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MULTIPLICITY OF SOLUTIONS FOR QUASILINEAR EQUATIONS INVOLVING CRITICAL ORLICZ-SOBOLEV NONLINEAR TERMS

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Abstract. In this work, we study the existence and multiplicity of solutions for a class of problems involving the ϕ-Laplacian operator in a bounded domain, where the nonlinearity has a critical growth. Our main tool is the variational method combined with the genus theory for even functionals.

1. Introduction

In this article, we consider the existence and multiplicity of solutions for the quasilinear problem

$$\begin{gathered} -\operatorname{div}\big(\phi(|\nabla u|)\nabla u\big) = \lambda\phi_*(|u|)u + f(x,u), \quad \text{in } \Omega \\ u = 0, \quad \text{on } \partial\Omega \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary, λ is a positive parameter and $\phi : (0, +\infty) \to \mathbb{R}$ is a continuous function satisfying

$$(\phi(t)t)' > 0 \quad \forall t > 0. \tag{1.2}$$

There exist $l, m \in (1, N)$ such that

$$l \le \frac{\phi(|t|)t^2}{\Phi(t)} \le m \quad \forall t \neq 0, \tag{1.3}$$

where

$$\Phi(t) = \int_0^{|t|} \phi(s)s\,ds, \quad l \le m < l^*, \quad l^* = \frac{lN}{N-l}, \quad m^* = \frac{mN}{N-m}.$$

Moreover, $\phi_*(t)t$ is such that Sobolev conjugate function Φ_* of Φ is its primitive; that is, $\Phi_*(t) = \int_0^{|t|} \phi_*(s)sds$.

Related to function $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, we assume the following:

(F1) $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ is odd with respect t and

$$f(x,t) = o\big(\phi(|t|)|t|\big), \quad \text{as } |t| \to 0 \text{ uniformly in } x;$$
$$f(x,t) = o\big(\phi_*(|t|)|t|\big), \quad \text{as } |t| \to +\infty \text{ uniformly in } x;$$

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(F2) There is $\theta \in (m, l^*)$ such that $F(x,t) \leq \frac{1}{\theta} f(x,t)t$, for all $t > 0$ and a.e. in Ω, where $F(x,t) = \int_0^t f(x,s)ds$.

Problem (1.1) associated with nonhomogenous nonlinear Φ arises in various fields of physics [16]:

(i) in nonlinear elasticity, $\Phi(t) = (1+|t|^2)^\gamma - 1$ for $\gamma \in (1, \frac{N}{N-2})$.
(ii) in plasticity, $\Phi(t) = |t|^p ln(1+|t|)$ for $1 < p_0 < p < N-1$ with $p_0 = \frac{-1+\sqrt{1+4N}}{2}$.
(iii) in generalized Newtonian fluids, $\Phi(t) = \int_0^t s^{1-\alpha}(\sinh^{-1} s)^\beta ds$, $0 \leq \alpha \leq 1$, $\beta > 0$.

Our main result reads as follows.

Theorem 1.1. *Assume that* (1.2), (1.3), (F1) *and* (F2) *are satisfied. Then, there exist a sequence* $\{\lambda_k\} \subset (0, +\infty)$ *with* $\lambda_{k+1} < \lambda_k$, *such that, for* $\lambda \in (\lambda_{k+1}, \lambda_k)$, *problem* (1.1) *has at least* k *pairs of nontrivial solutions.*

The main difficulty to prove Theorem 1.1 is related to the fact that the nonlinearity f has a critical growth. In this case, it is not clear that functional energy associated with (1.1) satisfies the well known (PS) condition, once that the embedding $W^{1,\Phi}(\Omega) \hookrightarrow L_{\Phi_*}(\Omega)$ is not compact. To overcome this difficulty, we use a version of the concentration compactness lemma due to Lions for Orlicz-Sobolev space found in Fukagai, Ito and Narukawa [14]. We would like to mention that Theorem 1.1 improves the main result found in [25].

We cite the papers of Alves and Barreiro [3], Alves, Gonçalves and Santos [4], Bonano, Bisci and Radulescu [5], Cerny [7], Clément, Garcia-Huidobro and Manásevich [9], Donaldson [10], Fuchs and Li [12], Fuchs and Osmolovski [13], Fukagai, Ito and Narukawa [14, 15], Gossez [17], Mihailescu and Raduslescu [19, 20], Mihailescu and Repovs [21], Pohozaev [22] and references therein, where quasilinear problems like (1.1) have been considered in bounded and unbounded domains of $\mathbb{R}^N$. In some those papers, the authors have mentioned that this class of problem arises in applications, such as, nonlinear elasticity, plasticity and non-Newtonian fluids.

This paper is organized as follows: In Section 2, we collect some preliminaries on Orlicz-Sobolev spaces that will be used throughout the paper, which can be found in [1], [2], [11] and [23]. In Section 3, we recall an abstract theorem involving genus theory that will use in the proof of Theorem 1.1 and prove some technical lemmas, and then we prove Theorem 1.1.

2. Preliminaries on Orlicz-Sobolev spaces

First of all, we recall that a continuous function $A : \mathbb{R} \to [0, +\infty)$ is a N-function if:

(A1) A is convex.
(A2) $A(t) = 0$ if and only if $t = 0$.
(A3) $\frac{A(t)}{t} \to 0$ as $t \to 0$, and $A(t)/t \to \infty$ as $t \to +\infty$.
(A4) A is an even function.

