$ satisfies the estimate in (1.16). In fact, we can find a homeomorphic solution of the last equation in (1.18) in the form $\chi(z) = z + T_0 h$ such that $[\chi(z)]_z, [\chi(z)]_{\bar z} \in L_{p_0}(\bar D)$ (see [3]). Next, we find a univalent analytic function $\zeta = \Psi(\chi)$, which maps $\chi(D)$ onto G, hence $\zeta = \zeta(z) = \Psi[\chi(z)]$. By the result on conformal mappings, applying the method of [6, Theorem 1.1,Chapter III] or [13, Theorem 1.1.1, Chapter I], we can prove that (1.16) is true. It is easy to see that the function $\Phi[\zeta(z)]$ satisfies the boundary conditions

$$\mathrm{Re}[\overline{\lambda(z)}e^{\phi(z)}\Phi(\zeta(z))] = c(z) + h(z)\overline{\lambda(z)}X(z) - \mathrm{Re}[\overline{\lambda(z)}e^{\phi(z)}\psi(z)], \quad z \in \Gamma^*. \tag{1.20}$$

On the basis of the estimates (1.14) and (1.16), and use the methods of [13, Theorem 1.1.1, Chapter I], we can prove that $\Phi[\zeta(z)]$ satisfies the estimate (1.17). $\square$

2. Estimates for discontinuous Riemann-Hilbert problems

Now, we derive a priori estimates of solutions for Problem B for the complex equation (1.1).

Theorem 2.1. *Under the same conditions as in Theorem 1.1, any solution $w(z)$ of Problem* B *for* (1.1) *satisfies the estimates*

$$\hat{C}_\delta[w(z),\overline{D}] = C_\delta[R(z)w(z),\overline{D}] \le M_1 = M_1(q_0,p_0,\delta,k,D), \tag{2.1}$$

$$\hat{L}^1_{p_0}[w(z),\overline{D}] = L_{p_0}[|RSw_{\bar z}| + |RSw_z|,\overline{D}] \le M_2 = M_2(q_0,p_0,\delta,k,D), \tag{2.2}$$

where $S(z)=\prod_{j=1}^m |z-t_j|^{1/\tau^2}$, $k=k(k_0,k_1,k_2,k_3)$, $\delta<\min(\beta_1,\dots,\beta_m,\tau)$, p_0,p, $(2<p_0\le p)$, M_j $(j=1,2)$ are positive constant only depending on q_0,p_0,δ,k,D.

Proof. On the basis of Theorem 1.1, the solution $w(z)$ of Problem B can be expressed the formula as in (1.13), hence the boundary value problem B can be transformed into the boundary value problem (Problem $\tilde{\mathrm{B}}$) for analytic functions

$$\operatorname{Re}[\overline{\Lambda(\zeta)}\Phi(\zeta)] = \hat{r}(\zeta) + H(\zeta)\overline{\lambda(z(\zeta))}X[z(\zeta)], \quad \zeta\in L^*=\zeta(\Gamma^*), \tag{2.3}$$

$$H(\zeta)=\begin{cases} 0, & \zeta\in L_0,\ K\ge 0,\\ h_j, & \zeta\in L_j,\ j=1,\dots,N,\ K\ge 0,\\ h_j, & \zeta\in L_j,\ j=1,\dots,N,\ K<0,\\ [1+(-1)^{2K}]h_0 & \\ +\operatorname{Re}\sum_{m=1}^{[|K|+1/2]-1}(h_m^+ + ih_m^-)\zeta^m, & \zeta\in L_0,\ K<0, \end{cases} \tag{2.4}$$

$$\operatorname{Im}[\overline{\Lambda(a_j')}\Phi(a_j')] = b_j', \quad j\in J, \tag{2.5}$$

where

$$\overline{\Lambda(\zeta)} = \overline{\lambda[z(\zeta)]}e^{\phi[z(\zeta)]}, \quad \hat{r}(\zeta) = r[z(\zeta)] - \operatorname{Re}\{\overline{\lambda[z(\zeta)]}\psi[z(\zeta)]\},$$
$$a_j'=\zeta(a_j), \quad b_j' = b_j - \operatorname{Im}[\overline{\lambda(a_j)}e^{\phi(a_j)}\psi(a_j)], \quad j\in J.$$

By (1.5) and (1.14)–(1.16), it can be seen that $\Lambda(\zeta)$, $\hat{r}(\zeta)$, b_j' $(j\in J)$ satisfy the conditions

$$C_{\alpha\beta}[R[z(\zeta)]\Lambda(\zeta),L]\le M_3, \quad C_{\alpha\beta}[R[z(\zeta)]\hat{r}(\zeta),L]\le M_3, \quad |b_j'|\le M_3,\ j\in J, \tag{2.6}$$

where $M_3=M_3(q_0,p_0,\beta,k,D)$. If we can prove that the solution $\Phi(\zeta)$ of Problem $\tilde{\mathrm{B}}$ satisfies the estimates

$$C_{\delta\beta}[R(z(\zeta))\Phi(\zeta),\overline{G}]\le M_4, \quad C[R(z(\zeta))S(z(\zeta))\Phi'(\zeta),\tilde{G}]\le M_5, \tag{2.7}$$

where β is the constant as defined in (1.14), $\tilde{G}=\zeta(\tilde{D})$, $M_j=M_j(q_0,p_0,\delta,k,D)$, $j=4,5$, then from the representation (1.13) of the solution $w(z)$ and the estimates (1.14)-(1.16) and (2.7), the estimates (2.1) and (2.2) can be derived.

It remains to prove that (2.7) holds. For this, we first verify the boundedness of $\Phi(\zeta)$; i.e.,

$$C[R(z(\zeta))\Phi(\zeta),\overline{G}]\le M_6 = M_6(q_0,p_0,\beta,k,D). \tag{2.8}$$

Suppose that (2.8) is not true. Then there exist sequences of functions $\{\Lambda_n(\zeta)\}$, $\{\hat{r}_n(\zeta)\}$, $\{b_{jn}'\}$ satisfying the same conditions as $\Lambda(\zeta)$, $\hat{r}(\zeta)$, b_j', which converge uniformly to $\Lambda_0(\zeta)$, $\hat{r}_0(\zeta)$, b_{j0}' $(j\in J)$ on L respectively. For the solution $\Phi_n(\zeta)$ of the boundary value problem (Problem $\mathrm{B_n}$) corresponding to $\Lambda_n(\zeta)$, $\hat{r}_n(\zeta)$, b_{jn}' $(j\in J)$, we have $I_n=C[R(z(\zeta))\Phi_n(\zeta),\overline{G}]\to\infty$ as $n\to\infty$. There is no harm in assuming that $I_n \geq 1$, $n = 1, 2, \dots$. Obviously $\tilde{\Phi}_n(\zeta) = \Phi_n(\zeta)/I_n$ satisfies the boundary conditions

$$\mathrm{Re}[\overline{\Lambda_n(\zeta)}\tilde{\Phi}_n(\zeta)] = [\hat{r}_n(\zeta) + H(\zeta)\overline{\lambda(z(\zeta))}X[z(\zeta)]]/I_n, \quad \zeta \in L^*, \tag{2.9}$$

$$\mathrm{Im}[\overline{\Lambda_n(a'_n)}\tilde{\Phi}_n(a'_n)] = b'_{jn}/I_n, \quad j \in J. \tag{2.10}$$

Applying the Schwarz formula, the Cauchy formula and the method of symmetric extension (see [5, Theorem 4.3, Chapter IV]), the estimates

$$C_{\delta\beta}[R(z(\zeta))\tilde{\Phi}_n(\zeta), \overline{G}] \leq M_7, \quad C[R(z(\zeta))S(z(\zeta))\tilde{\Phi}'_n(\zeta), \overline{G}] \leq M_8, \tag{2.11}$$

can be obtained, where $\tilde{G} = \zeta(\tilde{D})$, and $M_j = M_j(q_0, p_0, \delta, k, D)$, $j = 7, 8$. Thus we can select a subsequence of $\{\tilde{\Phi}_n(\zeta)\}$, which converge uniformly to an analytic function $\tilde{\Phi}_0(\zeta)$ in G, and $\tilde{\Phi}_0(\zeta)$ satisfies the homogeneous boundary conditions

$$\mathrm{Re}[\overline{\Lambda_0(\zeta)}\tilde{\Phi}_0(\zeta)] = H(\zeta)\overline{\lambda(z(\zeta))}X[z(\zeta)], \quad \zeta \in L^*, \tag{2.12}$$

$$\mathrm{Im}[\overline{\Lambda_0(a'_j)}\tilde{\Phi}_0(a'_j)] = 0, \quad j \in J. \tag{2.13}$$

