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LAVRENT'EV PROBLEM FOR SEPARATED FLOWS WITH AN EXTERNAL PERTURBATION

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Abstract. We study the Lavrent'ev mathematical model for separated flows with an external perturbation. This model consists of a differential equation with discontinuous nonlinearity and a boundary condition. Using a variational method, we show the existence of a semiregular solution. As a particular case, we study the one-dimensional model.

1. Introduction and statement of the problem

The Lavrent'ev model for separated flows as a main tool for hydrodynamics is discussed in [1]. Separated flows are constructed with a scheme of some "mixed" ideal fluid motion that is potential outside a separation zone and has a constant vorticity inside. The mathematical model of the Lavrent'ev problem is given in [2]. The resonance case and a nonlinear perturbation in a general form for this problem are presented in [3].

In the present article the Lavrent'ev model under an external continuous perturbation is studied. Unlike [3], here the coercive case is considered and the external perturbation is given in a concrete form. Actually the external perturbations such as a jump, an exponential, a polynomial or a sine are simplifying and are not quite adequate to real perturbations. So, we consider the model of the external perturbation in the special analytical form that has not been studied for the Lavrent'ev problem.

In a bounded domain $\Omega \subset \mathbb{R}^2$ with a boundary Γ of class $C_{2,\alpha}$, where $0 < \alpha \le 1$, we solve the Dirichlet problem for an elliptic equation with discontinuous nonlinearity

$$-\Delta u(x) = \mu \operatorname{sign}(u(x)) + f(\|x\|), \quad x \in \Omega, \tag{1.1}$$

$$u(x)|_{\Gamma} = 0. \tag{1.2}$$

Here Δ is the Laplace operator, a parameter $\mu > 0$ is the vorticity, a function $f \in C(\overline{\Omega})$.

We study the model of the external perturbation in the form

$$f(t) = e^{\alpha t} \sin(\omega t + \varphi), \tag{1.3}$$

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where α, ω, φ are real constants. Here ω is a frequency, φ is a phase angle that allows us to define a deviation at $t = 0$. The external perturbation of (1.3) is considered in [4, 5]. Function of (1.3) describes a fading oscillatory process at $\alpha < 0$ and an accruing oscillatory process at $\alpha > 0$.

In this article we study the existence of solutions for the Lavrent'ev problem (1.1), (1.2) under (1.3).

Let (1.1) be exposed to the nonperiodic external perturbation of (1.3) with a decreasing amplitude at $\alpha < 0$. For example, a shock sea wave that arises as a result of explosion may be described by the function $f(t)$ with a strongly decreasing amplitude. On the other hand, to describe a calming down storm that is accompanied with the fading fluctuations of waves it is possible to use the function $f(t)$ with a poorly decreasing amplitude. Also, we notice that

$$|f(\|x\|)| = |e^{\alpha\cdot\|x\|}\cdot\sin(\omega\cdot\|x\| + \varphi)| \le e^{\alpha\cdot\|x\|} \le 1$$

as $\alpha < 0$, $\|x\| \ge 0$. So f is bounded.

2. Preliminaries

In this section we recall some definitions and a basic result to control problems for the distributed systems of the elliptic type with a spectral parameter and discontinuous nonlinearity under an external perturbation (see [6]).

In a bounded domain $\Omega \subset \mathbb{R}^n$ ($n \ge 2$) with a boundary Γ of class $C_{2,\alpha}$ ($0 < \alpha \le 1$), we consider the controlled system with an external perturbation in the form

$$\begin{aligned} Lu(x) &\equiv -\sum_{i,j=1}^{n}\left(a_{ij}(x)u_{x_i}\right)_{x_j} + c(x)u(x) \\ &= \lambda g(x, u(x)) + Bv(x) + Dw(x), \quad x \in \Omega, \end{aligned} \tag{2.1}$$

$$Gu\big|_{\Gamma} = 0. \tag{2.2}$$

Here L is a uniformly elliptic and formally self-adjoint differential operator with coefficients $a_{ij} \in C_{1,\alpha}(\overline{\Omega})$ and $c \in C_{0,\alpha}(\overline{\Omega})$; λ is a positive parameter; the function $g : \Omega\times\mathbb{R} \to \mathbb{R}$ is superpositionally measurable and for almost all $x \in \Omega$ the section $g(x, \cdot)$ has only discontinuities of the first kind on $\mathbb{R}$, $g(x, u) \in [g_-(x, u), g_+(x, u)]$ for all $u \in \mathbb{R}$, where

$$g_-(x, u) = \liminf_{\eta\to u} g(x, \eta), \quad g_+(x, u) = \limsup_{\eta\to u} g(x, \eta),$$

