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ASYMPTOTIC BEHAVIOR OF POSITIVE SOLUTIONS OF THE NONLINEAR DIFFERENTIAL EQUATION $t^2u'' = u^n$

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Abstract. In this article we study properties of positive solutions of the ordinary differential equation $t^2u'' = u^n$ for $1 < n \in \mathbb{N}$, we obtain conditions for their blow-up in finite time, and some properties for global solutions. Equations containing more general nonlinear terms are also considered.

1. Introduction

Some interesting results on the blow-up, blow-up rates, and estimates for the life-span of solutions of the Emden-Fowler equation and the semi-linear wave equation $\Box u + f(u) = 0$ have been obtained, as shown in the references.

Here we wish to study the Emden-Fowler type wave equation, i.e. solutions, independent of the space variable x, of the equation $t^2u_{tt} - \Delta u = u^n$ for $n > 1$.

The existence and uniqueness of local solutions of the initial-value problem

$$\begin{gathered} t^2u'' = u^n, \quad 1 < n \in \mathbb{N}, \\ u(1) = u_0, \quad u'(1) = u_1, \end{gathered} \tag{1.1}$$

follow by standard arguments. Considering the transformation $t = e^s$, $u(t) = v(s)$, we have $t^2u''(t) = -v_s(s) + v_{ss}(s)$, $v(s)^n = -v_s(s) + v_{ss}(s)$ and $v(0) = u(1) = u_0$; $v_s(0) = u'(1) = u_1$, the problem (1.1) can be transformed into

$$\begin{gathered} v_{ss}(s) - v_s(s) = v^n(s), \\ v(0) = u_0, \quad v_s(0) = u_1. \end{gathered} \tag{1.2}$$

Thus, the existence of local solutions u for (1.1) in $(1, T)$ is equivalent to the existence of local solutions v for (1.2) in $(0, \ln T)$. In this article, we give estimates for the life-span T^* of positive solutions u of (1.1) in three different cases. The main results are as follows:

- (a) $u_1 = 0$, $u_0 > 0$: $T^* \le e^{k_1}$, $k_1 := s_0 + \frac{2(n+3)}{8-\epsilon}\frac{2}{n-1}v(s_0)^{\frac{1-n}{2}}$, $\varepsilon \in (0,1)$;
- (b) $u_1 > 0$, $u_0 > 0$:
  - (i) $E(0) \ge 0$, $T^* \le e^{k_2}$, $k_2 := \frac{2}{n-1}\sqrt{\frac{n+1}{2}}u_0^{\frac{1-n}{2}}$;
  - (ii) $E(0) < 0$, $T^* \le e^{k_3}$, $k_3 := \frac{2}{n-1}\frac{u_0}{u_1}$;
- (c) $u_1 < 0$, $u_0 \in (0, (-u_1)^{\frac{1}{n}})$: $u(t) \le (u_0 - u_1 - u_0^n) + (u_1 + u_0^n)t - u_0^n \ln t$.

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where $E(0)$ is defined in the next section and s_0 is given by (3.3).

In Section 6, we replace the nonlinear term v^n by a more general increasing function $f(v)$.

2. Notation and some lemmas

For a given function v, we use the following functions

$$a(s) := v(s)^2, \quad E(0) := u_1^2 - \frac{2}{n+1} u_0^{n+1}, \quad J(s) := a(s)^{-\frac{n-1}{4}},$$

where u_0 and u_1 are the given initial conditions.

By an easy calculation we can obtain the following two Lemmas; we shall omit the proof of the first lemma.

Lemma 2.1. *Suppose that $v \in C^2[0,T]$ is the solution of (1.2), then*

$$E(s) = v_s(s)^2 - 2\int_0^s v_s(r)^2 dr - \frac{2}{n+1} v(s)^{n+1} = E(0), \tag{2.1}$$

$$(n+3)v_s(s)^2 = (n+1)E(0) + a''(s) - a'(s) + 2(n+1)\int_0^s v_s(r)^2 dr, \tag{2.2}$$

$$J''(s) = \frac{n^2-1}{4} J(s)^{\frac{n+3}{n-1}} (E(0) - \frac{a'(s)}{n+1} + 2\int_0^s v_s(r)^2 dr), \tag{2.3}$$

$$\begin{aligned} J'(s)^2 &= J'(0)^2 + \frac{(n-1)^2}{4} E(0)(J(s)^{\frac{2(n+1)}{n-1}} - J(0)^{\frac{2(n+1)}{n-1}}) \\ &\quad + \frac{(n-1)^2}{2} J(s)^{\frac{2(n+1)}{n-1}} \int_0^s v_s(r)^2 dr. \end{aligned} \tag{2.4}$$