In what follows, we say that a N-function A satisfies the Δ_2-condition if, there exists $t_0 \geq 0$ and $k > 0$ such that

$$A(2t) \leq kA(t) \quad \forall t \geq t_0.$$

This condition can be rewritten of the following way: For each $s > 0$, there exists $M_s > 0$ and $t_0 \geq 0$ such that

$$A(st) \leq M_s A(t), \quad \forall t \geq t_0. \tag{2.1}$$

Fixed an open set $\Omega \subset \mathbb{R}^N$ and a N-function A satisfying Δ_2-condition, the space $L_A(\Omega)$ is the vectorial space of the measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\int_\Omega A(u) < \infty.$$

The space $L_A(\Omega)$ endowed with Luxemburg norm,

$$|u|_A = \inf\Big\{\alpha > 0 : \int_\Omega A\big(\frac{u}{\alpha}\big) \leq 1\Big\},$$

is a Banach space. The complement function of A, denoted by $\widetilde{A}(s)$, is given by the Legendre transformation,

$$\widetilde{A}(s) = \max_{t\geq 0}\{st - A(t)\} \quad \text{for } s \geq 0.$$

The functions A and $\widetilde{A}$ are complementary each other. Moreover, we have the Young's inequality

$$st \leq A(t) + \widetilde{A}(s), \quad \forall t, s \geq 0. \tag{2.2}$$

Using this inequality, it is possible to prove the Hölder type inequality

$$\Big|\int_\Omega uv\Big| \leq 2|u|_A |v|_{\widetilde{A}}, \quad \forall u \in L_A(\Omega) \text{ and } v \in L_{\widetilde{A}}(\Omega). \tag{2.3}$$

Another important function related to function A is the Sobolev's conjugate function A_* of A, defined by

$$A_*^{-1}(t) = \int_0^t \frac{A^{-1}(s)}{s^{(N+1)/N}} ds, \quad \text{for } t > 0.$$

When $A(t) = |t|^p$ for $1 < p < N$, we have $A_*(t) = p^{*p^*}|t|^{p^*}$, where $p^* = \frac{pN}{N-p}$.

Hereafter, we denote by $W_0^{1,A}(\Omega)$ the Orlicz-Sobolev space obtained by the completion of $C_0^\infty(\Omega)$ with respect to norm

$$\|u\| = |\nabla u|_A + |u|_A.$$

An important property is that: If A and $\widetilde{A}$ satisfy the Δ_2-condition, then the spaces $L_A(\Omega)$ and $W^{1,A}(\Omega)$ are reflexive and separable. Moreover, the Δ_2-condition also implies that

$$u_n \to u \text{ in } L_A(\Omega) \Longleftrightarrow \int_\Omega A(|u_n - u|) \to 0 \tag{2.4}$$

and

$$u_n \to u \text{ in } W^{1,A}(\Omega) \Longleftrightarrow \int_\Omega A(|u_n - u|) \to 0 \text{ and } \int_\Omega A(|\nabla u_n - \nabla u|) \to 0. \tag{2.5}$$

Another important inequality was proved by Donaldson and Trudinger [10], which establishes that for all open $\Omega \subset \mathbb{R}^N$ and there is a constant $S_N = S(N) > 0$ such that

$$|u|_{A_*} \leq S_N |\nabla u|_A, \quad u \in W_0^{1,A}(\Omega). \tag{2.6}$$

Moreover, exist $C_0 > 0$ such that

$$\int_\Omega A(u) \le C_0 \int_\Omega A(|\nabla u|), \ u \in W_0^{1,A}(\Omega). \tag{2.7}$$

This inequality shows the following embedding is continuous

$$W_0^{1,A}(\Omega) \hookrightarrow L_{A_*}(\Omega).$$

If Ω is a bounded domain and the two limits hold

$$\limsup_{t\to 0} \frac{B(t)}{A(t)} < +\infty, \quad \limsup_{|t|\to+\infty} \frac{B(t)}{A_*(t)} = 0, \tag{2.8}$$

then the embedding

$$W_0^{1,A}(\Omega) \hookrightarrow L_B(\Omega) \tag{2.9}$$

is compact.

The next four lemmas involving the functions $\Phi, \widetilde{\Phi}$ and Φ_* and theirs proofs can be found in [14]. Hereafter, Φ is the N-function given in the introduction and $\widetilde{\Phi}, \Phi_*$ are the complement and conjugate functions of Φ respectively.

Lemma 2.1. *Assume* (1.2) *and* (1.3). *Then*

$$\Phi(t) = \int_0^{|t|} s\phi(s)ds,$$

is a N-function with $\Phi, \widetilde{\Phi} \in \Delta_2$. Hence, $L_\Phi(\Omega), W^{1,\Phi}(\Omega)$ and $W_0^{1,\Phi}(\Omega)$ are reflexive and separable spaces.

Lemma 2.2. *The functions Φ, Φ_*, $\widetilde{\Phi}$ and $\widetilde{\Phi}_*$ satisfy the inequality*

$$\widetilde{\Phi}(\phi(|t|)t) \le \Phi(2t), \quad \widetilde{\Phi}_*(\phi_*(|t|)t) \le \Phi_*(2t), \quad \forall t \ge 0. \tag{2.10}$$

Lemma 2.3. *Assume that* (1.2) *and* (1.3) *hold and let $\xi_0(t) = \min\{t^l, t^m\}$, $\xi_1(t) = \max\{t^l, t^m\}$, for all $t \ge 0$. Then*

$$\xi_0(\rho)\Phi(t) \le \Phi(\rho t) \le \xi_1(\rho)\Phi(t) \quad \textit{for } \rho, t \ge 0,$$

$$\xi_0(|u|_\Phi) \le \int_\Omega \Phi(u) \le \xi_1(|u|_\Phi) \quad \textit{for } u \in L_\Phi(\Omega).$$