$|g(x, u)| \le a(x)$ for all $u \in \mathbb{R}$, $a \in L_q(\Omega)$, $q > \frac{2n}{n+2}$; the operator $B : U \to L_q(\Omega)$ is linear and bounded, U is the Banach space of controls, the function $v(x)$ in (2.1) is viewed as a control, the control $v \in U_{\rm ad} \subset U$, $U_{\rm ad}$ is the set of all admissible controls for system (2.1), (2.2); the operator $D : W \to L_q(\Omega)$ is linear and bounded, W is the Banach space of perturbations, the function $w(x)$ in (2.1) describes a perturbation, the perturbation $w \in W$. The boundary condition (2.2) is either the Dirichlet condition $u(x)|_\Gamma = 0$, or the Neumann condition $\frac{\partial u}{\partial \mathbf{n}_L}(x)|_\Gamma = 0$ with the conormal derivative $\frac{\partial u}{\partial \mathbf{n}_L}(x) \equiv \sum_{i,j=1}^{n} a_{ij}(x)u_{x_i}\cos(\mathbf{n}, x_j)$, where $\mathbf{n}$ is the outward normal to Γ and $\cos(\mathbf{n}, x_j)$ are the direction cosines of the normal $\mathbf{n}$, or the Robin condition $\frac{\partial u}{\partial \mathbf{n}_L}(x) + \sigma(x)u(x)|_\Gamma = 0$, where the function $\sigma \in C_{1,\alpha}(\Gamma)$ is nonnegative and does not identically vanish on Γ.

Such eigenvalue problems for elliptic equations with discontinuous nonlinearities but without control and perturbation ($v(x) \equiv 0$ and $w(x) \equiv 0$) was established earlier (see [7]–[10]).

Definition 2.1. A strong solution of problem (2.1), (2.2) at the fixed control v and the fixed perturbation w is a function $u \in \mathbf{W}_r^2(\Omega)$, $r > 1$, satisfying (2.1) for almost all $x \in \Omega$ and such that the trace $Gu(x)$ on Γ equals zero.

Definition 2.2. A semiregular solution of problem (2.1), (2.2) at the fixed control v and the fixed perturbation w is a strong solution u such that $u(x)$ is a point of continuity of the function $g(x, \cdot)$ for almost all $x \in \Omega$.

Definition 2.3. A jump discontinuity of a function $f : \mathbb{R} \to \mathbb{R}$ is a point $u \in \mathbb{R}$ such that $f(u-) < f(u+)$, where $f(u\pm) = \lim_{s \to u\pm} f(s)$.

Semiregular solutions for equations with discontinuous nonlinearities were introduced in [11]. Such solutions are significant in applications, for example, in the problem of separated flows of an incompressible fluid (see [12]). Semiregularness of solutions is provided with a restriction to discontinuities of the nonlinearity (for example, the jumping discontinuities).

Let $X = H_\circ^1(\Omega)$ if (2.2) is the Dirichlet condition, and $X = H^1(\Omega)$ if (2.2) is the Neumann or Robin condition. Put

$$J_1(u) = \frac{1}{2} \sum_{i,j=1}^{n} \int_\Omega a_{ij}(x) u_{x_i} u_{x_j} dx + \frac{1}{2} \int_\Omega c(x) u^2(x) dx$$

in the case of the Dirichlet or Neumann condition, and

$$J_1(u) = \frac{1}{2} \sum_{i,j=1}^{n} \int_\Omega a_{ij}(x) u_{x_i} u_{x_j} dx + \frac{1}{2} \int_\Omega c(x) u^2(x) dx + \frac{1}{2} \int_\Gamma \sigma(s) u^2(s) ds$$

in the case of the Robin condition. The following theorem shows the solvability for problem (2.1), (2.2) and corresponds to which is [6, Theorem 2].