Lemma 2.2. *For $u_0 > 0$, the positive solution v of (1.2) satisfies:*

$$\text{(i) If } u_1 \geq 0, \text{ then } v_s(s) > 0 \text{ for all } s > 0 \tag{2.5}$$

$$\text{(ii) If } u_1 < 0,\ u_0 \in (0, (-u_1)^{\frac{1}{n}}), \text{ then } v_s(s) < 0 \text{ for all } s > 0. \tag{2.6}$$

Proof. (i) Since $v_{ss}(0) = u_1 + u_0^n > 0$, we know that $v_{ss}(s) > 0$ in $[0, s_1)$ and $v_s(s)$ is increasing in $[0, s_1)$ for some $s_1 > 0$. Moreover, since v and v_s are increasing in $[0, s_1)$, $v_{ss}(s_1) = v_s(s_1) + v(s_1)^n > v_s(0) + v(0)^n > 0$ for all $s \in [0, s_1)$ and $v_s(s_1) > v_s(s) > 0$ for all $s \in [0, s_1)$, we know that there exists a positive number $s_2 > 0$, such that $v_s(s) > 0$ for all $s \in [0, s_1 + s_2)$. Continuing this process, we obtain $v_s(s) > 0$ for all $s > 0$, for which the solution exists.

(ii) Since $v_{ss}(0) = v_s(0) + v(0)^n = u_1 + u_0^n < 0$, there exists a positive number $s_1 > 0$ such that $v_{ss}(s) < 0$ in $[0, s_1), v_s(s)$ is decreasing in $[0, s_1)$; therefore, $v_s(s) < v_s(0) = u_1 < 0$ for all $s \in [0, s_1)$ and $v(s)$ is decreasing in $[0, s_1)$. Moreover, since v and v_s are decreasing in $[0, s_1)$, $v_{ss}(s) = v_s(s) + v(s)^n < v_s(0) + v(0)^n < 0$ for all $s \in [0, s_1)$ and $v_s(s_1) < v_s(s) < 0$ for all $s \in [0, s_1)$, we know that there exists a positive number $s_2 > 0$, such that $v_s(s) < 0$ for all $s \in [0, s_1 + s_2)$. Continuing this process, we obtain $v_s(s) < 0$ for all $s > 0$ in the interval of existence. □

3. Life-span of positive solutions of (1.1) when $u_1 = 0$, $u_0 > 0$

In this section we want to estimate the life-span of a positive solution u of (1.1) if $u_1 = 0$, $u_0 > 0$. Here the life-span T^* of u means that u is the solution of equation (1.1) and u exists only in $[0, T^*)$ so that the problem (1.1) possesses a positive solution $u \in C^2[0, T^*)$.

Theorem 3.1. *For $u_1 = 0$, $u_0 > 0$, the positive solution u of* (1.1) *blows up in finite time; that is, there exists $T^* < \infty$ so that*
$$u(t)^{-1} \to 0 \quad \text{as } t \to T^*.$$

