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IMPULSIVE NEUTRAL FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH STATE DEPENDENT DELAYS AND INTEGRAL CONDITION

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Abstract. In this article, we establish the existence of a solution for an impulsive neutral fractional integro-differential state dependent delay equation subject to an integral boundary condition. The existence results are proved by applying the classical fixed point theorems. An example is presented to demonstrate the application of the results established.

1. Introduction

Let X be a Banach space and $PC_t := PC([-d,t];X), d > 0, 0 \le t \le T < \infty$, be a Banach space of all such functions $\phi : [-d,t] \to X$, which are continuous everywhere except for a finite number of points t_i, $i = 1,2,\dots,m$, at which $\phi(t_i^+)$ and $\phi(t_i^-)$ exists and $\phi(t_i) = \phi(t_i^-)$, endowed with the norm

$$\|\phi\|_t = \sup_{-d\le s\le t} \|\phi(s)\|_X,\ \phi \in PC_t,$$

where $\|\cdot\|_X$ is the norm in X.

In this article we study an impulsive neutral fractional integro-differential equation of the form

$$\begin{aligned} &D_t^\alpha\Big[x(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s, x_{\rho(s,x_s)})ds\Big] \\ &= f(t, x_{\rho(t,x_t)}, B(x)(t)), \quad t \in J = [0,T],\ T < \infty,\ t \ne t_k, \end{aligned} \tag{1.1}$$

$$\Delta x(t_k) = I_k(x(t_k^-)), \quad \Delta x'(t_k) = Q_k(x(t_k^-)), \quad k = 1,2,\dots,m, \tag{1.2}$$

$$x(t) = \phi(t), \quad t \in [-d,0], \tag{1.3}$$

$$ax'(0) + bx'(T) = \int_0^T q(x(s))ds, \quad a+b \ne 0,\ b \ne 0, \tag{1.4}$$

where x' denotes the derivative of x with respect to t and D_t^α, $\alpha \in (1,2)$ is Caputo's derivative. The functions $f : J \times PC_0 \times X \to X$, $g : J \times PC_0 \to X$, and $q : X \to X$ are given continuous functions where $PC_0 = PC([-d,0],X)$ and for any $x \in PC_T = PC([-d,T],X)$, $t \in J$, we denote by x_t the element of PC_0 defined by

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$x_t(\theta) = x(t+\theta)$, $\theta \in [-d, 0]$. In the impulsive conditions for $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $Q_k, I_k \in C(X, X)$, $(k = 1, 2, \dots, m)$, are continuous and bounded functions. We have $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ and $\Delta x'(t_k) = x'(t_k^+) - x'(t_k^-)$. The term $Bx(t)$ is given by

$$Bx(t) = \int_0^t K(t,s)x(s)ds, \tag{1.5}$$

where $K \in C(D, \mathbb{R}^+)$, is the set of all positive functions which are continuous on $D = \{(t,s) \in \mathbb{R}^2 : 0 \le s \le t < T\}$ and $B^* = \sup_{t\in[0,t]} \int_0^t K(t,s)ds < \infty$.

The study of fractional differential equations has been gaining importance in recent years due to the fact that fractional order derivatives provide a tool for the description of memory and hereditary properties of various phenomena. Due to this fact, the fractional order models are capable to describe more realistic situation than the integer order models. Fractional differential equations have been used in many field like fractals, chaos, electrical engineering, medical science, etc. In recent years, we have seen considerable development on the topics of fractional differential equations, for instance, we refer to the articles [8, 10, 26, 27].

The theory of impulsive differential equations of integer order is well developed and has applications in mathematical modelling, especially in dynamics of populations subject to abrupt changes as well as other phenomena such as harvesting, disease, and so forth. For general theory and applications of fractional order differential equations with impulsive conditions, we refer the reader to the references [1, 7, 11, 17, 21, 22, 28, 29, 30].