On the basis of the uniqueness theorem (see [5, Theorems 3.2–3.4, Chapter IV]), we conclude that $\tilde{\Phi}_0(\zeta) = 0$, $\zeta \in \bar{G}$. However, from $C[R(z(\zeta))\tilde{\Phi}_n(\zeta), \bar{G}] = 1$, it follows that there exists a point $\zeta_* \in \overline{G}$, such that $|R(z(\zeta_*))\tilde{\Phi}_0(\zeta_*)| = 1$. This contradiction proves that (2.8) holds. Afterwards using the method which leads from (2.8) to (2.11), the estimate (2.7) can be derived. □

Theorem 2.2. *Under the same conditions as in Theorem 2.1, any solution $w(z)$ of Problem* B *for* (1.1) *satisfies*

$$\begin{gathered} \hat{C}_\delta[w(z), \overline{D}] = C_\delta[R(z)w(z), \overline{D}] \leq M_9 k_*, \\ \hat{L}^1_{p_0}[w, \overline{D}] = L_{p_0}[|RSw_{\bar{z}}| + |RSw_z|, \overline{D}] \leq M_{10} k_*, \end{gathered} \tag{2.14}$$

where δ, p_0 are as stated in Theorem 2.1, $k_* = k_1 + k_2 + k_3$, $M_j = M_j(q_0, p_0, \delta, k_0, D)$ $(j = 9, 10)$.

Proof. If $k_* = 0$, i.e. $k_1 = k_2 = k_3 = 0$, from Theorem 2.3 below, it follows that $w(z) = 0$, $z \in D$. If $k_* > 0$, it is easy to see that $W(z) = w(z)/k_*$ satisfies the complex equation and boundary conditions

$$W_{\bar{z}} - Q_1 W_z - Q_2 \overline{W_z} - A_1 W - A_2 \overline{W} = A_3/k_*, \quad z \in D, \tag{2.15}$$

$$\mathrm{Re}[\overline{\lambda(z)}W(z)] = [r(z) + h(z)\overline{\lambda(z)}X(z)]/k_*, \quad z \in \Gamma^*, \tag{2.16}$$

$$\mathrm{Im}[\overline{\lambda(a_j)}W(a_j)] = b_j/k_*, \quad j \in J, \tag{2.17}$$

Noting that $L_p[A_3/k_*, \overline{D}] \leq 1$, $C_\alpha[R(z)r(z)/k_*, \Gamma] \leq 1$, $|b_j/k_*| \leq 1$, $j \in J$ and according to the proof of Theorem 2.1, we have

$$\hat{C}_\delta[W(z), \overline{D}] \leq M_9, \quad \hat{L}^1_{p_0}[W(z), \overline{D}] \leq M_{10}. \tag{2.18}$$

From the above estimates, it follows that (2.14) holds. □

Next, we prove the uniqueness of solutions of Problem B for the complex equation (1.1). For this, we need to add the following condition: For any continuous functions $w_1(z), w_2(z)$ in D^* and $U(z)$ $(R(z)S(z)U(z) \in L_{p_0}(\overline{D})$, there is

$$F(z, w_1, U) - F(z, w_2, U) = Q(z, w_1, w_2, U)U_z + A(z, w_1, w_2, U)(w_1 - w_2), \tag{2.19}$$

in which $|Q(z, w_1, w_2, U)| \leq q_0 (< 1)$, $A(z, w_1, w_2, U) \in L_{p_0}(\overline{D})$. When (1.1) is linear, (2.19) obviously holds.

Theorem 2.3. *If Condition* C1–C3 *and* (2.19) *hold, then the solution of Problem* B *for* (1.1) *is unique.*

Proof. Let $w_1(z), w_2(z)$ be two solutions of Problem B for (1.1). By Condition (C1)–(C3) and (2.19), we see that $w(z) = w_1(z) - w_2(z)$ is a solution of the boundary value problem

$$w_{\bar z} - \tilde{Q} w_z = \tilde{A} w, \quad z \in D, \tag{2.20}$$

$$\operatorname{Re}[\overline{\lambda(z)} w(z)] = h(z)\overline{\lambda(z)} X(z), \quad z \in \Gamma^*, \tag{2.21}$$

$$\operatorname{Im}[\overline{\lambda(a_j)} w(a_j)] = 0, \quad j \in J, \tag{2.22}$$

where

$$\tilde{Q} = \begin{cases} [F(z, w_1, w_{1z}) - F(z, w_1, w_{2z})]/(w_1 - w_2)_z & \text{for } w_{1z} \neq w_{2z}, \\ 0 & \text{for } w_{1z} = w_{2z},\ z \in D, \end{cases}$$

$$\tilde{A} = \begin{cases} [F(z, w_1, w_{2z}) - F(z, w_2, w_{2z})]/(w_1 - w_2) & \text{for } w_1(z) \neq w_2(z), \\ 0 & \text{for } w_1(z) = w_2(z),\ z \in D, \end{cases}$$

and $|\tilde{Q}| \le q_0 < 1$, $z \in D$, $L_{p_0}(\tilde{A}, \overline{D}) < \infty$. According to the representation (1.13), we have

$$w(z) = \Phi[\zeta(z)] e^{\phi(z)}, \tag{2.23}$$

where $\phi(z), \zeta(z), \Phi(\zeta)$ are as stated in Theorem 2.1. It can be seen that the analytic function $\Phi(z)$ satisfies the boundary conditions of Problem B_0:

$$\operatorname{Re}[\overline{\Lambda(\zeta)} \Phi(\zeta)] = H(\zeta) \overline{\lambda[z(\zeta)]} X[z(\zeta)], \quad \zeta \in L^* = \zeta(\Gamma^*), \tag{2.24}$$

$$\operatorname{Im}[\overline{\Lambda(a_j')} \Phi(a_j')] = 0, \quad j \in J, \tag{2.25}$$

where $\Lambda(\zeta)$, $H(\zeta)$ ($\zeta \in L$), a_j' ($j \in J$) are as stated in (2.3)–(2.5). According to the method in the proof of [13, Theorem 1.2.4], we can derive that $\Phi(\zeta) = 0$, $\zeta \in G = \zeta(D)$. Hence, $w(z) = \Phi[\zeta(z)] e^{\phi(z)} = 0$; i.e., $w_1(z) = w_2(z)$, $z \in D$. □

3. Solvability of discontinuous Riemann-Hilbert problems

Now we prove the existence of solutions of Problem B for equation (1.1) by the Leray-Schauder theorem.

Theorem 3.1. *Suppose that* (1.1) *satisfies Conditions* (C1)–(C3) *and* (2.19). *Then the discontinuous boundary value problem, Problem* B*, for* (1.1) *has a solution.*

Proof. We discuss the complex equation (1.1); i.e.,

$$w_{\bar z} = F(z, w, w_z),\ F(z, w, w_z) = Q_1 w_z + Q_2 \overline{w}_{\bar z} + A_1 w + A_2 \overline{w} + A_3 \quad \text{in } D. \tag{3.1}$$