Proof. By (2.5), we know that $v_s(s) > 0$, $a'(s) > 0$ for all $s > 0$ provided that $u_1 = 0$, $u_0 > 0$. By Lemma 2.1,
$$\begin{gathered} a''(s) - a'(s) = 2(v_s(s)^2 + v(s)^{n+1}), \\ (a'(s)e^{-s})' = e^{-s}(a''(s) - a'(s)) = 2e^{-s}(v_s(s)^2 + v(s)^{n+1}), \\ a'(s)e^{-s} = 2\int_0^s e^{-r}(v_s(r)^2 + v(r)^{n+1})dr \geq 4\int_0^s e^{-r}v_s(r)v(r)^{\frac{n+1}{2}}dr, \end{gathered}$$
and $a'(0) = 0$, hence we have
$$\begin{aligned} a'(s)e^{-s} &\geq \frac{8}{n+3}(v(r)^{(n+3)/2}e^{-r}\mid_{r=0}^{s} + \int_0^s v(r)^{(n+3)/2}e^{-r}dr) \\ &= \frac{8}{n+3}(v(s)^{(n+3)/2}e^{-s} - v(0)^{(n+3)/2}) + \frac{8}{n+3}\int_0^s v(r)^{(n+3)/2}e^{-r}dr. \end{aligned}$$
Since $a'(s) > 0$ for all $s > 0$, v is increasing on $(0, \infty)$ and
$$\begin{gathered} a'(s)e^{-s} \geq \frac{8}{n+3}(v(s)^{(n+3)/2}e^{-s} - v(0)^{(n+3)/2}) + \frac{8}{n+3}\int_0^s v(0)^{(n+3)/2}e^{-r}dr \\ = \frac{8}{n+3}(v(s)^{(n+3)/2}e^{-s} - v(0)^{(n+3)/2}) + \frac{8}{n+3}v(0)^{(n+3)/2}(1 - e^{-s}), \\ a'(s) \geq \frac{8}{n+3}(v(s)^{(n+3)/2} - v(0)^{(n+3)/2}) = \frac{8}{n+3}(v(s)^{(n+3)/2} - u_0^{(n+3)/2}). \end{gathered} \tag{3.1}$$
Using $u_1 = 0$ and integrating (1.2), we obtain
$$\begin{gathered} v_s(s) = v(s) - u_0 + \int_0^s v(r)^n dr, \\ v_s(s) \geq v(s) - u_0 + \int_0^s v(0)^n dr = v(s) - u_0 + u_0^n s, \\ (e^{-s}v(s))_s = e^{-s}(v_s(s) - v(s)) \geq e^{-s}(u_0^n s - u_0), \\ a'(s) \geq \frac{8}{n+3}(v(s)^{(n+3)/2} - v(0)^{(n+3)/2}) = \frac{8}{n+3}(v(s)^{(n+3)/2} - u_0^{(n+3)/2}). \end{gathered} \tag{3.2}$$
According to (3.2), and since $v'(s) > 0$, $v(s)^{(n+3)/2} \geq (u_0 + u_0^n(e^s - 1 - s))^{(n+3)/2}$, for all $\epsilon \in (0, 1)$, we obtain
$$\begin{aligned} \epsilon v(s)^{(n+3)/2} &\geq \epsilon(u_0 + u_0^n(e^s - 1 - s))^{(n+3)/2}, \\ \epsilon v(s)^{(n+3)/2} - 8u_0^{(n+3)/2} &\geq \epsilon(u_0 + u_0^n(e^s - 1 - s))^{(n+3)/2} - 8u_0^{(n+3)/2} \\ &\geq \epsilon(u_0^{(n+3)/2} + u_0^{\frac{n(n+3)}{2}}(e^s - 1 - s)^{(n+3)/2}) - 8u_0^{\frac{n+3}{2}} \\ &= (\epsilon - 8)u_0^{(n+3)/2} + \epsilon u_0^{\frac{n(n+3)}{2}}(e^s - 1 - s)^{(n+3)/2}. \end{aligned}$$
Now, we want to find a number $s_0 > 0$ such that
$$e^{s_0} - s_0 = 1 + \big(\frac{8-\epsilon}{\epsilon}u_0^{\frac{n+3}{2}(1-n)}\big)^{2/(n+3)}. \tag{3.3}$$

This means that there exists a number $s_0 > 0$ satisfying (3.3) with $\epsilon \in (0,1)$ such that

$$\epsilon v(s)^{(n+3)/2} - 8u_0^{(n+3)/2} \geq 0 \quad \text{for all } s \geq s_0.$$