Lemma 2.4. *The function Φ_* satisfies the inequality*

$$l^* \le \frac{\Phi_*'(t)t}{\Phi_*(t)} \le m^* \quad \textit{for } t > 0.$$

As an immediate consequence of the Lemma 2.4, we have the following result

Lemma 2.5. *Assume that* (1.2) *and* (1.3) *hold and let $\xi_2(t) = \min\{t^{l^*}, t^{m^*}\}$, $\xi_3(t) = \max\{t^{l^*}, t^{m^*}\}$ for all $t \ge 0$. Then*

$$\xi_2(\rho)\Phi_*(t) \le \Phi_*(\rho t) \le \xi_3(\rho)\Phi_*(t) \quad \textit{for } \rho, t \ge 0,$$

$$\xi_2(|u|_{\Phi_*}) \le \int_\Omega \Phi_*(u)dx \le \xi_3(|u|_{\Phi_*}) \quad \textit{for } u \in L_{A_*}(\Omega).$$

Lemma 2.6. *Let $\widetilde{\Phi}$ be the complement of Φ and put*

$$\xi_4(s) = \min\{s^{\frac{l}{l-1}}, s^{\frac{m}{m-1}}\}, \quad \xi_5(s) = \max\{s^{\frac{l}{l-1}}, s^{\frac{m}{m-1}}\}, \quad s \ge 0.$$

Then the following inequalities hold

$$\xi_4(r)\widetilde{\Phi}(s) \le \widetilde{\Phi}(rs) \le \xi_5(r)\widetilde{\Phi}(s), \ r, s \ge 0;$$

$$\xi_4(|u|_{\widetilde{\Phi}}) \le \int_\Omega \widetilde{\Phi}(u)dx \le \xi_5(|u|_{\widetilde{\Phi}}),\ u \in L_{\widetilde{\Phi}}(\Omega).$$

3. An abstract theorem and technical lemmas

In this section we recall an important abstract theorem involving genus theory, which will use in the proof of Theorem 1.1. After, we prove some technical lemmas that will use to show that the energy functional associated with problem (1.1) satisfies the hypotheses of the abstract theorem.

3.1. **An abstract theorem.** Let E be a real Banach space and Σ the family of sets $Y \subset E\backslash\{0\}$ such that Y is closed in E and symmetric with respect to 0; that is,

$$\Sigma = \{Y \subset E\backslash\{0\}; Y \text{ is closed in } E \text{ and } Y = -Y\}.$$

Hereafter, let us denote by $\gamma(Y)$ the genus of $Y \in \Sigma$ (see [24, pp. 45]). Moreover, we set

$$K_c = \{u \in E; I(u) = c \text{ and } I'(u) = 0\},$$
$$A_c = \{u \in E; I(u) \le c\}.$$

Next, we recall a version of the Mountain Pass Theorem for even functionals, whose proof can be found in [24].

Theorem 3.1. *Let E be an infinite dimensional Banach space with $E = V \oplus X$, where V is finite dimensional and let $I \in C^1(E, \mathbb{R})$ be a even function with $I(0) = 0$, and satisfying:*

- (I1) *there are constants $\beta, \rho > 0$ such that $I(u) \ge \beta > 0$, for each $u \in \partial B_\rho \cap X$;*
- (I2) *there is $\Upsilon > 0$ such that I satisfies the $(PS)_c$ condition, for $0 < c < \Upsilon$;*
- (I3) *for each finite dimensional subspace $\widetilde{E} \subset E$, there is $R = R(\widetilde{E}) > 0$ such that $I(u) \le 0$ for all $u \in \widetilde{E}\backslash B_R(0)$.*

Suppose V is k dimensional and $V = \operatorname{span}\{e_1, \dots, e_k\}$. For $m \ge k$, inductively choose $e_{m+1} \notin E_m := \operatorname{span}\{e_1, \dots, e_m\}$. Let $R_m = R(E_m)$ and $D_m = B_{R_m} \cap E_m$. Define

$$G_m := \big\{h \in C(D_m, E); h \text{ is odd and } h(u) = u, \forall u \in \partial B_{R_m} \cap E_m\big\}, \tag{3.1}$$

$$\Gamma_j := \big\{h(\overline{D_m\backslash Y}); h \in G_m, m \ge j, Y \in \Sigma, \text{ and } \gamma(Y) \le m - j\big\}. \tag{3.2}$$

For each $j \in \mathbb{N}$, let

$$c_j = \inf_{K\in\Gamma_j} \max_{u\in K} I(u). \tag{3.3}$$

Then, $0 < \beta \le c_j \le c_{j+1}$ for $j > k$, and if $j > k$, $c_j < \Upsilon$ and c_j is critical value of I. Moreover, if $c_j = c_{j+1} = \dots = c_{j+l} = c < \Upsilon$ for $j > k$, then $\gamma(K_c) \ge l + 1$.

3.2. **Technical lemmas.** Associated with problem (1.1), we have the energy functional $J_\lambda : W_0^{1,\Phi}(\Omega) \to \mathbb{R}$ defined by

$$J_\lambda(u) = \int_\Omega \Phi(|\nabla u|) - \lambda \int_\Omega \Phi_*(u) - \int_\Omega F(x, u).$$

By conditions (F1) and (F2), $J_\lambda \in C^1\Big(W_0^{1,\Phi}(\Omega), \mathbb{R}\Big)$ with

$$J'_\lambda(u) \cdot v = \int_\Omega \phi(|\nabla u|)\nabla u \nabla v - \lambda \int_\Omega \phi_*(|u|)uv - \int_\Omega f(x, u)v,$$

for any $u, v \in W_0^{1,\Phi}(\Omega)$. Thus, critical points of J_λ are weak solutions of problem (1.1).

Lemma 3.2. *Under the conditions* (F1) *and* (F2)*, the functional* J_λ *satisfies* (I1).