Integral boundary conditions have various applications in applied fields such as blood flow problems, chemical engineering, thermoelasticity, underground water flow, population dynamics, etc. For a detailed description of the integral boundary conditions, we refer the reader to some recent papers [4, 5, 6, 13, 16, 17] and the references therein. On the other hand, we know that the delay arises naturally in systems due to the transmission of signal or the mechanical transmission. Moreover, the study of fractional order problems involving various types of delay (finite, infinite and state dependant) considered in Banach spaces has been receiving attention, see [2, 3, 8, 12, 14, 15, 18, 19, 20, 23, 24, 25] and references cited in these articles.

In [17] authors have established the existence and uniqueness of a solution for the following system

$$\begin{gathered}
D_t^\alpha x(t) = f(t, x_t, Bx(t)), \quad t \in J = [0,T],\ t \neq t_k, \\
\Delta x(t_k) = Q_k(x(t_k^-)), \quad k = 1, 2, \dots, m, \\
\Delta x'(t_k) = I_k(x(t_k^-)), \quad k = 1, 2, \dots, m, \\
x(t) = \phi(t), \quad t \in (-\infty, 0], \\
ax'(0) + bx'(T) = \int_0^T q(x(s))ds,
\end{gathered} \tag{1.6}$$

the results are proved by using the contraction and Krasnoselkii's fixed point theorems. This paper is motivated from some recent papers treating the boundary value problems for impulsive fractional differential equations [4, 5, 13, 17, 30].

To the best of our knowledge, there is no work available in literature on impulsive neutral fractional integro-differential equation with state dependent delay and with an integral boundary condition. In this article, we first establish a general framework to find a solution to system (1.1)–(1.4) and then by using classical fixed point theorems we proved the existence and uniqueness results.

2. Preliminaries

In this section, we shall introduce notations, definitions, preliminary results which are required to establish our main results. We continue to use the function spaces introduced in the earlier section. For the following definitions we refer to the reader to see the monograph of Podlunby [27].

Definition 2.1. Caputo's derivative of order α for a function $f : [0,\infty) \to \mathbb{R}$ is defined as

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds = J^{n-\alpha} f^{(n)}(t), \tag{2.1}$$

for $n-1 \le \alpha < n$, $n \in N$. If $0 \le \alpha < 1$, then

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f^{(1)}(s) ds. \tag{2.2}$$

Definition 2.2. The Riemann-Liouville fractional integral operator for order $\alpha > 0$, of a function $f : \mathbb{R}^+ \to \mathbb{R}$ and $f \in L^1(\mathbb{R}^+, X)$ is defined by

$$J_t^0 f(t) = f(t),\ J_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad \alpha > 0,\ t > 0, \tag{2.3}$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Lemma 2.3 ([30])**.** *For $\alpha > 0$, the general solution of fractional differential equations $D_t^\alpha x(t) = 0$ is given by $x(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \cdots + c_{n-1} t^{n-1}$ where $c_i \in \mathbb{R}$, $i = 0, 1, \dots, n-1$, $n = [\alpha] + 1$ and $[\alpha]$ represent the integral part of the real number α.*

Lemma 2.4 ([21, Lemma 2.6])**.** *Let $\alpha \in (1,2), c \in \mathbb{R}$ and $h : J \to \mathbb{R}$ be continuous function. A function $x \in C(J,\mathbb{R})$ is a solution of the following fractional integral equation*

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds - \int_0^w \frac{(w-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds + x_0 - c(t-w), \tag{2.4}$$

if and only if x is a solution of the following fractional Cauchy problem

$$D_t^\alpha x(t) = h(t), \quad t \in J,\ x(w) = x_0,\ w \ge 0. \tag{2.5}$$

As a consequence of Lemma 2.3 and Lemma 2.4 we have the following result.