From (3.1), it follows that

$$\begin{aligned} a'(s) &\geq \frac{8}{n+3} v(s)^{(n+3)/2} - \frac{8}{n+3} u_0^{(n+3)/2} \\ &= \frac{8-\epsilon}{n+3} v(s)^{(n+3)/2} + \frac{\epsilon v(s)^{(n+3)/2} - 8u_0^{(n+3)/2}}{n+3} \\ &\geq \frac{8-\epsilon}{n+3} v(s)^{(n+3)/2}, \quad \text{for all } s \geq s_0. \end{aligned}$$

For all $s \geq s_0$, $\epsilon \in (0,1)$, we obtain that

$$\begin{gathered} 2v(s)v_s(s) \geq \frac{8-\epsilon}{n+3} v(s)^{(n+3)/2}, \\ v(s)^{-\frac{n+1}{2}} v_s(s) \geq \frac{8-\epsilon}{2(n+3)}, \\ \frac{2}{1-n} (v(s)^{\frac{1-n}{2}})_s \geq \frac{8-\epsilon}{2(n+3)} \end{gathered}$$

and hence

$$(v(s)^{\frac{1-n}{2}})_s \leq \frac{8-\epsilon}{2(n+3)} \frac{1-n}{2}.$$

Integrating the above inequality, we conclude that

$$v(s)^{\frac{1-n}{2}} \leq v(s_0)^{\frac{1-n}{2}} - \frac{8-\epsilon}{2(n+3)} \frac{n-1}{2} (s - s_0).$$

Thus, there exists a number

$$s_1^* \leq s_0 + \frac{2(n+3)}{8-\epsilon} \frac{2}{n-1} v(s_0)^{\frac{1-n}{2}} =: k_1$$

such that $v(s)^{-1} \to 0$ for $s \to s_1^*$, that is, $u(t)^{-1} \to 0$ as $t \to e^{k_1}$, which implies that the life-span T^* of a positive solution u is finite and $T^* \leq e^{k_1}$. □

4. Life-span of positive solutions of (1.1) when $u_1 > 0$, $u_0 > 0$

In this section we estimate the life-span of a positive solution u of (1.1) whenever $u_1 > 0$, $u_0 > 0$.

Theorem 4.1. *For $u_1 > 0$, $u_0 > 0$, the positive solution u of* (1.1) *blows up in finite time; that is, there exists a number $T^* < \infty$ so that*

$$u(t)^{-1} \to 0 \quad \text{as } t \to T^*.$$

Proof. We separate the proof into two parts depending on whether $E(0) \geq 0$ or $E(0) < 0$.

(i) Assume that $E(0) \geq 0$. By (2.1) and (2.5) we have

$$v_s(s)^2 - \frac{2}{n+1} v(s)^{n+1} \geq E(0),$$

$$v_s(s)^2 \geq \frac{2}{n+1} v(s)^{n+1} + E(0), v_s(s) \geq \sqrt{\frac{2}{n+1} v(s)^{n+1} + E(0)}.$$

Since $E(0) \geq 0$, we obtain

$$
\begin{gathered}
v_s(s) \geq \sqrt{\frac{2}{n+1}} v(s)^{\frac{n+1}{2}}, \\
v(s)^{-\frac{n+1}{2}} \cdot v_s(s) \geq \sqrt{\frac{2}{n+1}}, \\
(v(s)^{\frac{1-n}{2}})_s \leq \frac{1-n}{2} \sqrt{\frac{2}{n+1}}.
\end{gathered}
$$

Integrating the above inequality, we obtain

$$
v(s)^{\frac{1-n}{2}} \leq u_0^{\frac{1-n}{2}} + \frac{1-n}{2} \sqrt{\frac{2}{n+1}} s.
$$

Thus, there exists

$$
s_2^* \leq \frac{2}{n-1} \sqrt{\frac{n+1}{2}} u_0^{\frac{1-n}{2}} =: k_2
$$

such that $v(s)^{-1} \to 0$ for $s \to s_2^*$; that is, $u(t)^{-1} \to 0$ as $t \to e^{k_2}$, which means that the life-span T^* of a positive solution u is finite and $T^* \leq e^{k_2}$.