To find a solution $w(z)$ of Problem B for equation (3.1) by the Leray-Schauder theorem, we consider the equation (3.1) with the parameter $t \in [0, 1]$

$$w_{\bar z} = tF(z, w, w_z),\ F(z, w, w_z) = Q_1 w_z + Q_2 \overline{w}_{\bar z} + A_1 w + A_2 \overline{w} + A_3 \quad \text{in } D, \tag{3.2}$$

and introduce a bounded open set B_M of the Banach space $B = \hat{C}(\overline{D}) \cap \hat{L}^1_{p_0}(\overline{D})$, whose elements are functions $w(z)$ satisfying the condition

$$\begin{aligned} w(z) \in \hat{C}(\overline{D}) \cap \hat{L}^1_{p_0}(\overline{D}) &: \hat{C}[w, \overline{D}] + \hat{L}^1_{p_0}[w, \overline{D}] \\ &= C[R(z) w(z), \overline{D}] + L_{p_0}[|RS w_{\bar z}| + |RS w_z|, \overline{D}] < M_{11}, \end{aligned} \tag{3.3}$$

where $M_{11} = 1 + M_1 + M_2$, M_1, M_2, δ are constants as stated in (2.1) and (2.2). We choose an arbitrary function $W(z) \in \overline{B_M}$ and substitute it in the position of w in $F(z, w, w_z)$. By using the method in the proof of [5, Theorem 6.6, Chapter V] and [13, Theorem 1.2.5], a solution $w(z) = \Phi(z) + \Psi(z) = W(z) + T_0(tF)$ of Problem B for the complex equation

$$w_{\overline{\overline{z}}} = tF(z, W, W_z) \tag{3.4}$$

can be found. Noting that $tR(z)S(z)F[z, W(z), W_z] \in L_\infty(\overline{D})$, the above solution of Problem B for (3.4) is unique. Denote by $w(z) = T[W, t]\,(0 \le t \le 1)$ the mapping from $W(z)$ to $w(z)$. From Theorem 2.2, we know that if $w(z)$ is a solution of Problem B for the equation

$$w_{\overline{\overline{z}}} = tF(z, w, w_z) \quad \text{in } D, \tag{3.5}$$

then the function $w(z)$ satisfies the estimate

$$\hat{C}[w, \overline{D})] < M_{11}. \tag{3.6}$$

Set $B_0 = B_M \times [0, 1]$. Now we verify the three conditions of the Leray-Schauder theorem:

(1) For every $t \in [0, 1]$, $T[W, t]$ continuously maps the Banach space B into itself, and is completely continuous in $\overline{B_M}$. In fact, we arbitrarily select a sequence $W_n(z)$ in $\overline{B_M}$, $n = 0, 1, 2, \dots$, such that $\hat{C}[W_n - W_0, \overline{D}] \to 0$ as $n \to \infty$. By Condition C, we see that $L_\infty[RS(F(z, W_n, W_{nz}) - F(z, W_0, W_{0z})), \overline{D}] \to 0$ as $n \to \infty$. Moreover, from $w_n = T[W_n, t]$, $w_0 = T[W_0, t]$, it is easy to see that $w_n - w_0$ is a solution of Problem B for the following complex equation

$$(w_n - w_0)_{\overline{\overline{z}}} = t[F(z, W_n, W_{nz}) - F(z, W_0, W_{0z})] \quad \text{in } D, \tag{3.7}$$

and then we can obtain the estimate

$$\hat{C}[w_n - w_m, \overline{D})] \le 2k_0 \hat{C}[W_n(z) - W_0(z), \overline{D}]. \tag{3.8}$$

Hence $\hat{C}[w_n - w_0, \overline{D}] \to 0$ as $n \to \infty$. In addition for $W_n(z) \in \overline{B_M}$, $n = 1, 2, \dots$, we have $w_n = T[W_n, t]$, $w_m = T[W_m, t]$, $w_n, w_m \in B_M$, and then

$$(w_n - w_m)_{\overline{\overline{z}}} = t[F(z, W_n, W_{nz}) - F(z, W_m, W_{mz})] \quad \text{in } D, \tag{3.9}$$

where $L_\infty[RS(F(z, W_n, W_{nz}) - F(z, W_m, W_{mz})), \overline{D}] \le 2k_0 M_5$. Hence similarly to the proof of Theorem 2.2, we can obtain the estimate

$$\hat{C}[w_n - w_m, \overline{D}] \le 2M_9 k_0 M_{11}. \tag{3.10}$$

Thus there exists a function $w_0(z) \in B_M$, from $\{w_n(z)\}$ we can choose a subsequence $\{w_{n_k}(z)\}$ such that $\hat{C}[w_{n_k} - w_0, \overline{D}] \to 0$ as $k \to \infty$. This shows that $w = T[W, t]$ is completely continuous in $\overline{B_M}$. Similarly we can prove that for $W(z) \in \overline{B_M}$, $T[W, t)$ is uniformly continuous with respect to $t \in [0, 1]$.

(2) For $t = 0$, it is evident that $w = T[W, 0] = \Phi(z) \in B_M$.

(3) From the estimate (2.14), we see that $w = T[W, t]$ $(0 \le t \le 1)$ does not have a solution $w(z)$ on the boundary $\partial B_M = \overline{B_M} \backslash B_M$.

Hence by the Leray-Schauder theorem, we know that there exists a function $w(z) \in B_M$, such that $w(z) = T[w(z), t]$, and the function $w(z) \in \hat{C}_\delta(\overline{D})$ is just a solution of Problem B for the complex equation (1.1). □

Moreover, we can derive the solvability result of Problem A for (1.1) as follows.

Theorem 3.2. *Under the same conditions as in Theorem 3.1, the following statements hold.*

(1) *If the index $K \geq N$, then Problem* A *for* (1.1) *is solvable, if N solvability conditions hold, under these conditions, its general solution includes $2K+1$ arbitrary real constants.*

(2) *If $K < 0$, then Problem* A *for* (1.1) *is solvable under $-2K-1$ solvability conditions.*

Proof. Let the solution $w(z)$ of Problem B for (1.1) be substituted into the boundary condition (1.8). If the function $h(z) = 0$, $z \in \Gamma$; i.e.,

$$\begin{gathered} h_j = 0, \quad j = 1, \dots, N, \quad \text{if } K \geq 0, \\ h_j = 0, \quad j = [1-(-1)^{2K}]/2, , \dots, N, \quad \text{if } K < 0, \\ h_m^{\pm} = 0, \quad m = 1, \dots, [|K| + 1/2] - 1, \quad \text{if } K < 0, \end{gathered}$$

then the function $w(z)$ is just a solution of Problem A for (1.1). Hence the total number of above equalities is just the number of solvability conditions as stated in this theorem. Also note that the real constants $b_j (j \in J)$ in (1.10) are arbitrarily chosen. This shows that the general solution of Problem A for (1.1) includes the number of arbitrary real constants as stated in the theorem. □

The above theorem shows that the general solution of Problem A for (1.1) includes the number of arbitrary real constants as stated in the above theorem. In fact, for the linear case of the complex equation (1.1) satisfying Conditions (C1)–(C3), namely

$$w_{\bar{z}} = Q_1(z) w_z + Q_2(z) \bar{w}_{\bar{z}} + A_1(z) w + A_2(z) \bar{w} + A_3(z) \quad \text{in } D, \tag{3.11}$$

the general solution of Problem A with the index $K \geq 0$ can be written as

$$w(z) = w_0(z) + \sum_{n=1}^{2K+1} d_n w_n(z), \tag{3.12}$$

where $w_0(z)$ is a solution of nonhomogeneous boundary value problem (Problem A), and d_n $(n = 1, \dots, 2K+1)$ are the arbitrary real constants, $w_n(z)$ $(n = 1, \dots, 2K+1)$ are linearly independent solutions of homogeneous boundary value problem (Problem A_0), which can be satisfied the point conditions

$$\operatorname{Im}[\overline{\lambda(a_j)} w_n(a_j)] = \delta_{jn}, \quad j, n = 1, \dots, 2K+1, \ K \geq 0,$$

where $\delta_{jn} = 1$, if $j = n = 1, \dots, 2K+1$ and $\delta_{jn} = 0$, if $j \neq n$, $1 \leq j, n \leq 2K+1$.

4. Formulation of the general discontinuous Poincaré problem

Now we discuss the general discontinuous Poincaré problem for some nonlinear elliptic equations of second order in multiply connected domains and its new well-posedness.

Let D be a bounded $(N+1)$-connected domain point with the boundary $\Gamma = \cup_{j=0}^{N} \Gamma_j$ in $\mathbb{C}$ as stated in Section 1. We consider the nonlinear elliptic equation of second order in the complex form

$$\begin{gathered} u_{z\bar{z}} = F(z, u, u_z, u_{zz}), \quad F = \operatorname{Re}[Q u_{zz} + A_1 u_z] + A_2 u + A_3, \\ Q = Q(z, u, u_z, u_{zz}), \quad A_j = A_j(z, u, u_z), \quad j = 1, 2, 3, \end{gathered} \tag{4.1}$$

satisfying the following conditions.

(C4) $Q(z,u,w,U), A_j(z,u,w)$ $(j=1,2,3)$ are continuous in $u \in \mathbb{R}$, $w \in \mathbb{C}$ for almost every point $z \in D$, $U \in \mathbb{C}$, and $Q=0$, $A_j = 0$ $(j=1,2,3)$ for $z \in \mathbb{C}\backslash D$.