Lemma 2.5. *Let $\alpha \in (1,2)$ and $f : J \times PC_0 \times X \to X$ be continuously differentiable function. A piecewise continuous differential function $x(t) : (-d, T] \to X$ is a* *solution of system* (1.1)–(1.4) *if and only if* x *satisfied the integral equation*

$$x(t)=\begin{cases}\phi(t), \quad t\in[-d,0],\\ \phi(0)-\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,x_{\rho(s,x_s)})ds+\frac{bt}{a+b}\Big\{\frac{1}{b}\int_0^T q(x(s))ds\\ -\sum_{i=1}^k Q_i(x(t_i^-))+\int_0^T\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}g(s,x_{\rho(s,x_s)})ds\\ -\int_0^T\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,x_{\rho(s,x_s)},B(x)(s))ds\Big\}\\ +\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x_{\rho(s,x_s)},B(x)(s))ds, \quad t\in[0,t_1],\\ \dots\\ \phi(0)+\sum_{i=1}^k I_i(x(t_i^-))+\sum_{i=1}^k(t-t_i)Q_i(x(t_i^-))\\ -\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,x_{\rho(s,x_s)})ds+\frac{bt}{a+b}\Big\{\frac{1}{b}\int_0^T q(x(s))ds\\ -\sum_{i=1}^k Q_i(x(t_i^-))+\int_0^T\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}g(s,x_{\rho(s,x_s)})ds\\ -\int_0^T\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,x_{\rho(s,x_s)},B(x)(s))ds\Big\}\\ +\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x_{\rho(s,x_s)},B(x)(s))ds, \quad t\in(t_k,t_{k+1}].\end{cases}\tag{2.6}$$

Proof. If $t\in[0,t_1]$, then

$$\begin{gathered}D_t^\alpha[x(t)+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,x_{\rho(s,x_s)})ds]=f(t,x_{\rho(t,x_t)},B(x)(t)),\\ x(t)=\phi(t),\ t\in[-d,0].\end{gathered}\tag{2.7}$$

Taking the Riemann-Liouville fractional integral of (2.7) and using the Lemma 2.4, we have

$$\begin{aligned}&x(t)+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,x_{\rho(s,x_s)})ds\\ &=a_0+b_0t+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x_{\rho(s,x_s)},B(x)(s))ds,\end{aligned}\tag{2.8}$$

using the initial condition, we get $a_0=\phi(0)$, then (2.8) becomes

$$\begin{aligned}&x(t)+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,x_{\rho(s,x_s)})ds\\ &=\phi(0)+b_0t+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x_{\rho(s,x_s)},B(x)(s))ds.\end{aligned}\tag{2.9}$$

Similarly, if $t\in(t_1,t_2]$, then

$$D_t^\alpha[x(t)+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,x_{\rho(s,x_s)})ds]=f(t,x_{\rho(t,x_t)},B(x)(t))\tag{2.10}$$

$$x(t_1^+)=x(t_1^-)+I_1(x(t_1^-)),\tag{2.11}$$

$$x'(t_1^+)=x'(t_1^-)+Q_1(x(t_1^-)).\tag{2.12}$$

Again apply the Riemann-Liouville fractional integral operator on (2.10) and using the lemma 2.4, we obtain

$$\begin{aligned}&x(t)+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,x_{\rho(s,x_s)})ds\\ &=a_1+b_1t+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x_{\rho(s,x_s)},B(x)(s))ds,\end{aligned}\tag{2.13}$$

rewrite (2.13) as

$$\begin{aligned} &x(t_1^+)+\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,x_{\rho(s,x_s)})ds \\ &=a_1+b_1t_1+\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x_{\rho(s,x_s)},B(x)(s))ds, \end{aligned} \tag{2.14}$$

due to impulsive condition (2.11) and the fact that $x(t_1)=x(t_1^-)$, we may write (2.14) as

$$\begin{aligned} &x(t_1)+I_1(x(t_1^-))+\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,x_{\rho(s,x_s)})ds \\ &=a_1+b_1t_1+\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x_{\rho(s,x_s)},B(x)(s))ds. \end{aligned} \tag{2.15}$$