(ii) Assume that $E(0) < 0$. From (2.1) and (2.5) we obtain that $J'(s) = -\frac{n-1}{4} a(s)^{-\frac{n+3}{4}} a'(s)$, $a'(s) > 0$, $v_s(s) > 0$ for all $s > 0$ and

$$
\begin{aligned}
J'(s) &= -\frac{n-1}{2} \sqrt{\frac{2}{n+1} + E(0) a(s)^{-\frac{n+1}{2}} + 2a(s)^{-\frac{n+1}{2}} \int_0^s v_s(r)^2 dr} \\
&\leq -\frac{n-1}{2} \sqrt{\frac{2}{n+1} + E(0) a(s)^{-\frac{n+1}{2}}}, \\
J(s) &\leq J(0) - \frac{n-1}{2} \int_0^s \sqrt{\frac{2}{n+1} + E(0) a(r)^{-\frac{n+1}{2}}} dr.
\end{aligned}
$$

Since $E(0) < 0$ and $a'(s) > 0$ for all $s > 0$,

$$
\begin{aligned}
J(s) &\leq J(0) - \frac{n-1}{2} \int_0^s \sqrt{\frac{2}{n+1} + E(0) a(0)^{-\frac{n+1}{2}}} dr \\
&= a(0)^{-\frac{n-1}{4}} - \frac{n-1}{2} \sqrt{\frac{2}{n+1} + E(0) a(0)^{-\frac{n+1}{2}}} s.
\end{aligned}
$$

Thus, there exists a number

$$
s_3^* \leq \frac{2}{n-1} a(0)^{-\frac{n-1}{4}} (\frac{2}{n+1} + E(0) a(0)^{-\frac{n+1}{2}})^{-\frac{1}{2}} =: k_3
$$

such that $J(s_3^*) = 0$ and $a(s)^{-1} \to 0$ for $s \to s_3^*$; that is, $u(t)^{-1} \to 0$ as $t \to e^{k_3}$. This means that the life-span T^* of u is finite and $T^* \leq e^{k_3}$. $\square$

5. Life-span of positive solutions of (1.1) when $u_1 < 0$

Finally, we estimate the life-span of a positive solution u of (1.1) when $u_1 < 0$.

Theorem 5.1. *For $u_1 < 0$, $u_0 \in (0, (-u_1)^{\frac{1}{n}})$ we have*

$$
u(t) \leq (u_0 - u_1 - u_0^n) + (u_1 + u_0^n) t - u_0^n \ln t,
$$

and in particular, if $E(0) \ge 0$, we have

$$u(t) \le (u_0^{\frac{1-n}{2}} + \frac{n-1}{2}\sqrt{\frac{2}{n+1}}\ln t)^{\frac{2}{1-n}}.$$

Proof. (i) By (1.2) and integrating this equation with respect to s, we get $v_s(s) = (u_1 - u_0) + v(s) + \int_0^s v(r)^n dr$. By (2.6), we have that v is decreasing and

$$v_s(s) \le (u_1 - u_0) + v(s) + \int_0^s v(0)^n dr = (u_1 - u_0) + v(s) + u_0^n s,$$

$$\begin{aligned} e^{-s}v(s) - u_0 &\le (u_1 - u_0)\int_0^s e^{-r}dr + u_0^n \int_0^s re^{-r}dr \\ &= (u_1 - u_0)(1 - e^{-s}) + u_0^n(-se^{-s} - e^{-s} + 1); \end{aligned}$$

that is,

$$\begin{aligned} u(t) &\le (u_0 - u_1) + u_1 t + u_0^n(t - 1 - \ln t) \\ &= (u_0 - u_1 - u_0^n) + (u_1 + u_0^n)t - u_0^n \ln t. \end{aligned}$$

(ii) If $E(0) \ge 0$, by (2.1), we have

$$\begin{gathered} v_s(s)^2 - \frac{2}{n+1}v(s)^{n+1} = E(0) + 2\int_0^s v_s(r)^2 dr \ge E(0), \\ v_s(s)^2 \ge E(0) + \frac{2}{n+1}v(s)^{n+1} \ge \frac{2}{n+1}v(s)^{n+1}. \end{gathered}$$