Proof. On the one hand, from (F1) and (F2), for a given $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$|F(x,t)| \le \epsilon\Phi(t) + C_\epsilon\Phi_*(t), \quad \forall (x,t) \in \bar{\Omega} \times \mathbb{R}. \tag{3.4}$$

Combining (2.7) with (3.4),

$$J_\lambda(u) \ge (1 - \epsilon C_0)\int_\Omega \Phi(|\nabla u|) - (1 + C_\epsilon)\int_\Omega \Phi_*(u).$$

For ϵ is small enough and $\|u\| = \rho \simeq 0$, from (2.6) and Lemma 2.4, it follows that

$$J_\lambda(u) \ge C_1|\nabla u|_\Phi^m - C_2 S_N^{l^*}|\nabla u|_\Phi^{l^*}$$

for some positive constants C_1 and C_2. For $m < l^*$, if ρ is small enough, there is $\beta > 0$ such that

$$J_\lambda(u) \ge \beta > 0 \quad \forall u \in \partial B_\rho(0),$$

which completes the proof. □

Lemma 3.3. *Under conditions* (F1) *and* (F2)*, the functional* J_λ *satisfies* (I3).

Proof. Suppose (I3) does not hold. Then, there is a finite dimensional subspace $\widetilde{E} \subset W_0^{1,\Phi}(\Omega)$ and a sequence $(u_n) \subset \widetilde{E}\backslash B_n(0)$ satisfying

$$J_\lambda(u_n) > 0, \quad \forall n \in \mathbb{N}. \tag{3.5}$$

A direct computation shows that given $\epsilon > 0$, there is a constant $M > 0$ such that

$$-M - \epsilon\Phi_*(t) \le F(x,t) \le M + \epsilon\Phi_*(t), \quad \forall (x,t) \in \overline{\Omega} \times \mathbb{R}. \tag{3.6}$$

Consequently,

$$J_\lambda(u_n) \le \int_\Omega \Phi(|\nabla u_n|)dx - \lambda\int_\Omega \Phi_*(u_n) + \epsilon\int_\Omega \Phi_*(u_n) + M|\Omega|.$$

Fixing $\epsilon = \lambda/2$, and using Lemma 2.5, we obtain

$$J_\lambda(u_n) \le \int_\Omega \Phi(|\nabla u_n|) - \frac{\lambda}{2}\xi_3(|u_n|_{\Phi_*}) + M|\Omega|. \tag{3.7}$$

Using that $\dim\tilde{E} < \infty$, we know that any two norms are equivalent in $\tilde{E}$. Then, using that $\|u_n\| \to \infty$, we can assume that $|u_n|_{\Phi_*} > 1$. Thereby, from Lemmas 2.3 and 2.5,

$$J_\lambda(u_n) \le |\nabla u_n|_\Phi^m - \frac{\lambda}{2}|u_n|_{\Phi_*}^{l^*} + M|\Omega|.$$

Using again the equivalence of the norms in $\tilde{E}$, there is $C > 0$ such that

$$J_\lambda(u_n) \le \|u_n\|^m - \frac{\lambda}{2}C\|u_n\|^{l^*} + M|\Omega|.$$

Recalling that $m < l^*$, the above inequality implies that there is $n_0 \in \mathbb{N}$ such that

$$J_\lambda(u_n) < 0, \quad \forall n \ge n_0,$$

which contradicts (3.5). □

Lemma 3.4. *Under conditions* (F1)–(F2)*, any (PS) sequence for* J_λ *is bounded in* $W_0^{1,\Phi}(\Omega)$.

Proof. Let $\{u_n\}$ be a $(PS)_d$ sequence of J_λ. Then,

$$J_\lambda(u_n) \to d, \quad J'_\lambda(u_n) \to 0 \quad \text{as } n \to +\infty.$$

We claim that $\{u_n\}$ is bounded. Indeed, note that

$$\begin{aligned} J_\lambda(u_n) - \frac{1}{\theta} J'_\lambda(u_n)u_n = &\int_\Omega \Phi(|\nabla u_n|) - \frac{1}{\theta}\int_\Omega \phi(|\nabla u_n|)|\nabla u_n|^2 \\ &- \lambda \int_\Omega \Phi_*(u_n) + \frac{\lambda}{\theta}\int_\Omega \phi_*(|u_n|)u_n^2 \\ &- \int_\Omega F(x,u_n) + \frac{1}{\theta}\int_\Omega f(x,u_n)u_n. \end{aligned}$$

Consequently,

$$\begin{aligned} \lambda \int_\Omega \big(\frac{1}{\theta}\phi_*(|u_n|)u_n^2 - \Phi_*(u_n)\big) = &J_\lambda(u_n) - \frac{1}{\theta}J'_\lambda(u_n)u_n - \int_\Omega \Phi(|\nabla u_n|) \\ &+ \frac{1}{\theta}\int_\Omega \phi(|\nabla u_n|)|\nabla u_n|^2 \\ &+ \int_\Omega \big(F(x,u_n) - \frac{1}{\theta}f(x,u_n)u_n\big). \end{aligned}$$

Then, by (1.3), (F2) and Lemma 2.4, for n sufficiently large,

$$\lambda(\frac{l^*}{\theta} - 1)\int_\Omega \Phi_*(u_n) \le C + 1 + \|u_n\| + (\frac{m}{\theta} - 1)\int_\Omega \Phi(|\nabla u_n|),$$

which implies that

$$[\lambda(\frac{l^*}{\theta} - 1)]\int_\Omega \Phi_*(u_n) \le C + \|u_n\|,$$

where C is a positive constant, and so

$$\int_\Omega \Phi_*(u_n)dx \le C(1 + \|u_n\|). \tag{3.8}$$

By (3.6) and (3.8),

$$\begin{aligned} \int_\Omega \Phi(|\nabla u_n|) &\le J_\lambda(u_n) + \lambda\int_\Omega \Phi_*(u_n) + \int_\Omega F(x,u_n)dx \\ &\le C + o_n(1) + (\lambda + \epsilon)\int_\Omega \Phi_*(u_n) \\ &\le C(1 + \|u_n\|) + o_n(1). \end{aligned}$$