(C5) The above functions are measurable in $z \in D$ for all continuous functions $u(z), w(z)$ in D, and satisfy

$$L_p[A_1(z,u,w),\overline{D}] \le k_0, \quad L_p[A_1(z,u,w),\overline{D}] \le \varepsilon k_0, \quad L_p[A_3(z,u,w),\overline{D}] \le k_1, \tag{4.2}$$

in which p, p_0, k_0, k_1 are non-negative constants with $2 < p_0 \le p$, ε is a sufficiently small positive constant.

(C6) Equation (4.1) satisfies the uniform ellipticity condition, namely for any number $u \in \mathbb{R}$ and $w, U_1, U_2 \in \mathbb{C}$, the inequality

$$|F(z,u,w,U_1) - F(z,u,w,U_2)| \le q_0|U_1 - U_2|, \tag{4.3}$$

holds for almost every point $z \in D$ holds, where $q_0 < 1$ is a non-negative constant.

Now, we formulate the general discontinuous boundary value problem as follows.

Problem P. Find a solution $u(z)$ of (4.1), which is continuously differentiable in $D^* = \overline{D}\backslash Z$, and satisfies the boundary condition

$$\frac{1}{2}\frac{\partial u}{\partial \nu} + c_1(z)u = c_2(z), \text{ i.e. } \operatorname{Re}[\overline{\lambda(z)}u_z] + c_1(z)u = c_2(z), \quad z \in \Gamma^* = \Gamma\backslash Z, \tag{4.4}$$

in which $\lambda(z) = a(z) + ib(z)$, $|\lambda(z)| = 1$ on Γ, and $Z = \{t_1, t_2, \dots, t_m\}$ are the first kind of discontinuous points of $\lambda(z)$ on Γ, and $\lambda(z), c(z)$ satisfies the conditions

$$\begin{gathered} C_\alpha[\lambda(z), \hat{\Gamma}_j] \le k_0, \quad C_\alpha[|z-t_{j-1}|^{\beta_{j-1}}|z-t_j|^{\beta_j} c_1(z), \hat{\Gamma}_j] \le \varepsilon k_0, \\ C_\alpha[|z-t_{j-1}|^{\beta_{j-1}}|z-t_j|^{\beta_j} c_2(z), \hat{\Gamma}_j] \le k_2, \quad j = 1, \dots, m, \end{gathered} \tag{4.5}$$

in which $\hat{\Gamma}_j$ is an arc from the point t_{j-1} to t_j on $\hat{\Gamma}$, $\hat{\Gamma}_j$, $(j=1,2,\dots,m)$ does not include the end points, and $\alpha, \varepsilon, \beta_j$ are positive constants with $1/2 < \alpha < 1$ and $\beta_j < 1$, $j = 1, \dots, m$. Denote by $\lambda(t_j - 0)$ and $\lambda(t_j + 0)$ the left limit and right limit of $\lambda(z)$ as $z \to t_j$ $(j = 1, 2, \dots, m)$ on Γ, and

$$\begin{gathered} \mathrm{e}^{i\phi_j} = \frac{\lambda(t_j - 0)}{\lambda(t_j + 0)}, \quad \gamma_j = \frac{1}{\pi i}\ln[\frac{\lambda(t_j - 0)}{\lambda(t_j + 0)}] = \frac{\phi_j}{\pi} - K_j, \\ K_j = [\frac{\phi_j}{\pi}] + J_j, \quad J_j = 0 \text{ or } 1, \ j = 1, \dots, m, \end{gathered} \tag{4.6}$$

in which $0 \le \gamma_j < 1$ when $J_j = 0$, and $-1 < \gamma_j < 0$ when $J_j = 1$, $j = 1, \dots, m$. The number

$$K = \frac{1}{2\pi}\Delta_\Gamma \arg\lambda(z) = \sum_{j=1}^{m} \frac{K_j}{2} \tag{4.7}$$

is called the index of Problem P. Let $\beta_j + |\gamma_j| < 1$ for $j = 1, \dots, m$, we require that the solution $u(z)$ possess the property

$$\begin{gathered} R(z)u_z \in C_\delta(\overline{D}), \quad R(z) = \prod_{j=1}^{m} |z - t_j|^{\eta_j/\tau^2}, \\ \eta_j = \begin{cases} \beta_j + \tau, & \text{for } \gamma_j \ge 0, \ \gamma_j < 0, \beta_j \ge |\gamma_j|, \\ |\gamma_j| + \tau, & \text{for } \gamma_j < 0, \ \beta_j < |\gamma_j|, \ j = 1, \dots, m, \end{cases} \end{gathered} \tag{4.8}$$

in the neighborhood ($\subset D$) of z_j, where $\tau \le \min(\alpha, 1-2/p_0)$, $\delta < \min(\beta_1, \dots, \beta_m, \tau$ are two small positive constants.

We mention that the first boundary value problem, second boundary value problem and third boundary value problem; i.e., regular oblique derivative problem are the special cases of Problem P, because their boundary conditions are the continuous boundary conditions, and their indexes are equal to $K = N - 1$. Now $2K$ can be equal to any positive or negative integer, hence Problem P is a very general boundary value problem. Because Problem P is not certainly solvable, In the following, we introduce a new well-posedness of discontinuous Poincaré boundary value problem for the nonlinear elliptic equations of second order, namely

Problem Q. Find a continuous solution $[w(z), u(z)]$ of the complex equation

$$\begin{gathered} w_{\bar z} = F(z, u, w, w_z), \quad z \in D, \\ F = \operatorname{Re}[Qw_z + A_1 w] + A_2 u + A_3, \end{gathered} \tag{4.9}$$

satisfying the boundary condition

$$\operatorname{Re}[\overline{\lambda(z)} w(z)] + c_1(z) u = c_2(z) + h(z)\overline{\lambda(z)} X(z), \quad z \in \Gamma^*, \tag{4.10}$$

and the relation

$$u(z) = \operatorname{Re} \int_{a_0}^{z} [w(z) + \sum_{j=1}^{N} \frac{i d_j}{z - z_j} \mathrm{d}z] + b_0, \tag{4.11}$$

where $a_0 = 1$, b_0 is a real constant, d_j $(j = 1, \dots, N)$ are appropriate real constants such that the function determined by the integral in (4.11) is single-valued in D, and the undetermined function $h(z)$ is

$$h(z) = \begin{cases} 0, & z \in \Gamma_0,\ K \ge 0, \\ h_j, & z \in \Gamma_j, j = 1, \dots, N,\ K \ge 0, \\ h_j, & z \in \Gamma_j,\ j = 1, \dots, N,\ K < 0, \\ [1 + (-1)^{2K}] h_0 & \\ + \operatorname{Re} \sum_{m=1}^{[|K|+1/2]-1} (h_m^+ + i h_m^-) z^m, & z \in \Gamma_0,\ K < 0, \end{cases} \tag{4.12}$$

in which h_j $(j = [1 - (-1)^{2K}]/2, \dots, N+1)$ are unknown real constants to be determined appropriately, and $h_{N+1} (= h_0) = 0$, if $2|K|$ is an odd integer. And

$$\begin{aligned} \Pi(z) = {} & \prod_{j=1}^{m_0} (z - t_j)^{\gamma_j} \prod_{l=l}^{N} (z - z_l)^{-[\tilde K_l]} \prod_{j=m_0+1}^{m_1} \Big(\frac{z - t_j}{z - z_1} \Big)^{\gamma_j} \dots \\ & \times \prod_{j=m_{N_0-1}+1}^{m_{N_0}} \Big(\frac{z - t_j}{z - z_{N_0}} \Big)^{\gamma_j} \prod_{j=m_{N_0}+1}^{m_{N_0+1}} \Big(\frac{z - t_j}{z - z_{N_0+1}} \Big)^{\gamma_j} \Big(\frac{z - t'_{N_0+1}}{z - z_{N_0+1}} \Big) \dots \\ & \times \prod_{j=m_{N-1}+1}^{m} \Big(\frac{z - t_j}{z - z_N} \Big)^{\gamma_j} \Big(\frac{z - t'_N}{z - z_N} \Big), \end{aligned} \tag{4.13}$$