By (2.6), we obtain that $-v_s(s) \ge \sqrt{\frac{2}{n+1}}v(s)^{\frac{n+1}{2}}$, $\frac{2}{n-1}(v(s)^{\frac{1-n}{2}})_s \ge \sqrt{\frac{2}{n+1}}$ and

$$\begin{gathered} \sqrt{\frac{2}{n+1}}s \le \frac{2}{n-1}(v(s)^{\frac{1-n}{2}} - v(0)^{\frac{1-n}{2}}), \\ v(s)^{\frac{1-n}{2}} \ge (u_0^{\frac{1-n}{2}} + \frac{n-1}{2}\sqrt{\frac{2}{n+1}}s). \end{gathered}$$

Then, we know that

$$v(s) \le (u_0^{\frac{1-n}{2}} + \frac{n-1}{2}\sqrt{\frac{2}{n+1}}s)^{\frac{2}{1-n}}, \quad \text{for all } s \ge 0;$$

that is,

$$u(t) \le (u_0^{\frac{1-n}{2}} + \frac{n-1}{2}\sqrt{\frac{2}{n+1}}\ln t)^{\frac{2}{1-n}} \quad \text{for all } t \ge 1.$$

□

6. A generalization of Theorem 4.1

In this section we want to extent the blow-up result for the following generalization of (1.2),

$$\begin{gathered} v_{ss}(s) - v_s(s) = f(v), \\ v(0) = v_0, \quad v_s(0) = v_1, \end{gathered} \tag{6.1}$$

where f is an increasing continuous function with $f(0) = 0$. We have the following result.

Theorem 6.1. *Suppose that f is an increasing function with $f(0)=0$ and suppose v is a positive solution of (6.1). If $F(v) := \int_0^v f(r)dr$, then*

$$\bar{E}(s) := v_s(s)^2 - 2\int_0^s v_s(r)^2 dr - 2F(v(s)) \tag{6.2}$$

is constant. Furthermore, if there exists a positive constant k such that $F(s) \geq ks^{p+1}$, $p>1$ for all $s \geq 0$, and $v_1 > 0$, then the life span of v is finite.

Proof. By an argument similar to that used in proving (2.1), we easily obtain that $\bar{E}(s)$ is a constant. Since f is increasing, we have

$$vf(v) = (v-0)\cdot(f(v)-f(0)) \geq 0 \quad \text{for } v \geq 0,$$

thus

$$(v^2)_s - 2v^2(s) \geq 2v_0(v_1 - v_0) + 2\int_0^s v_s^2(r)dr. \tag{6.3}$$

By (6.2) and (6.3), we have $\bar{E}(s) = v_1^2 - 2F(v_0) := \bar{E}$, and

$$v^2(s) \geq v_0 v_1 e^{2s} - v_0(v_1 - v_0), \tag{6.4}$$

$$\begin{aligned} (v^2)_s - 2v^2(s) &\geq 2v_0(v_1 - v_0) + 2\int_0^s v_s^2(r)dr \\ &= 2v_0(v_1 - v_0) + 2\int_0^s (\bar{E} + 2F(v(r)) + 2\int_0^r v_s(\eta)^2 d\eta)dr \\ &\geq 2v_0(v_1 - v_0) + 2\bar{E}s + 4k\int_0^s v^{p+1}(r)dr \\ &\geq 2v_0(v_1 - v_0) + 2\bar{E}s + 4ks^{1-p}(\int_0^s v^2(r)dr)^{\frac{p+1}{2}}. \end{aligned} \tag{6.5}$$

Let $\int_0^s v^2(r)dr := b(s)$, $e^{-s}b(s) = B(s)$. Then

$$b(s)'' - 2b(s)' \geq 2v_0(v_1 - v_0) + 2\bar{E}s + 4ks^{1-p}b(s)^{\frac{p+1}{2}} \quad \text{for } s > 0$$

and by (6.5), we have

$$\begin{aligned} &(e^{-s}(b(s)' - b(s)))' \\ &= e^{-s}(b(s)'' - 2b(s)' + b(s)) \\ &\geq 2v_0(v_1 - v_0) + 2\bar{E}s + 4ks^{1-p}e^{-s}b^{\frac{p+1}{2}} + e^{-s}b(s) \\ &= 2v_0(v_1 - v_0) + 2\bar{E}s + 4ks^{1-p}(e^{-s}b(s))^{\frac{p+1}{2}}e^{\frac{p-1}{2}s} + e^{-s}b(s) \geq 0, \end{aligned} \tag{6.6}$$