Therefore, for n sufficiently large,

$$\int_\Omega \Phi(|\nabla u_n|) \le C\left(1 + \|u_n\|\right).$$

If $\|u_n\| > 1$, from Lemma 2.5, it follows that

$$\|u_n\|^l \le C(1 + \|u_n\|).$$

Using that $l > 1$, the above inequality gives that $\{u_n\}$ is bounded in $W_0^{1,\Phi}(\Omega)$. $\square$

As a consequence of the above result, if $\{u_n\}$ is a (PS) sequence for J_λ, we can extract a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$ and $u \in W_0^{1,\Phi}(\Omega)$, such that

- $u_n \rightharpoonup u$ in $W_0^{1,\Phi}(\Omega)$;
- $u_n \rightharpoonup u$ in $L_{\Phi_*}(\Omega)$;
- $u_n \to u$ in $L_\Phi(\Omega)$;
- $u_n(x) \to u(x)$ a.e. in Ω;

From the concentration compactness lemma of Lions in Orlicz-Sobolev space found in [14], there exist two nonnegative measures $\mu, \nu \in \mathcal{M}(\mathbb{R}^N)$, a countable set $\mathcal{J}$, points $\{x_j\}_{j\in\mathcal{J}}$ in $\overline{\Omega}$ and sequences $\{\mu_j\}_{j\in\mathcal{J}}, \{\nu_j\}_{j\in\mathcal{J}} \subset [0, +\infty)$, such that

$$\Phi(|\nabla u_n|) \to \mu \geq \Phi(|\nabla u|) + \sum_{j\in\mathcal{J}} \mu_j \delta_{x_j} \quad \text{in } \mathcal{M}(\mathbb{R}^N) \tag{3.9}$$

$$\Phi_*(u_n) \to \nu = \Phi_*(u) + \sum_{j\in\mathcal{J}} \nu_j \delta_{x_j} \quad \text{in } \mathcal{M}(\mathbb{R}^N) \tag{3.10}$$

$$\nu_j \leq \max\{S_N^{l^*} \mu_j^{\frac{l^*}{l}}, S_N^{m^*} \mu_j^{\frac{m^*}{l}}, S_N^{l^*} \mu_j^{\frac{l^*}{m}}, S_N^{m^*} \mu_j^{\frac{m^*}{m}}\}, \tag{3.11}$$

where S_N satisfies (2.6).

Next, we will show an important estimate for $\{\nu_i\}$, from below. Firs, we prove a technical lemma.

Lemma 3.5. *Under the conditions of Lemma 3.4. If $\{u_n\}$ is a (PS) sequence for J_λ and $\{\nu_j\}$ as above, then for each $j \in \mathcal{J}$,*

$$\nu_j \geq \big(\frac{l}{\lambda m^*}\big)^{\frac{\beta}{\beta-1}} S_N^{-\frac{\alpha}{\beta-1}} \quad \textit{or} \quad \nu_j = 0,$$

for some $\alpha \in \{l^*, m^*\}$ and $\beta \in \{\frac{l^*}{l}, \frac{m^*}{l}, \frac{l^*}{m}, \frac{m^*}{m}\}$.

Proof. Let $\psi \in C_0^\infty(\mathbb{R}^N)$ such that

$$\psi(x) = 1 \text{ in } B_{1/2}(0), \quad \operatorname{supp}\psi \subset B_1(0), \quad 0 \leq \psi(x) \leq 1\ \forall x \in \mathbb{R}^N.$$

For each $j \in \Gamma$ and $\epsilon > 0$, let us define

$$\psi_\epsilon(x) = \psi\big(\frac{x - x_j}{\epsilon}\big), \quad \forall x \in \mathbb{R}^N.$$

Then $\{\psi_\epsilon u_n\}$ is bounded in $W_0^{1,\Phi}(\Omega)$. Since $J'_\lambda(u_n) \to 0$, we have

$$J'_\lambda(u_n)(\psi_\epsilon u_n) = o_n(1),$$

or equivalently,

$$\begin{aligned} \int_\Omega \phi(|\nabla u_n|)\nabla u_n \nabla\,(u_n\psi_\epsilon) &= o_n(1) + \lambda \int_\Omega \phi_*(|u_n|) u_n^2 \psi_\epsilon + \int_\Omega f(x, u_n) u_n \psi_\epsilon \\ &\leq o_n(1) + \lambda m^* \int_\Omega \Phi_*(u_n)\psi_\epsilon + \int_\Omega f(x, u_n) u_n \psi_\epsilon. \end{aligned} \tag{3.12}$$

Using that

$$\lim_{t\to+\infty} \frac{f(x,t)t\psi_\epsilon(x)}{\Phi_*(t)} = 0, \quad \text{uniformly in } x \in \overline{\Omega}$$

and that $\lim_{n\to+\infty} f(x, u_n)u_n\psi_\epsilon = 0$ a.e. on $\overline{\Omega}$, we have by compactness Lemma of Strauss [8] (note that this result is still true when we replace $\mathbb{R}^N$ by $\overline{\Omega}$)

$$\lim_{n\to\infty} \int_\Omega f(x, u_n) u_n \psi_\epsilon = \int_\Omega f(x, u) u \psi_\epsilon. \tag{3.13}$$

On the other hand, by (1.3)

$$
\begin{aligned}
\int_\Omega \phi(|\nabla u_n|)\nabla u_n \nabla(u_n\psi_\epsilon) &= \int_\Omega \phi(|\nabla u_n|)|\nabla u_n|^2\psi_\epsilon + \int_\Omega \phi(|\nabla u_n|)(\nabla u_n\nabla\psi_\epsilon)u_n \\
&\geq l\int_\Omega \Phi(|\nabla u|)\psi_\epsilon + \int_\Omega \phi(|\nabla u_n|)(\nabla u_n\nabla\psi_\epsilon)u_n.
\end{aligned}
\tag{3.14}
$$