Now from (2.9), we have

$$\begin{aligned} &x(t_1)+\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,x_{\rho(s,x_s)})ds \\ &=\phi(0)+b_0t_1+\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x_{\rho(s,x_s)},B(x)(s))ds. \end{aligned} \tag{2.16}$$

From (2.15) and (2.16), we get $a_1=\phi(0)+b_0t_1-b_1t_1+I_1(x(t_1^-))$, hence (2.14) can be written as

$$\begin{aligned} &x(t)+\int_0^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,x_{\rho(s,x_s)})ds \\ &=\phi(0)+b_0t_1+b_1(t-t_1)+I_1(x(t_1^-))+\int_0^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x_{\rho(s,x_s)},B(x)(s))ds. \end{aligned} \tag{2.17}$$

On differentiating (2.13) with respect to t at $t=t_1$, and incorporate second impulsive condition (2.12), we obtain

$$\begin{aligned} &x'(t_1^-)+Q_1(x(t_1^-))+\int_0^{t_1}\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}g(s,x_{\rho(s,x_s)})ds \\ &=b_1+\int_0^{t_1}\frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,x_{\rho(s,x_s)},B(x)(s))ds, \end{aligned} \tag{2.18}$$

Now differentiating (2.9), with respect to t at $t=t_1$, we get

$$\begin{aligned} &x'(t_1)+\int_0^{t_1}\frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)}g(s,x_{\rho(s,x_s)})ds \\ &=b_0+\int_0^{t_1}\frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,x_{\rho(s,x_s)},B(x)(s))ds. \end{aligned} \tag{2.19}$$

From (2.18) and (2.19), we obtain $b_1=b_0+Q_1(x(t_1^-))$. Thus, (2.17) become

$$\begin{aligned} &x(t)+\int_0^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,x_{\rho(s,x_s)})ds \\ &=\phi(0)+b_0t+I_1(x(t_1^-))+(t-t_1)Q_1(x(t_1^-)) \\ &\quad+\int_0^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x_{\rho(s,x_s)},B(x)(s))ds. \end{aligned} \tag{2.20}$$

Similarly, for $t \in (t_2, t_3]$, we can write the solution of the problem as

$$\begin{aligned} x(t) &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s, x_{\rho(s,x_s)}) ds \\ &= \phi(0) + b_0 t + I_1(x(t_1^-)) + I_2(x(t_2^-)) + (t-t_1)Q_1(x(t_1^-)) + (t-t_2)Q_2(x(t_2^-)) \\ &\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_{\rho(s,x_s)}, B(x)(s)) ds. \end{aligned}$$

In general, if $t \in (t_k, t_{k+1}]$, then we have the result

$$\begin{aligned} x(t) &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s, x_{\rho(s,x_s)}) ds \\ &= \phi(0) + b_0 t + \sum_{i=1}^{k} I_i(x(t_i^-)) + \sum_{i=1}^{k} (t-t_i) Q_i(x(t_i^-)) \\ &\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_{\rho(s,x_s)}, B(x)(s)) ds. \end{aligned} \tag{2.21}$$

Finally, we use the integral boundary condition $ax'(0) + bx'(T) = \int_0^T q(x(s))ds$, where $x'(0)$ calculated from (2.9) and $x'(T)$ from (2.20). On simplifying, we get the following value of the constant b_0,

$$\begin{aligned} b_0 = \frac{b}{a+b} \Big\{ &\frac{1}{b} \int_0^T q(x(s)) ds - \sum_{i=1}^{m} Q_i(x(t_i^-)) + \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} g(s, x_{\rho(s,x_s)}) ds \\ &- \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, x_{\rho(s,x_s)}, B(x)(s)) ds \Big\}. \end{aligned}$$

On summarizing, we obtain the desired integral equation (2.6). Conversely, assuming that x satisfies (2.6), by a direct computation, it follows that the solution given in (2.6) satisfies system (1.1)–(1.4). This completes the proof of the lemma. □

3. Existence result

The function $\rho : J \times PC_0 \to [-d, T]$ is continuous and $\phi(0) \in PC_0$. Let the function t$\to \varphi_t$ be well defined and continuous from the set $\Re(\rho^-) = \{\rho(s,\psi) : (s,\psi) \in [0,T] \times PC_0\}$ into PC_0. Further, we introduce the following assumptions to establish our results.