Theorem 2.4 ([6]). *Suppose that the following conditions are satisfied:*

1. *the inequality $J_1(u) \geq 0$ holds for each $u \in X$;*
2. *for almost all $x \in \Omega$ the function $g(x, \cdot)$ has only jump discontinuities; $g(x, 0) = 0$ and $|g(x, u)| \leq a(x)$ for all $u \in \mathbb{R}$, where $a \in L_q(\Omega)$ ($q > \frac{2n}{n+2}$) is fixed;*
3. *there exists a $u_0 \in X$ such that*
$$\int_\Omega dx \int_0^{u_0(x)} g(x, s) ds > 0;$$
4. *if the solution space $N(L)$ of the problem*
$$Lu = 0, \quad x \in \Omega,$$
$$Gu|_\Gamma = 0$$
is nonzero (the resonance case), then it is additionally assumed that
$$\lim_{u \in N(L), \|u\| \to +\infty} \int_\Omega dx \int_0^{u(x)} g(x, s) ds = -\infty;$$
5. *the operator $B : U \to L_q(\Omega)$ is linear and bounded, the control space U is Banach, the set of admissible controls $U_{\mathrm{ad}} \subset U$ is nonempty;*

(6) *the operator* $D : W \to L_q(\Omega)$ *is linear and bounded, the perturbation space* W *is Banach.*

Then for any $v \in U_{\rm ad}$ *and* $w \in W$ *there exists a semiregular solution of problem* (2.1)*,* (2.2)*.*

Under conditions (1)–(4) of the above theorem, when control and perturbation are absent, and using a variational method, existence results were obtained in [7, Theorems 3 and 4], and [10, Theorem 3].

3. Solution of the problem

Let us verify that all the conditions of Theorem 2.4 are fulfilled for the Lavrent'ev problem (1.1), (1.2) under (1.3). We have

$$J_1(u) = \frac{1}{2}\sum_{i=1}^{2}\int_\Omega u_{x_i}^2\,dx = \frac{1}{2}\int_\Omega (u_{x_1}^2 + u_{x_2}^2)dx = \frac{1}{2}\|u\|^2 \ge 0 \quad \forall u \in H^1_\circ(\Omega).$$

Condition (1) is satisfied.

For almost all $x \in \Omega$ the function $\operatorname{sign}(\cdot)$ has only jump discontinuity $u = 0$ as $-1 = \operatorname{sign}(0-) < \operatorname{sign}(0+) = 1$; $\operatorname{sign}(0) = 0$ and $|\operatorname{sign}(u)| \le 1$ for all $u \in \mathbb{R}$, $1 \in L_q(\Omega)$, $q > \frac{2\cdot 2}{2+2} = 1$ are valid. Therefore condition (2) of Theorem 2.4 is fulfilled.

As in [13], it can be shown that there exists a $u_0 \in H^1_\circ(\Omega)$ such that

$$\int_\Omega dx \int_0^{u_0(x)} \operatorname{sign}(s)ds > 0.$$

Condition (3) of Theorem 2.4 holds.

Since the space $N(-\Delta)$ of solutions for the problem

$$-\Delta u = 0, u\big|_\Gamma = 0$$

is zero, it follows that no additional assumption in condition (4) of Theorem 2.4 is needed.

Clearly, condition (5) of Theorem 2.4 is not required as the control in (1.1) is absent.

We see that at the perturbation f in (1.1) there is the identical operator I, i.e., $If = f$. The operator I is linear and bounded. The space $C(\overline{\Omega})$ of the perturbations is a Banach space. Condition (6) of Theorem 2.4 is satisfied.

Thus all the conditions of Theorem 2.4 for the Lavrent'ev problem (1.1), (1.2) under (1.3) are fulfilled. This implies that the Lavrent'ev problem has a semiregular solution.

In the present paper we show the existence of the semiregular solution of the Dirichlet problem for the elliptic equation with the discontinuous nonlinearity by the variational method unlike in [2].

If, in addition, the variational functional corresponding to problem (1.1), (1.2) has no more than a countable number of points of a global minimum, then, according to [14, 15], there is the regular solution of problem (1.1), (1.2); i.e., the semiregular solution with the property of correctness. Earlier (see [1]–[3]) the regular solutions for the Lavrent'ev problem were not investigated.

We note that other theoretical results for the Lavrent'ev problem are received similarly to results for the Gol'dshtik mathematical model for separated flows of incompressible fluid [12], which are analyzed in [13, 16, 17].

4. One-dimensional model

Further we consider the one-dimensional analog of model (1.1), (1.2). We have

$$-u''(x) = \mu \operatorname{sign}(u(x)) + f(x), \quad x \in [0,1], \tag{4.1}$$

$$u(0) = u(1) = 0. \tag{4.2}$$

A system of ordinary differential equations that contains a hysteresis nonlinearity such as a relay and the external perturbation of (1.3) is studied in [4, 5]. By replacement of variables, this system can be reduced to model (4.1), (4.2). Solvability for this problem was established earlier. Arguing as above, we see that other results for problem (4.1), (4.2) can be obtained as well as for the one-dimensional analog of Gol'dshtik's model that is considered in [16, 18].

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