By Lemmas 2.2 and 2.6, the sequence $\{|\phi(|\nabla u_n|)\nabla u_n|_{\widetilde{\Phi}}\}$ is bounded. Thus, there is a subsequence $\{u_n\}$ such that

$$
\phi(|\nabla u_n|)\nabla u_n \rightharpoonup \widetilde{w}_1 \quad \text{weakly in } L_{\widetilde{\Phi}}(\Omega,\mathbb{R}^N),
$$

for some $\widetilde{w}_1 \in L_{\widetilde{\Phi}}(\Omega,\mathbb{R}^N)$. Since $u_n \to u$ in $L_\Phi(\Omega)$,

$$
\int_\Omega \phi(|\nabla u_n|)(\nabla u_n\nabla\psi_\epsilon)u_n \to \int_\Omega (\widetilde{w}_1\nabla\psi_\epsilon)u.
$$

Thus, combining (3.12), (3.13), (3.14), and letting $n\to\infty$, we have

$$
l\int_\Omega \psi_\epsilon d\mu + \int_\Omega (\widetilde{w}_1\nabla\psi_\epsilon)u \leq \lambda m^*\int_\Omega \psi_\epsilon d\nu + \int_\Omega f(x,u)u\psi_\epsilon. \tag{3.15}
$$

Now we show that the second term of the left-hand side converges 0 as $\epsilon\to 0$.
Claim 1: $\{f(x,u_n)\}$ is bounded in $L_{\widetilde{\Phi}_*}(\Omega)$. In fact, by (F1) and Lemma 2.2 we have

$$
\begin{aligned}
\int_\Omega \widetilde{\Phi}_*(f(x,u_n)) &\leq c_1\int_\Omega \widetilde{\Phi}_*(\phi_*(|u_n|)u_n) + c_2\int_{[|u_n|>1]} \widetilde{\Phi}_*(\phi(|u_n|)u_n) \\
&\quad + c_3\int_{[|u_n|\leq 1]} \widetilde{\Phi}_*(\phi(|u_n|)u_n) \\
&\leq c_1\int_\Omega \Phi_*(u_n) + c_2\int_{[|u_n|>1]} \widetilde{\Phi}_*(\phi(|u_n|)u_n) + c_3|\Omega|.
\end{aligned}
$$

Hence, by (1.3), Lemma 2.3 and $m<l^*$,

$$
\begin{aligned}
\int_\Omega \widetilde{\Phi}_*(f(x,u_n)) &\leq C_1\int_\Omega \widetilde{\Phi}_*(\phi_*(|u_n|)u_n) + C_2\int_{[|u_n|>1]} \widetilde{\Phi}_*(|u_n|^{m-1}) + C_3|\Omega| \\
&\leq C_1\int_\Omega \Phi_*(u_n) + C_2\int_{[|u_n|>1]} \widetilde{\Phi}_*(|u_n|^{l^*-1}) + C_3|\Omega|.
\end{aligned}
$$

Now, by Lemmas 2.4 and 2.2,

$$
\int_\Omega \widetilde{\Phi}_*(f(x,u_n)) \leq K_1\int_\Omega \Phi_*(u_n) + K_2|\Omega| < +\infty.
$$

From Claim 1, there is a subsequence $\{u_n\}$ such that

$$
\phi_*(|u_n|)u_n + f(x,u_n) \rightharpoonup \widetilde{w}_2 \quad \text{weakly in } L_{\widetilde{\Phi}_*}(\Omega),
$$

for some $\widetilde{w}_2 \in L_{\widetilde{\Phi}_*}(\Omega)$. Since

$$
J'_\lambda(u_n)v = \int_\Omega \phi(|\nabla u_n|)\nabla u_n\nabla v - \int_\Omega (\phi_*(|u_n|)u_n + f(x,u_n))\,v \to 0,
$$

as $n\to\infty$ for any $v\in W_0^{1,\Phi}(\Omega)$,

$$
\int_\Omega (\widetilde{w}_1\nabla v - \widetilde{w}_2 v) = 0,
$$

for any $v \in W_0^{1,\Phi}(\Omega)$. Substituting $v = u\psi_\epsilon$ we have

$$\int_\Omega (\widetilde{w}_1 \nabla(u\psi_\epsilon) - \widetilde{w}_2 u\psi_\epsilon) = 0.$$

Namely,

$$\int_\Omega (\widetilde{w}_1 \nabla\psi_\epsilon)u = -\int_\Omega (\widetilde{w}_1 \nabla u - \widetilde{w}_2 u)\psi_\epsilon.$$

Noting $\widetilde{w}_1 \nabla u - \widetilde{w}_2 u \in L^1(\Omega)$, we see that right-hand side tends to 0 as $\epsilon \to 0$. Hence we have

$$\int_\Omega (\widetilde{w}_1 \nabla\psi_\epsilon)u \to 0,$$

as $\epsilon \to 0$.

Letting $\epsilon \to 0$ in (3.15), we obtain $l\mu_j \le \lambda m^* \nu_j$. Hence,

$$S_N^{-\alpha}\nu_j \le \mu_j^\beta \le \big(\frac{l\lambda}{m^*}\big)^\beta \nu_j^\beta,$$

for some $\alpha \in \{l^*, m^*\}$, $\beta \in \{\frac{l^*}{l}, \frac{m^*}{l}, \frac{l^*}{m}, \frac{m^*}{m}\}$, and so

$$\nu_j \ge \big(\frac{l}{\lambda m^*}\big)^{\frac{\beta}{\beta-1}} S_N^{-\frac{\alpha}{\beta-1}} \quad \text{or} \quad \nu_j = 0.$$

□

Lemma 3.6. *Assume that* (F1)–(F2)*. Then, J_λ satisfies $(PS)_d$ for $d \in (0, d_\lambda)$ where*

$$d_\lambda = \min\Big\{\frac{l^* - \theta}{\theta S_N^{\frac{\alpha}{\beta-1}} \lambda^{\frac{1}{\beta-1}}}\big(\frac{l}{m^*}\big)^{\frac{\beta}{\beta-1}}; \alpha \in \{l^*, m^*\},\ \beta \in \{\frac{l^*}{l}, \frac{m^*}{l}, \frac{l^*}{m}, \frac{m^*}{m}\}\Big\}.$$