- (H1) There exist positive constants L_{f1}, L_{f2}, L_q and L_g, such that

$$\begin{gathered} \|f(t,\psi,x) - f(t,\chi,y)\|_X \le L_{f1}\|\psi - \chi\|_{PC_0} + L_{f2}\|x-y\|_X, \\ \|g(t,\psi) - g(t,\chi)\|_X \le L_g \|\psi-\chi\|_{PC_0}, t \in J, \forall\ \psi,\chi \in PC_0,\ \forall\ x,y \in X, \\ \|q(x) - q(y)\|_X \le L_q\|x-y\|_X,\ \forall x,y \in X. \end{gathered}$$

- (H2) There exist positive constants L_Q, L_I, L_q, such that

$$\|Q_k(x) - Q_k(y)\|_X \le L_Q\|x-y\|_X, \quad \|I_k(x) - I_k(y)\|_X \le L_I\|x-y\|_X.$$

- (H3) The functions Q_k, I_k, q are bounded continuous and there exist positive constants C_1, C_2, C_3, such that

$$\|Q_k(x)\|_X \le C_1, \quad \|I_k(x)\|_X \le C_2, \quad \|q(x)\|_X \le C_3, \quad \forall x \in X.$$

Our first result is based on the Banach contraction theorem.

Theorem 3.1. *Let the assumptions* (H1)–(H2) *are satisfied with*

$$\begin{aligned}\triangle = \Big\{ & m(L_I + TL_Q) + \frac{T^\alpha L_g}{\Gamma(\alpha+1)} + \frac{bT}{a+b}\Big(\frac{TL_q}{b} + mL_Q \\ & + \frac{T^{\alpha-1}L_g}{\Gamma(\alpha)} + \frac{T^{\alpha-1}(L_{f1} + L_{f2}B^*)}{\Gamma(\alpha)}\Big) + \frac{T^\alpha(L_{f1}+L_{f2}B^*)}{\Gamma(\alpha+1)}\Big\} < 1.\end{aligned}$$

Then (1.1)–(1.4) *has a unique solution.*

Proof. We transform problem (1.1)–(1.4) into a fixed point problem. Consider the operator $P : PC_T \to PC_T$ defined by

$$Px(t) = \begin{cases} \phi(t), \quad t \in [-d, 0], \\ \phi(0) - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s, x_{\rho(s,x_s)})ds + \frac{bt}{a+b}\Big\{\frac{1}{b}\int_0^T q(x(s))ds \\ -\sum_{i=1}^m Q_i(x(t_i^-)) + \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} g(s, x_{\rho(s,x_s)})ds \\ -\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, x_{\rho(s,x_s)}, B(x)(s))ds\Big\} \\ +\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_{\rho(s,x_s)}, B(x)(s))ds, \quad t \in [0, t_1] \\ \dots \\ \phi(0) + \sum_{i=1}^k I_i(x(t_i^-)) + \sum_{i=1}^k (t-t_i)Q_i(x(t_i^-)) \\ -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s, x_{\rho(s,x_s)})ds + \frac{bt}{a+b}\Big\{\frac{1}{b}\int_0^T q(x(s))ds - \sum_{i=1}^m Q_i(x(t_i^-)) \\ +\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} g(s, x_{\rho(s,x_s)})ds - \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, x_{\rho(s,x_s)}, B(x)(s))ds\Big\} \\ +\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_{\rho(s,x_s)}, B(x)(s))ds, \quad t \in (t_k, t_{k+1}]. \end{cases}$$