where $\tilde K_l = \sum_{j=m_{l-1}+1}^{m_l} K_j$ are denoted the partial indexes on Γ_l $(l = 1, \dots, N)$; $t_j \in \Gamma_0$ $(j = 1, \dots, m_0)$, $t_j \in \Gamma_1$ $(j = m_0 + 1, \dots, m_1)$,..., $t_j \in \Gamma_N$, $(j = m_{N-1} + 1 \dots, m)$ are all discontinuous points of $\lambda(z)$ on Γ. If $\lambda(z)$ on Γ_l $(0 \le l \le N)$ has no discontinuous point, then we can choose a point $t_j \in \Gamma_l\,(0 \le l \le N)$ as a discontinuous point of $\lambda(z)$ on Γ_l $(0 \le l \le N)$, in this case $t_j = t_{j+1}$. There is in no harm assuming that the partial indexes K_l of $\lambda(z)$ on Γ_l $(l = 0, 1, \dots, N_0\,(\le N))$

are integers, and the partial indexes K_l of $\lambda(z)$ on Γ_l $(j = N_0 + 1, \dots, N)$ are no integers, and we choose the points t_l' $(\in \Gamma_l,\ l = N_0+1, \dots, N)$ are not discontinuous points on Γ_l $(l = N_0 + 1, \dots, N)$ respectively. Similarly to (1.7)-(1.12), [5, Chapter V], we see that

$$\frac{\lambda(t_j-0)}{\lambda(t_j+0)}\overline{\Big[\frac{Y(t_j-0)}{Y(t_j+0)}\Big]} = \frac{\lambda(t_j-0)}{\lambda(t_j+0)}e^{-i\pi\gamma_j} = \pm 1,$$

it only needs to charge the symbol on some arcs on Γ, then $\lambda(z)\overline{Y(z)}/|Y(z)|$ on Γ is continuous. In this case, the new index

$$\kappa = \frac{1}{2\pi}\Delta_\Gamma[\lambda(z)\overline{Y(z)}] = K - \frac{N-N_0}{2}$$

is an integer; and

$$\begin{gathered}
X(z) = \begin{cases} iz^{[\kappa]}e^{iS(z)}Y(z), & z \in \Gamma_0, \\ ie^{i\theta_j}e^{iS(z)}Y(z), & z \in \Gamma_j,\ j = 1, \dots, N, \end{cases} \\
\operatorname{Im}[\overline{\lambda(z)}X(z)] = 0, \quad z \in \Gamma, \\
\operatorname{Re} S(z) = \begin{cases} \arg\lambda(z) - [\kappa]\arg z - \arg Y(z), & z \in \Gamma_0, \\ \arg\lambda(z) - \arg Y(z) - \theta_j, & z \in \Gamma_j,\ j = 1, \dots, N, \end{cases} \\
\operatorname{Im}[S(1)] = 0,
\end{gathered} \tag{4.14}$$

where $S(z)$ is a solution of the modified Dirichlet problem with the above boundary condition for analytic functions, θ_j $(j = 1, \dots, N)$ are real constants.

If $K \ge 0$, we require that the solution $w(z) = u_z$ satisfy the point conditions

$$\operatorname{Im}[\overline{\lambda(a_j)}w(a_j)] = b_j, \quad j \in J = \{1, \dots, 2K+1\}, \quad \text{if } K \ge 0, \tag{4.15}$$

in which $a_j \in \Gamma_0$ $(j \in J)$ are distinct points; and b_j $(j \in J)$, b_0 are real constants satisfying the conditions

$$|b_j| \le k_3, \quad j \in J \cup \{0\} \tag{4.16}$$

with the a positive constant k_3. This is the well-posedness of Problem P for equation (4.1).

Problem Q with the conditions $A_3(z) = 0$ in (4.1), $c_2(z) = 0$ in (4.10) and $b_j = 0$ $(j \in J \cup \{0\})$ in (4.11), (4.15) will be called Problem Q_0.

The undetermined real constants d_j, h_j $(j = [1-(-1)^{2K}]/2, \dots, N)$, $h_m^{\pm}$ $(m = 1 \dots, -K-1)$ in (4.11), (4.12) are for ensuring the existence of continuous solutions, and b_j $(j = 0, 1, \dots, 2K+1)$ in (4.11), (4.15) are for ensuring the uniqueness of continuous solutions in $\overline{D}$.

Now we introduce the previous well-posedness of the discontinuous Poincaré problem of elliptic complex equations.

Problem R. Find a continuous solution $w(z)$ in D of (4.9) with the modified boundary condition (4.10) and the relation (4.11), where

$$h(z)=\begin{cases}0, & z\in\Gamma,\ K>N-1,\\ 0, & z\in\Gamma_j,\ j=1,\dots,[K]+1,\ 0\le K\le N-1,\\ h_j, & z\in\Gamma_j,\ j=[K]+2,\dots,[K]+1+N-K',\\ & \quad 0\le K\le N-1,\\ h_j, & z\in\Gamma_j,\, j=1,\dots,N,\ K<0,\\ [1+(-1)^{2K}]h_0 & \\ +\operatorname{Re}\sum_{m=1}^{[|K|+1/2]-1}(h_m^+ & \\ +ih_m^-)z^m, & z\in\Gamma_0,\ K<0,\end{cases} \tag{4.17}$$

in which $K'=[K+1/2]$, $[K]$ denotes the integer part of K, h_0,h_m^+,h_m^-, ($m=1,\dots,[|K|+1/2]-1$) are unknown real constants to be determined appropriately, and $h_{N+1}(=h_0)=0$, if $2|K|$ is an odd integer; and the solution $w(z)$ satisfies the point conditions

$$\operatorname{Im}[\overline{\lambda(a_j)}w(a_j)]=b_j,\quad j\in J=\begin{cases}1,\dots,2K-N+1, & \text{if } K>N-1,\\ 1,\dots,[K]+1, & \text{if } 0\le K\le N-1,\end{cases} \tag{4.18}$$

in which where $a_j\in\Gamma_j$ $(j=1,\dots,N_0)$, $a_j\in\Gamma_0$ $(j=N_0+1,\dots,2K-N+1, \text{if } K\ge N)$ are distinct points; and when $[K]+1>N_0$, $a_j\in\Gamma_j$ $(j=1,\dots,N_0)$, $a_j\in\Gamma_0$ $(j=N_0+1,\dots,[K]+1,\ \text{if } 0\le K<N)$, otherwise $a_j\in\Gamma_j$ $(j=1,\dots,[K]+1,\ \text{if } 0\le K<N)$ are distinct points; and

$$|b_j|\le k_3,\quad j\in J$$

with a non-negative constant k_3.

The equivalence of Problem Q for equation (4.9) and Problem R for (4.9) can be verified. We can see that the advantages of the new well-posedness. We mention the equivalence of these well-posedness, i.e. if there exists the unique solvability of one well-posedness, then we can derive that another well-posedness possesses the unique solution. Hence it is best to choose the most simple well-posedness.

5. Estimates for solutions of discontinuous Poincaré problems

First of all, we prove the following result.

Theorem 5.1. *Suppose that* (4.1) *satisfies Conditions* (C4)–(C6) *and* ε *in* (4.2), (4.5) *is small enough. Then Problem* Q_0 *for equation* (4.1) *in* D *has only the trivial solution.*