Proof. Using that $J_\lambda(u_n) = d + o_n(1)$ and $J'_\lambda(u_n) = o_n(1)$, we have

$$\begin{aligned}
d = \lim_{n\to\infty} I(u_n) &= \lim_{n\to\infty} \big(J_\lambda(u_n) - \frac{1}{\theta} J'_\lambda(u_n)u_n\big) \\
&\ge \lim_{n\to\infty} \Big[(1 - \frac{m}{\theta}) \int_\Omega \Phi(|\nabla u_n|) + \lambda(\frac{l^*}{\theta} - 1) \int_\Omega \Phi_*(u_n) \\
&\quad - \int_\Omega \big(F(x, u_n) - \frac{1}{\theta} f(x, u_n)u_n\big)\Big] \\
&\ge \lambda(\frac{l^*}{\theta} - 1) \int_\Omega \Phi_*(u_n).
\end{aligned}$$

Recalling that

$$\lim_{n\to\infty} \int_\Omega \Phi_*(u_n)dx = \Big[\int_\Omega \Phi_*(u) + \sum_{j\in\mathcal{J}} \nu_j\Big] \ge \nu_j,$$

we derive that

$$\begin{aligned}
d &\ge \lambda\big(\frac{l^*}{\theta} - 1\big)\big(\frac{l}{\lambda m^*}\big)^{\frac{\beta}{\beta-1}} S_N^{-\frac{\alpha}{\beta-1}} \\
&= \big(\frac{l^* - \theta}{\theta}\big)\big(\frac{l}{m^*}\big)^{\frac{\beta}{\beta-1}} S_N^{-\frac{\alpha}{\beta-1}} \lambda^{\frac{1}{1-\beta}},
\end{aligned}$$

for some $\alpha \in \{l^*, m^*\}$, $\beta \in \{\frac{l^*}{l}, \frac{m^*}{l}, \frac{l^*}{m}, \frac{m^*}{m}\}$, which is an absurd. From this, we must have $\nu_j = 0$ for any $j \in \mathcal{J}$, leading to

$$\int_\Omega \Phi_*(u_n) \to \int_\Omega \Phi_*(u). \tag{3.16}$$

Combining the last limit with Brézis and Lieb [6], we obtain

$$\int_\Omega \Phi_*(u_n-u) \to 0 \text{ as } \quad n\to\infty,$$

from where it follows by Lemma 2.5

$$u_n \to u \text{ in } L_{\Phi_*}(\Omega).$$

Now, as $J'_\lambda(u_n)u_n = o_n(1)$, the last limit gives

$$\int_\Omega \phi(|\nabla u_n|)|u_n|^2 = \lambda\int_\Omega \phi_*(|u_n|)u_n^2 + \int_\Omega f(x,u_n)u_n + o_n(1).$$

In what follows, let us denote by $\{P_n\}$ the sequence

$$P_n(x) = \langle \phi(|\nabla u_n(x)|)\nabla u_n(x) - \phi(|\nabla u(x)|)\nabla u(x), \nabla u_n(x)-\nabla u(x)\rangle.$$

Since Φ is convex in $\mathbb{R}$ and $\Phi(|.|)$ is C^1 class in $\mathbb{R}^N$, has $P_n(x)\geq 0$. From definition of $\{P_n\}$,

$$\int_\Omega P_n = \int_\Omega \phi(|\nabla u_n|)|\nabla u_n|^2 - \int_\Omega \phi(|\nabla u_n|)\nabla u_n\nabla u - \int_\Omega \phi(|\nabla u|)\nabla u\nabla(u_n-u).$$

Recalling that $u_n \rightharpoonup u$ in $W_0^{1,\Phi}(\Omega)$, we have

$$\int_\Omega \phi(|\nabla u|)\nabla u\nabla(u_n-u) \to 0 \quad \text{as } n\to\infty, \tag{3.17}$$

which implies that

$$\int_\Omega P_n = \int_\Omega \phi(|\nabla u_n|)|\nabla u_n|^2 - \int_\Omega \phi(|\nabla u_n|)\nabla u_n\nabla u + o_n(1).$$

On the other hand, from $J'_\lambda(u_n)u_n = o_n(1)$ and $J'_\lambda(u_n)u = o_n(1)$, we derive

$$\begin{aligned} 0\leq \int_\Omega P_n &= \lambda\int_\Omega \phi_*(|u_n|)|u_n|^2 - \lambda\int_\Omega \phi_*(|u_n|)u_n u \\ &\quad + \int_\Omega f(x,u_n)u_n - \int_\Omega f(x,u_n)u + o_n(1). \end{aligned}$$

Combining (3.16) with the compactness Lemma of Strauss [8], we deduce that

$$\int_\Omega P_n \to 0 \quad \text{as } n\to\infty.$$

Using that Φ is convex, from a result due to Dal Maso and Murat [18], it follows that

$$\nabla u_n(x) \to \nabla u(x) \text{ a.e. } \Omega.$$

Now, using Lebesgue's Theorem,

$$\int_\Omega \Phi(|\nabla u_n - \nabla u|)dx \to 0,$$

which shows that

$$u_n \to u \text{ in } W_0^{1,\Phi}(\Omega). \tag{3.18}$$

□

The next lemma is similar to [25, Lemma 5] and its proof will be omitted.