Let $x, x^* \in PC_T$ and $t \in [0, t_1]$. Then

$$\begin{aligned} & \|P(x) - P(x^*)\|_X \\ & \le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \|g(s, x_{\rho(s,x_s)}) - g(s, x^*_{\rho(s,x^*_s)})\|_X ds \\ & \quad + \frac{bt}{a+b}\Big\{\frac{1}{b}\int_0^T \|q(x(s)) - q(x^*(s))\|_X ds + \sum_{i=1}^m \|Q_i(x(t_i^-)) - Q_i(x^*(t_i^-))\|_X \\ & \quad + \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} \|g(s, x_{\rho(s,x_s)}) - g(s, x^*_{\rho(s,x^*_s)})\|_X ds \\ & \quad + \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} \|f(s, x_{\rho(s,x_s)}, B(x)(s)) - f(s, x^*_{\rho(s,x^*_s)}, B(x^*)(s))\|_X ds\Big\} \\ & \quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \|f(s, x_{\rho(s,x_s)}, B(x)(s)) - f(s, x^*_{\rho(s,x^*_s)}, B(x^*)(s))\|_X ds \\ & \le \Big\{\frac{T^\alpha}{\Gamma(\alpha+1)}L_g + \frac{bT}{a+b}\Big(\frac{T}{b}L_q + mL_Q + \frac{T^{\alpha-1}}{\Gamma(\alpha)}L_g \\ & \quad + \frac{T^{\alpha-1}}{\Gamma(\alpha)}(L_{f1} + L_{f2}B^*)\Big) + \frac{T^\alpha}{\Gamma(\alpha+1)}(L_{f1} + L_{f2}B^*)\Big\}\|x - x^*\|_{PC_T}. \end{aligned}$$

In a similar way for $t \in (t_k, t_{k+1}]$, we have

$$\|P(x) - P(x^*)\|_X$$

$$
\begin{aligned}
&\le \sum_{i=1}^{k} \|I_i(x(t_i^-)) - I_i(x^*(t_i^-))\|_X + \sum_{i=1}^{k} (t-t_i)\|Q_i(x(t_i^-)) - Q_i(x^*(t_i^-))\|_X \\
&\quad \times \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \|g(s, x_{\rho(s,x_s)}) - g(s, x^*_{\rho(s,x^*_s)})\|_X ds \\
&+ \frac{bt}{a+b}\Big\{ \frac{1}{b}\int_0^T \|q(x(s)) - q(x^*(s))\|_X ds + \sum_{i=1}^{m} \|Q_i(x(t_i^-)) - Q_i(x^*(t_i^-))\|_X \\
&\quad + \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} \|g(s, x_{\rho(s,x_s)}) - g(s, x^*_{\rho(s,x^*_s)})\|_X ds \\
&\quad + \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} \|f(s, x_{\rho(s,x_s)}, B(x)(s)) - f(s, x^*_{\rho(s,x^*_s)}, B(x^*)(s))\|_X ds \Big\} \\
&\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \|f(s, x_{\rho(s,x_s)}, B(x)(s)) - f(s, x^*_{\rho(s,x^*_s)}, B(x^*)(s))\|_X ds \\
&\le \Big\{ mL_I + mTL_Q + \frac{T^\alpha}{\Gamma(\alpha+1)} L_g + \frac{bT}{a+b}\Big(\frac{T}{b} L_q + mL_Q + \frac{T^{\alpha-1}}{\Gamma(\alpha)} L_g \\
&\quad + \frac{T^{\alpha-1}}{\Gamma(\alpha)} (L_{f1} + L_{f2}B^*)\Big) + \frac{T^\alpha}{\Gamma(\alpha+1)} (L_{f1} + L_{f2}B^*)\Big\} \|x - x^*\|_{PC_T} \\
&\le \Delta \|x - x^*\|_{PC_T}.
\end{aligned}
$$

Since $\Delta < 1$, implies that the map P is a contraction map and therefore has a unique fixed point $x \in PC_T$, hence system (1.1)–(1.4) has a unique solution on the interval $[-d, T]$. This completes the proof of the theorem. □

Our second result is based on Krasnoselkii's fixed point theorem.