Proof. Let $[u(z),w(z)]$ be any solution of Problem Q_0 for equation (4.9); i.e., $[w(z),u(z)]$ satisfies the complex equation with boundary conditions

$$w_{\bar z}+\operatorname{Re}[Qw_z+A_1w]=-A_2u\quad\text{in } D, \tag{5.1}$$

$$\begin{gathered}\operatorname{Re}[\overline{\lambda(z)}w(z)]+c_1(z)u=h(z)\overline{\lambda(z)}X(z),\quad z\in\Gamma^*,\\ \operatorname{Im}[\overline{\lambda(a_j)}w(a_j)]=0,\quad j\in J,\ u(a_0)=0.\end{gathered} \tag{5.2}$$

and the relation

$$u(z)=\operatorname{Re}\int_{a_0}^{z}[\,w(z)+\sum_{j=1}^{N}\frac{id_j}{z-z_j}\mathrm{d}z], \tag{5.3}$$

where $a_0 = 1$. From the three formulae in (5.3), we see that

$$\begin{gathered} d_j = \frac{1}{2\pi} \int_{\Gamma_j} w(z) d\theta, \quad j = 1, \dots, N, \\ C_\delta[R'(z)u(z), \overline{D}] \le M_{12} C_\delta[R(z)w(z), \overline{D}], \end{gathered} \tag{5.4}$$

where δ ia a positive constant as stated in (4.8), $M_{12} = M_{12}(R, D)$ is a non-negative constant. From the conditions (4.2) and (4.5), we can obtain

$$\begin{aligned} L_{p_0}[R' A_2 u, \overline{D}] &\le L_{p_0}[A_2, \overline{D}]\, C[R' u, \overline{D}] \le \varepsilon k_0 C[R'(z)u(z), \overline{D}] \\ &\le \varepsilon k_0 C_\delta[R'(z)u(z), \overline{D}], \\ C_\alpha[R(z)c_1(z)R'(z)u(z), \Gamma] &\le C_\alpha[R(z)c_1(z), \Gamma] C_\delta[R'(z)u(z), \overline{D}] \\ &\le \varepsilon k_0 C_\delta[R'(z)u(z), \overline{D}], \end{aligned} \tag{5.5}$$

where $R(z)$ is as stated in (4.8) and $|R(z)| \le 1$ in $\overline{D}$. Thus by using the result of the Riemann-Hilbert boundary value problem for the complex equation of first order (see [5, Theorems 3.2-3.4, Chapter V] and [6, Theorem 6.1, Chapter VI]), the following estimate of the solution $w(z)$ can be obtained, namely

$$C_\delta[R(z)w(z), \overline{D}] \le 2\varepsilon k_0 M_{13} C_\delta[R'(z)u(z), \overline{D}], \tag{5.6}$$

where $M_{13} = M_{13}(q_0, p_0, \delta, k_0, D)$ is a non-negative constant. From the estimate (5.4), it follows the estimate about $u(z)$:

$$C_\delta[R'(z)u(z), \overline{D}] \le 2\varepsilon k_0 M_{12} M_{13} C_\delta[R'(z)u(z), \overline{D}]. \tag{5.7}$$

Provided that the positive number ε in (4.2) and (4.5) is small enough, such that

$$2\varepsilon k_0 M_{12} M_{13} < 1, \tag{5.8}$$

it can be derived that $u(z) \equiv 0$ and then $w(z) \equiv 0$ in D. Hence Problem Q_0 for equation (5.1) has only the trivial solution. This completes the proof of Theorem 5.1. □

Theorem 5.2. *Let* (4.1) *satisfy Conditions* (C4)–(C6) *and* (4.2)*,* (4.5) *with the sufficiently small positive number ε. Then any solution $[u(z), w(z)]$ of Problem* Q *for* (4.9) *satisfies the estimates*

$$\begin{gathered} \hat{C}^1_\delta[u, \overline{D}] = C_\delta[R'(z)u, \overline{D}] + C_\delta[R(z)w(z), \overline{D}] \le M_{14}, \\ \hat{L}^1_{p_0}[w, \overline{D}] = L_{p_0}[|RSw_{\bar z}| + |RSw_z|, \tilde{D}] \le M_{15}, \end{gathered} \tag{5.9}$$

where $R(z)$ and , $S(z)$ are

$$\begin{gathered} R(z) = \prod_{j=1}^m |z - t_j|^{\eta_j/\tau^2}, \quad S(z) = \prod_{j=1}^m |z - t_j|^{1/\tau^2}, \\ \eta_j = \begin{cases} |\gamma_j| + \tau, & \text{if } \gamma_j < 0,\ \beta_j \le |\gamma_j|, \\ \beta_j + \tau, & \text{if } \gamma_j \ge 0,\ \gamma_j < 0,\ \beta_j > |\gamma_j|, \end{cases} \end{gathered} \tag{5.10}$$

where γ_j $(j = 1, \dots, m)$ are real constants as stated in (4.6)*, $\tau = \min(\alpha, 1 - 2/p_0)$, $\delta < \min[\beta_1, \dots, \beta_m, \tau]$ is a small positive constant, $k = k(k_0, k_1, k_2, k_3)$, $M_j = M_j(q_0, p_0, \delta, k, D)$ $(j = 14, 15)$ are non-negative constants only dependent on q_0, p_0, δ, k, D, $j = 3, 4$.*

Proof. By using the reduction to absurdity, we shall prove that any solution $u(z)$ of Problem Q satisfies the estimate of bounded-ness

$$\hat{C}^1[u,\overline{D}] = C[R'(z)u(z),\overline{D}] + C[R(z)w(z),\overline{D}] \le M_{16}, \tag{5.11}$$

in which $M_{16} = M_{16}(q_0,p_0,\delta,k,D)$ is a non-negative constant. Suppose that (5.11) is not true, then there exist sequences of coefficients $\{A_j^{(m)}\}$ $(j=1,2,3)$, $\{Q^{(m)}\}$, $\{\lambda^{(m)}(z)\}$, $\{c_j^{(m)}\}$ $(j=1,2)$, $b_j^{(m)}$ $(j\in J\cup\{0\})$, which satisfy Conditions (C4)–(C6) and (4.5), (4.16), such that $\{A_j^{(m)}\}$ $(j=1,2,3)$, $\{Q^{(m)}\}$, $\{\lambda^{(m)}(z)\}$, $\{|z-t_{j-1}|^{\beta_{j-1}}|z-t_j|^{\beta_j}c_j^{(m)}\}$ $(j=1,2)$ and $\{b_j^{(m)}\}$ $(j\in J\cup\{0\})$ in $\overline{D}$, Γ^* converge weakly or converge uniformly to $A_j^{(0)}$ $(j=1,2,3)$, $Q^{(0)}$, $\lambda^{(0)}(z)$, $|z-t_{j-1}|^{\beta_{j-1}}|z-t_j|^{\beta_j}c_j^{(0)}$ $(j=1,2)$, $b_j^{(0)}(j\in J\cup\{0\})$ respectively, and the corresponding boundary value problem

$$w_{\bar z} - \mathrm{Re}[Q^{(m)}w_z + A_1^{(m)}w] - A_2^{(m)}u = A_3^{(m)}, \tag{5.12}$$

and

$$\begin{gathered}\mathrm{Re}[\overline{\lambda(z)}w(z)] + c_1^{(m)}(z)u = c_2^{(m)}(z) + c(z)\overline{\lambda(z)}X(z) \quad \text{on } \Gamma^*,\\ \mathrm{Im}[\overline{\lambda(a_j)}w(a_j)] = b_j^{(m)}, \quad j\in J, \quad u(a_0) = b_0^{(m)}\end{gathered} \tag{5.13}$$

have the solutions $\{u^{(m)}(z),w^{(m)}(z)\}$, but $\hat{C}^1[u^{(m)}(z),\overline{D}]$ $(m=1,2,\dots)$ are unbounded. Thus we can choose a subsequence of $\{u^{(m)}(z),w^{(m)}(z)\}$ denoted by $\{u^{(m)}(z),w^{(m)}(z)\}$ again, such that $h_m = \hat{C}[u^{(m)}(z),\overline{D}] \to \infty$ as $m\to\infty$, and assume that $H_m \ge \max[k_1,k_2,k_3,1]$. It is easy to see that $\{\tilde u^{(m)}(z),\tilde w^{(m)}(z)\} = \{u^{(m)}(z)/H_m, \tilde w^{(m)}(z)/H_m\}$ $(m=1,2,\dots)$ are solutions of the boundary value problems

$$\tilde w_{\bar z} - \mathrm{Re}[Q^{(m)}\tilde w_z + A_1^{(m)}\tilde w_z] - A_2^{(m)}\tilde u = A_3^{(m)}/H_m, \tag{5.14}$$

$$\begin{gathered}\mathrm{Re}[\overline{\lambda(z)}\tilde w(z)] + c_1^{(m)}(z)\tilde u = [c_2^{(m)}(z) + h(z)\overline{\lambda(z)}X(z)]/H_m \quad \text{on } \Gamma^*,\\ \mathrm{Im}[\overline{\lambda(a_j)}\tilde w(a_j)] = b_j^{(m)}/H_m, \quad j\in J, \quad \tilde u(a_0) = b_0^{(m)}/H_m.\end{gathered} \tag{5.15}$$