Lemma 3.7. *Under conditions* (F1)–(F2)*, there is sequence* $\{M_m\} \subset (0, +\infty)$ *independent of* λ *with* $M_m \leq M_{m+1}$*, such that for any* $\lambda > 0$*,*

$$c_m^\lambda = \inf_{K \in \Gamma_m} \max_{u \in K} J_\lambda(u) < M_m. \tag{3.19}$$

Proof of Theorem 1.1. For each $k \in \mathbb{N}$, choose λ_k such that $M_k < d_{\lambda_k}$. Thus, for $\lambda \in (\lambda_{k+1}, \lambda_k)$,

$$0 < c_1^\lambda \leq c_2^\lambda \leq \cdots \leq c_k^\lambda < M_k \leq d_\lambda.$$

By Theorem 3.1, the levels $c_1^\lambda \leq c_2^\lambda \leq \cdots \leq c_k^\lambda$ are critical values of J_λ. Thus, if

$$c_1^\lambda < c_2^\lambda < \cdots < c_k^\lambda,$$

the functional J_λ has at least k critical points. Now, if $c_j^\lambda = c_{j+1}^\lambda$ for some $j = 1, 2, \dots, k$, it follows from Theorem 3.1 that $K_{c_j^\lambda}$ is an infinite set [24, Cap. 7]. Then, in this case, problem (1.1) has infinitely many solutions. □

References

[1] A. Adams, J. F. Fournier; *Sobolev spaces*, 2nd ed., Academic Press, (2003).

[2] A. Adams, L. I. Hedberg; *Function spaces and Potential Theory*, Grundlehren der Mathematischen Wissenschaften, vol. 314, Springer-Verlag, Berlin, 1996.

[3] C. O. Alves, J. L. P. Barreiro; *Existence and multiplicity of solutions for a $p(x)$-Laplacian equation with critical growth*, J. Math. Anal. Appl. (Print), 2013.

[4] C. O. Alves, J. V. Gonçalves, J. A. Santos; *Strongly Nonlinear Multivalued Elliptic Equations on a Bounded Domain*, J Glob Optim, DOI 10.1007/s10898-013-0052-3, 2013.

[5] G. Bonano, G. M. Bisci, V. Radulescu; *Quasilinear elliptic non-homogeneous Dirichlet problems through Orlicz-Sobolev spaces*, Nonl. Anal. 75 (2012), 4441-4456.

[6] H. Brezis, E. Lieb; *A relation between pointwise convergence of functions and convergence of functinals*, Proc. Amer. Math. Soc. 88 (1983), 486-490.

[7] R. Cerný; *Generalized Moser-Trudinger inequality for unbounded domains and its application,* Nonlinear Differ. Equ. Appl. DOI 10.1007/s00030-011-0143-0.

[8] J. Chabrowski; *Weak covergence methods for semilinear elliptic equations.* World Scientific Publishing Company, 1999.

[9] Ph. Clément, M. Garcia-Huidobro, R. Manásevich, K. Schmitt; *Mountain pass type solutions for quasilinear elliptic equations,* Calc. Var. 11 (2000), 33-62.

[10] T. K. Donaldson; *Nonlinear elliptic boundary value problems in Orlicz-Sobolev spaces,* J. Diff. Equat. 10 (1971), 507-528.

[11] T. K. Donaldson, N. S. Trudinger; *Orlicz-Sobolev spaces and imbedding theorems,* J. Funct. Anal. 8 (1971) 52-75.

[12] M. Fuchs, G. Li; *Variational inequalities for energy functionals with nonstandard growth conditions*, Abstr. Appl. Anal. 3 (1998), 405-412.

[13] M. Fuchs, V. Osmolovski; *Variational integrals on Orlicz-Sobolev spaces.* Z. Anal. Anwendungen 17, 393-415 (1998) 6.

[14] N. Fukagai, M. Ito, K. Narukawa; *Positive solutions of quasilinear elliptic equations with critical Orlicz-Sobolev nonlinearity on* $\mathbb{R}^N$, Funkcialaj Ekvacioj 49 (2006), 235-267.

[15] N. Fukagai, M. Ito, K. Narukawa; *Quasilinear elliptic equations with slowly growing principal part and critical Orlicz-Sobolev nonlinear term*, Proc. R. S. Edinburgh 139 A (2009), 73-106.

[16] N. Fukagai and K. Narukawa; *On the existence of multiple positive solutions of quasilinear elliptic eigenvalue problems*, Annali di Matematica, 186, 539-564, 2007.

[17] J. P. Gossez; *Nonlinear elliptic boundary value problem for equations with rapidly (or slowly) increasing coefficients,* Trans. Am. Math. Soc. 190 (1974), 163-205.

[18] G. Dal Maso, F. Murat; *Almost everywhere convergence of gradients of solutions to nonlinear elliptic systems*, Nonlinear Anal. 31 (1998), 405-412.

[19] M. Mihailescu, V. Radulescu; *Nonhomogeneous Neumann problems in Orlicz-Sobolev spaces*, C .R. Acad. Sci. Paris, Ser. I 346 (2008), 401-406.

[20] M. Mihailescu, V. Radulescu; *Existence and multiplicity of solutions for a quasilinear nonhomogeneous problems: An Orlicz-Sobolev space setting*, J. Math. Anal. Appl. 330 (2007), 416-432.

[21] M. Mihailescu, D. Repovs; *Multiple solutions for a nonlinear and non-homogeneous problems in Orlicz-Sobolev spaces,* Appl. Math. Comput. 217 (2011), 6624-6632.
[22] S. L. Pohozaev; *Eingenfunctions for the equation* $\Delta u + \lambda f(u) = 0$, Soviet Math. Dokl. 6 (1965) 1408-1411.
[23] R. O'Neill; *Fractional integration in Orlicz spaces*, Trans. Amer. Math. Soc. 115 (1965) 300-328.
[24] P. H. Rabinowitz; *Minimax methods in critical point theory with applications to diferential equations.* CBMS Reg. Conf. series in math 65, 1984.
[25] W. Zhihui, W. Xinmin; *A multiplicity result for quasilinear elliptic equations involving critical sobolev exponents.* Nonlinear Analysis, Theory, Methods and Applications 18 (1992), 559–567.

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