Theorem 3.2. *Let B be a closed convex and nonempty subset of a Banach space X. Let P and Q be two operators such that*

(i) *$Px + Qy \in B$, whenever $x, y \in B$. (ii) P is compact and continuous.*
(iii) *Q is a contraction mapping. Then there exists $z \in B$ such that $z = Pz + Qz$.*

Theorem 3.3. *Let the function f, g be continuous for every $t \in [0, T]$, and satisfy the assumptions (H1)–(H3) with*

$$
\begin{aligned}
\Delta = \Big\{ &\frac{T^\alpha}{\Gamma(\alpha+1)} L_g + \frac{bT}{a+b}\Big(\frac{T}{b} L_q + \frac{T^{\alpha-1}}{\Gamma(\alpha)} L_g + \frac{T^{\alpha-1}}{\Gamma(\alpha)} (L_{f1} + L_{f2}B^*)\Big) \\
&+ \frac{T^\alpha}{\Gamma(\alpha+1)} (L_{f1} + L_{f2}B^*)\Big\} < 1.
\end{aligned}
$$

Then system (1.1)–(1.4) *has at least one solution on* $[-d, T]$.

Proof. Choose

$$
\begin{aligned}
r \ge \Big[&\|\phi(0)\| + mL_I r + mTL_Q r + \frac{T^\alpha}{\Gamma(\alpha+1)} L_g r + \frac{bT}{a+b}(\frac{T}{b} L_q r + mL_Q r \\
&+ \frac{T^{\alpha-1}}{\Gamma(\alpha)} L_g r + \frac{T^{\alpha-1}}{\Gamma(\alpha)} (L_{f1} r + L_{f2}B^* r)) + \frac{T^\alpha}{\Gamma(\alpha+1)} (L_{f1} r + L_{f2} B^* r)\Big].
\end{aligned}
$$

Define $PC_T^r = \{x \in PC_T : \|x\|_{PC_T} \le r\}$, then PC_T^r is a bounded, closed convex subset in PC_T. Consider the operators $N : PC_T^r \to PC_T^r$ and $P : PC_T^r \to PC_T^r$ for

$t \in J_k = (t_k, t_{k+1}]$, defined by

$$N(x) = \phi(0) + \sum_{i=1}^{k} I_i(x(t_i^-)) + \sum_{i=1}^{k}(t-t_i)Q_i(x(t_i^-)) - \frac{bt}{a+b}\sum_{i=1}^{m} Q_i(x(t_i^-)) \tag{3.1}$$

$$\begin{aligned} P(x) = \frac{bt}{a+b}\Big\{\frac{1}{b}\int_0^T q(x(s))ds + \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} g(s, x_{\rho(s,x_s)})ds \\ - \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, x_{\rho(s,x_s)}, B(x)(s))ds\Big\} \\ - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s, x_{\rho(s,x_s)})ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_{\rho(s,x_s)}, B(x)(s))ds. \end{aligned} \tag{3.2}$$

We complete the proof in the following steps:
Step 1. Let $x, x^* \in PC_T^r$ then,