We can see that the functions in the above equation and the boundary conditions satisfy the condition (C4)–(C6), (4.5),(4.16) and

$$\begin{gathered}|R'(z)u^{(m)}|/H_m \le 1, \quad L_\infty[A_3^{(m)}/H_m,\overline{D}] \le 1,\\ |R(z)c_2^{(m)}/H_m| \le 1, \quad |b_j^{(m)}/H_m| \le 1, \quad j\in J\cup\{0\},\end{gathered} \tag{5.16}$$

hence by using a similar method as in the proof of [6, Theorem 6.1, Chapter IV], we can obtain the estimates

$$\hat{C}_\delta[\tilde u^{(m)}(z),\overline{D}] \le M_{17}, \quad \hat{L}^1_{p_0}[\tilde w^{(m)}(z),\overline{D}] \le M_{18}, \tag{5.17}$$

where $M_j = M_j(q_0,p_0,\delta,k_0,D)$ $(j=17,18)$ are non-negative constants. Moreover from the sequence $\{\tilde u^{(m)}(z),\ \tilde w^{(m)}(z)\}$, we can choose a subsequence denoted by $\{\tilde u^{(m)}(z),\tilde w^{(m)}(z)\}$ again, which in $\overline{D}$ uniformly converge to $\tilde u_0(z),\tilde w_0(z)$ respectively, and $R(z)S(z)(\tilde w^{(m)})_{\bar z}$, $R(z)S(z)(\tilde w^{(m)})_z$ in D are weakly convergent. This shows that $[\tilde u_0(z),\tilde w_0(z)]$ is a solution of the boundary-value problem

$$\tilde w_{0\bar z} - \mathrm{Re}[Q^{(0)}\tilde w_{0z} + A_1^{(0)}\tilde u_0] - A_2^{(0)}\tilde u_0 = 0, \tag{5.18}$$

$$\begin{gathered}\mathrm{Re}[\overline{\lambda(z)}\tilde w_0(z)] + 2c_1^{(0)}(z)\tilde u_0 = h(z)\overline{\lambda(z)}X(z) \quad \text{on } \Gamma^*,\\ \mathrm{Im}[\overline{\lambda(a_j)}\tilde w_0(a_j)] = 0, \quad j\in J,\ \tilde u_0(a_0) = 0.\end{gathered} \tag{5.19}$$

We see that (5.18) is a homogeneous equation, and (5.19) is a homogeneous boundary condition. On the basis of Theorem 5.1, the solution $\tilde{u}_0(z)=0$, $\tilde{w}_0(z)=0$ however, from $\hat{C}^1[\tilde{u}^{(m)}(z),\overline{D}]=1$, we can derive that there exists a point $z^*\in\overline{D}$, such that $|R'(z^*)\tilde{u}_0(z^*)|+|R(z^*)\tilde{w}_0(z^*)|\neq 0$. This is impossible. This shows that the first estimate in (5.9) are true. Moreover it is not difficult to verify the second estimate in (5.9). □

Now we prove the uniqueness of solutions of Problem Q for equation (4.1) as follows.

Theorem 5.3. *Suppose that* (4.1) *satisfies conditions* (C4)–(C6) *and the following condition: for any real functions* $R'(z)u_j(z)\in C(D^*), R(z)w_j(z)\in C(D^*)$, $R(z)S(z)U(z)\in L_{p_0}(\overline{D})$ $(j=1,2)$*, the equality*

$$F(z,u_1,w_1,U)-F(z,u_2,w_2,U)=\operatorname{Re}[\tilde{Q}U+\tilde{A}_1(w_1-w_2)]+\tilde{A}_2(u_1-u_2) \tag{5.20}$$

holds, where $|\tilde{Q}|\leq_0<1$ *in* D, $L_p[\tilde{A}_1,\overline{D}]\leq K_0$, $L_p[\tilde{A}_2,\overline{D}]\leq\varepsilon k_0$ *and* (4.5) *with the sufficiently small positive constant* ε*. Then Problem* Q *for equation* (4.1) *has at most one solution.*

Proof. Denote by $[u_j(z),w_j(z)](j=1,2)$ two solutions of Problem Q for (4.9), and substitute them into (4.9)-(4.11) and (4.15), we see that $[u(z),w(z)]=[u_1(z)-u_2(z),w_1(z)-w_2(z)]$ is a solution of the homogeneous boundary-value problem

$$\begin{gathered}
w_{\bar z}=\operatorname{Re}[\tilde{Q}w_z+\tilde{A}_1w]+\tilde{A}_2u,\quad z\in D,\\
\operatorname{Re}[\overline{\lambda(z)}w(z)]+c_1(z)u(z)=h(z)\overline{\lambda(z)}X(z),\quad z\in\Gamma^*,\\
\operatorname{Im}[\overline{\lambda(a_j)}w(a_j)]=0,\quad j\in J,\\
u(z)=\int_{a_0}^{z}[w(z)dz+\sum_{j=1}^{N}\frac{id_j}{z-z_j}dz]\quad\text{in } D,
\end{gathered}$$

the coefficients of which satisfy same conditions of (4.2),(4.3),(4.5) and (4.16), but $k_1=k_2=k_3=0$. On the basis of Theorem 5.1, provided that ε is sufficiently small, we can derive that $u(z)=w(z)=0$ in $\overline{D}$; i.e., $u_1(z)=u_2(z)$, $w_1(z)=w_2(z)$ in $\overline{D}$. □

6. Solvability results of discontinuous Poincaré problem

In this section, we shall prove the solvability of general discontinuous Poincaré boundary value problem by the the method of parameter extension.

Theorem 6.1. *Suppose that the nonlinear elliptic equation* (4.1) *satisfies condition* (C4)–(C6), (5.20)*, and* ε *in* (4.2), (4.5) *is small enough. Then there exists a solution* $[u(z),w(z)]$ *of Problem* Q *for* (4.9) *and* $[u(z),w(z)]\in B=\hat{C}^1_\delta(\overline{D})\cap\hat{L}^1_{p_0}(\overline{D})$*, where* $B=\hat{C}^1_\delta(\overline{D})\cap\hat{L}^1_{p_0}(\overline{D})$ *is a Banach space; i.e.,* $\hat{C}^1_\delta[u,\overline{D}]<\infty$, $\hat{L}^1_{p_0}[w,\overline{D}]<\infty$*, and* $p_0\,(>2)$ *is stated as in* (5.9).

Proof. We introduce the nonlinear elliptic equation with the parameter $t\in[0,1]$,

$$w_{\bar z}=tF(z,u,w,w_z)+A(z), \tag{6.1}$$

where $A(z)$ is any measurable function in D and $R(z)S(z)A(z)\in L_{p_0}(\overline{D})$, $2<p_0\leq p$. Let E be a subset of $0\leq t\leq 1$ such that Problem Q is solvable for (6.1) with any $t\in E$ and any $R(z)S(z)A(z)\in L_{p_0}(\overline{D})$. In accordance with the method in the

proof of [6, Theorem 6.1, Chapter VI], we can prove that when $t = 0$, Problem Q has a unique solution $[u(z), w(z)]$ satisfying the complex equation and boundary conditions; i.e.,

$$w_{\bar z} = A(z),\ z \in D, \tag{6.2}$$

$$\begin{gathered}\operatorname{Re}[\overline{\lambda(z)}w(z)] + c_1(z)u = c_2(z) + h(z)\overline{\lambda(z)}X(z), \quad z \in \Gamma^*,\\ \operatorname{Im}[\overline{\lambda(a_j)}w(a_j)] = b_j,\ j \in J,\end{gathered} \tag{6.3}$$

and the relation

$$u(z) = \operatorname{Re}\int_{a_0}^{z}[w(z) + \sum_{j=1}^{N}\frac{id_j}{z - z_j}\mathrm{d}z] + b_0. \tag{6.4}$$

This shows that the point set E is not empty.