$$\begin{aligned} (e^{-s}b(s))'' &= (e^{-s}(b(s)' - b(s)))' \\ &\geq 2v_0(v_1 - v_0) + 2\bar{E}s + 4k(\frac{e^s}{s^2})^{\frac{p+1}{2}}(e^{-s}b(s))^{\frac{p+1}{2}} + e^{-s}b(s) \\ &\geq 2v_0(v_1 - v_0) + 2\bar{E}s + 2^{2-\frac{p+1}{2}}k(e^{-s}b(s))^{\frac{p+1}{2}} + e^{-s}b(s), \end{aligned}$$

$$B(s)'' \geq 2v_0(v_1 - v_0) + 2\bar{E}s + 2^{2-\frac{p+1}{2}}kB(s)^{\frac{p+1}{2}} + B(s). \tag{6.7}$$

From (6.4) it follows that

$$\begin{aligned} b(s) &\geq \frac{v_0 v_1}{2}(e^{2s} - 1) - v_0(v_1 - v_0)s, \\ B(s) &\geq \frac{v_0 v_1}{2}(e^{s} - e^{-s}) - v_0(v_1 - v_0)se^{-s}, \end{aligned}$$

$$2v_0(v_1 - v_0) + 2\bar{E}s + \frac{B(s)}{2} \geq 0, \quad s \geq s_0$$

for some $s_0 > 0$. Therefore,

$$B(s)'' \geq \frac{B(s)}{2} \geq \frac{v_0 v_1}{5} e^s, \quad s \geq s_0,$$
$$B'(s) \geq \frac{v_0 v_1}{5}(e^s - e^{s_0}) + B'(s_0) > 0, \; s \geq s_1$$

for some $s_1 > s_0$.

By (6.7), for all $s \geq s_1$,

$$\begin{aligned} ((B(s)')^2)' &= 2B(s)'B(s)'' \\ &\geq 2^{2-\frac{p-1}{2}} k B(s)^{\frac{p+1}{2}} B(s)' \\ &= 2^{2-\frac{p-1}{2}} k \frac{2}{p+3} (B(s)^{\frac{p+3}{2}})', \end{aligned}$$

$$(B')^2 - B'(s_1)^2 \geq \frac{2^{3-\frac{p-1}{2}} k}{p+3} (B^{\frac{p+3}{2}} - B(s_1)^{\frac{p+3}{2}}),$$

$$\begin{aligned} (B')^2 &\geq \frac{2^{3-\frac{p-1}{2}} k}{p+3} (B^{\frac{p+3}{2}} - B(s_1)^{\frac{p+3}{2}}) + B'(s_1)^2 \\ &= \frac{2^{3-\frac{p-1}{2}} k}{2(p+3)} B^{\frac{p+3}{2}} + \Big(\frac{2^{3-\frac{p-1}{2}} k}{2(p+3)} B^{\frac{p+3}{2}} - 2B(s_1)^{\frac{p+3}{2}} \Big) + B'(s_1)^2, \end{aligned}$$

$$B' \geq \frac{2^{\frac{7-p}{4}}}{\sqrt{p+3}} B^{\frac{p+3}{4}} \quad \text{for } s \geq s_2,$$

for some $s_2 > s_1$; hence, for $s \geq s_2$,

$$\frac{4}{1-p}(B^{\frac{1-p}{4}})' = B^{\frac{p+3}{-4}} B'(s) \geq \frac{2^{\frac{7-p}{4}}}{\sqrt{p+3}},$$

$$B(s)^{\frac{1-p}{4}} \leq B(s_2)^{\frac{1-p}{4}} - \frac{p-1}{4} \frac{2^{\frac{7-p}{4}}}{\sqrt{p+3}} (s - s_2) \quad \text{for all } s \geq s_2 > 0.$$

Thus $B(s)$ blows up at a finite s^*. Since $b(s) = e^s B(s)$, $b(s)$ also blows up at s^*. Further, since $v^2(s) = b'(s) \geq 2b(s)$, $v(s)$ blows up at s^*, as well. □

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