From Theorem 5.3, We see that $[u(z), w(z)] \in B = \hat C_\delta^1(\overline{D}) \cap \hat L_{p_0}^1(\overline{D})$. Suppose that when $t = t_0\,(0 \le t_0 < 1)$, Problem Q for the complex equation (6.1) has a unique solution, we shall prove that there exists a neighborhood of t_0: $E = \{|t - t_0| \le \delta_0, 0 \le t \le 1, \delta_0 > 0\}$, so that for every $t \in E$ and any function $R(z)S(z)A(z) \in L_{p_0}(\overline{D})$, Problem Q for (6.1) is solvable. In fact, the complex equation (6.1) can be written in the form

$$w_{\bar z} - t_0F(z, u, w, w_z) = (t - t_0)F(z, u, w, w_z) + A(z). \tag{6.5}$$

We select an arbitrary function $[u_0(z), w_0(z)] \in B = \hat C_\delta^1(\overline{D}) \cap \hat L_{p_0}^1(\overline{D})$, in particular $[u_0(z), w_0(z)] = 0$ in $\overline{D}$. Let $[u_0(z), w_0(z)]$ be replaced into the position of $u(z), w(z)$ in the right hand side of (6.5). By condition (C4)–(C6), it is obvious that

$$B_0(z) = (t - t_0)RSF(z, u_0, w_{0z}, w_{0zz}) + R(z)S(z)A(z) \in L_{p_0}(\overline{D}).$$

Noting the (6.5) has a solution $[u_1(z), w_1(z)] \in B$. Applying the successive iteration, we can find out a sequence of functions: $[u_n(z), w_n(z)] \in B$, $n = 1, 2, \dots$, which satisfy the complex equations

$$w_{n+1z\bar z} - t_0F(z, u_{n+1}, w_{n+1}, w_{n+1z}) = (t - t_0)F(z, u_n, w_n, w_{nz}) + A(z), \tag{6.6}$$

for $n =, 2, \dots$. The difference of the above equations for $n + 1$ and n is as follows:

$$\begin{aligned}&(w_{n+1} - w_n)_{z\bar z} - t_0[F(z, u_{n+1}, w_{n+1}, w_{n+1z}) - F(z, u_n, w_n, w_{nz}]\\ &= (t - t_0)[F(z, u_n, w_n, w_{nz}) - F(z, u_{n-1}, w_{n-1}, w_{n-1z})], \quad n = 1, 2, \dots.\end{aligned} \tag{6.7}$$

From conditions (C4)–(C6), it can be seen that

$$\begin{gathered}\begin{aligned}&F(z, u_{n+1}, w_{n+1}, w_{n+1z}) - F(z, u_n, w_n, w_{nz}) = F(z, u_{n+1}, w_{n+1}, w_{n+1z})\\ &- F(z, u_{n+1}, w_{n+1}, w_{nz}) + [F(z, u_{n+1}, w_{n+1}, w_{nz}) - F(z, u_n, w_n, w_{nz})]\\ &= \operatorname{Re}[\tilde Q_{n+1}(w_{n+1} - w_n)_z + \tilde A_{1n+1}(w_{n+1} - w_n)] + \tilde A_{2n+1}(u_{n+1} - u_n),\end{aligned}\\ |\tilde Q_{n+1}| \le q_0 < 1, \quad L_{p_0}[\tilde A_{1n+1}, \overline{D}] \le k_0, \quad L_{p_0}[\tilde A_{2n+1}, \overline{D}] \le \varepsilon k_0,\end{gathered} \tag{6.8}$$

for $n = 1, 2, \dots$, and

$$\begin{aligned}&L_{p_0}[RS(F(z, u_n, w_n, w_{nz}) - F(z, u_{n-1}, w_{n-1}, w_{n-1z})), \overline{D}]\\ &\le q_0L_{p_0}[RS(w_n - w_{n-1})_z, \overline{D}] + k_0C_\delta[R(w_n - w_{n-1}), \overline{D}]\\ &\le (q_0 + k_0)[\hat C_\delta^1[u_n - u_{n-1}, \overline{D}] + \hat L_{p_0}^1[w_n - w_{n-1}, \overline{D}] = (q_0 + k_0)L_n.\end{aligned}$$

Moreover, $u_{n+1}(z)-u_n(z)$ satisfies the homogeneous boundary conditions

$$\begin{gathered}\operatorname{Re}[\overline{\lambda(z)}(w_{n+1}-w_n)]+c_1(z)[u_{n+1}(z)-u_n(z)]=h(z)\overline{\lambda(z)}X(z),\quad z\in\Gamma^*,\\ \operatorname{Im}[\overline{\lambda(a_j)}(w_{n+1}(a_j)-w_n(a_j))]=0,\quad j\in J,\quad u_{n+1}(a_0)-u_n(a_0)=0.\end{gathered}\tag{6.9}$$

On the basis of Theorem 5.2, we have

$$L_{n+1}=\hat{C}^1_\delta[u_{n+1}-u_n,\overline{D}]+\hat{L}^1_{p_0}[w_{n+1}-w_n,\overline{D}]\le M_{19}|t-t_0|(q_0+k_0)L_n,\tag{6.10}$$

where $M_{19}=(M_{17}+M_{18})k_*$, M_{17} and M_{18} are as stated in (5.17). Provided $\delta_0>0$ is small enough, so that $\sigma=\delta_0M_{19}(q_0+2k_0)<1$, it can be obtained that

$$L_{n+1}\le\sigma L_n\le\sigma^nL_1=\sigma^n[\hat{C}^1_\delta(u_1,\overline{D})+\hat{L}^1_{p_0}(w_1,\overline{D})]\tag{6.11}$$

for every $t\in E$. Thus

$$\begin{aligned}&\hat{C}^1_\delta[u_n-u_m,\overline{D}]+\hat{L}^1_{p_0}[w_n-w_m,\overline{D}]\\ &\le L_n+L_{n-1}+\dots+L_{m+1}\le(\sigma^{n-1}+\sigma^{n-2}+\dots+\sigma^m)L_1\\ &=\sigma^m(1+\sigma+\dots+\sigma^{n-m-1})L_1\\ &\le\sigma^{N+1}\frac{1-\sigma^{n-m}}{1-\sigma}L_1\le\frac{\sigma^{N+1}}{1-\sigma}L_1\end{aligned}\tag{6.12}$$

for $n\ge m>N$, where N is a positive integer. This shows that $S(u_n-u_m)\to0$ as $n,m\to\infty$. Following the completeness of the Banach space $B=\hat{C}^1_\delta(\overline{D})\cap\hat{L}^1_{p_0}(\overline{D})$, there is a function $w_*(z)\in B$, such that

$$\hat{C}^1_\delta[u_n-u_*,\overline{D}]+\hat{L}^1_{p_0}[w_n-w_*,\overline{D}]\to0,\quad\text{as } n\to\infty.$$

By conditions (C4)–(C6), from (5.17) it follows that $u_*(z)$ is a solution of Problem Q for (6.5); i.e., (6.1) for $t\in E$. It is easy to see that the positive constant δ_0 is independent of t_0 $(0\le t_0<1)$. Hence from Problem Q for the complex equation (6.5) with $t=t_0=0$ is solvable, we can derive that when $t=\delta_0,2\delta_0,\dots,[1/\delta_0]\delta_0,1$, Problem Q for (6.5) are solvable, especially Problem Q for (6.1) with $t=1$ and $A(z)=0$, namely Problem Q for (4.9) has a unique solution.

From the above theorem, the solvability results of Problem P for equation (4.1) can be derived. □

Theorem 6.2. *Under the same conditions as in Theorem 6.1, the following statements hold.*

(1) *When the index $K\ge0$, Problem P for (4.1) has $2N$ solvability conditions, and the solution of Problem P depends on $2K+2$ arbitrary real constants.*

(2) *When $K<0$, Problem P for (4.1) is solvable under $2N-2K-1$ conditions, and the solution of Problem P depends on one arbitrary real constant.*

Proof. Let the solution $[w(z),u(z)]$ of Problem Q for (4.9) be substituted into the boundary condition (4.10), (4.12) and the relation (4.11). If the function $h(z)=0$, $z\in\Gamma$; i.e.,

$$\begin{gathered}h_j=0,\quad j=1,\dots,N,\quad\text{if } K\ge0,\\ h_j=0,\quad j=[1-(-1)^{2K}]/2,\dots,N,\quad\text{if } K<0,\\ h^\pm_m=0,\quad m=1,\dots,[|K|+1/2]-1,\quad\text{if } K<0,\end{gathered}$$

and $d_j=0$, $j=1,\dots,N$, then we have $w(z)=u_z$ in D and the function $w(z)$ is just a solution of Problem P for (4.1). Hence the total number of above equalities is just the number of solvability conditions as stated in this theorem. Also note that the real constants b_0 in (4.11) and b_j ($j \in J$) in (4.15) and (4.16) are arbitrarily chosen. This shows that the general solution of Problem P for (4.1) includes the number of arbitrary real constants as stated in the theorem. $\square$

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