$$\begin{aligned} \|N(x) + P(x^*)\|_X &\le \|\phi(0)\|_X + \sum_{i=1}^{k} \|I_i(x(t_i^-))\|_X + \sum_{i=1}^{k}(t-t_i)\|Q_i(x(t_i^-))\|_X \\ &\quad + \frac{bt}{a+b}\sum_{i=1}^{m} \|Q_i(x(t_i^-))\|_X + \frac{bt}{a+b}\Big\{\frac{1}{b}\int_0^T \|q(x^*(s))\|_X ds \\ &\quad + \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} \|g(s, x^*_{\rho(s,x^*_s)})\|_X ds \\ &\quad + \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} \|f(s, x^*_{\rho(s,x^*_s)}, B(x^*)(s))\|_X ds\Big\} \\ &\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \|g(s, x^*_{\rho(s,x^*_s)})\|_X ds \\ &\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \|f(s, x^*_{\rho(s,x^*_s)}, B(x^*)(s))\|_X ds \\ &\le \Big[\|\phi(0)\| + mC_2 + mTC_1 + \frac{T^\alpha}{\Gamma(\alpha+1)} L_g r + \frac{bT}{a+b}(\frac{T}{b}C_3 \\ &\quad + mC_1 + \frac{T^{\alpha-1}}{\Gamma(\alpha)} L_g r + \frac{T^{\alpha-1}}{\Gamma(\alpha)}(L_{f1} r + L_{f2} B^* r)) \\ &\quad + \frac{T^\alpha}{\Gamma(\alpha+1)}(L_{f1} r + L_{f2} B^* r)\Big] \le r. \end{aligned}$$

Which shows that PC_T^r is closed with respect to both the maps.
Step 2. N is continuous. Let $x_n \to x$ be sequence in PC_T^r, then for each $t \in J_k$

$$\begin{aligned} &\|N(x_n) - N(x)\|_X \\ &\le \sum_{i=1}^{k} \|I_i(x_n(t_i^-)) - I_i(x(t_i^-))\|_X + \sum_{i=1}^{k}(t-t_i)\|Q_i(x_n(t_i^-)) - Q_i(x(t_i^-))\|_X \\ &\quad + \frac{bt}{a+b}\sum_{i=1}^{m} \|Q_i(x_n(t_i^-)) - Q_i(x(t_i^-))\|_X. \end{aligned}$$

Since the functions Q_k and I_k, $k = 1, \dots, m$, are continuous, hence $\|N(x_n) - N(x)\| \to 0$, as $n \to \infty$. Which implies that the mapping N is continuous on PC_T^r.

Step 3. The fact that the mapping N is uniformly bounded is a consequence of the following inequality. For each $t \in J_k$, $k = 0, 1, \dots, m$ and for each $x \in PC_T^r$, we have

$$\begin{aligned}\|N(x)\|_X &\le \|\phi(0)\|_X + \sum_{i=1}^{k} \|I_i(x(t_i^-))\|_X + \sum_{i=1}^{k}(t - t_i)\|Q_i(x(t_i^-))\|_X \\ &\quad + \frac{bt}{a+b} \sum_{i=1}^{m} \|Q_i(x(t_i^-))\|_X \\ &\le \|\phi(0)\| + mC_2 + mTC_1 + \frac{bT}{a+b} mC_1.\end{aligned}$$

Step 4. Now, to show that N is equi-continuous, let $l_1, l_2 \in J_k$, $t_k \le l_1 < l_2 \le t_{k+1}$, $k = 1, \dots, m$, $x \in PC_T^r$, we have

$$\|N(x)(l_2) - N(x)(l_1)\|_X \le (l_2 - l_1) \sum_{i=1}^{k} \|Q_i(x(t_i^-))\|_X + \frac{b(l_2 - l_1)}{a+b} \sum_{i=1}^{m} \|Q_i(x(t_i^-))\|_X.$$

As $l_2 \to l_1$, then $\|N(x)(l_2) - N(x)(l_1)\| \to 0$ implies that N is an equi-continuous map. Combining the Steps 2 to 4, together with the Arzela Ascoli's theorem, we conclude that the operator N is compact.

Step 5. Now, we show that P is a contraction mapping. Let $x, x^* \in PC_T^r$ and $t \in J_k$, $k = 1, \dots, m$, we have

$$\begin{aligned}&\|P(x) - P(x^*)\|_X \\ &\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \|g(s, x_{\rho(s,x_s)}) - g(s, x^*_{\rho(s,x_s^*)})\|_X ds \\ &\quad + \frac{bt}{a+b} \Big\{ \frac{1}{b} \int_0^T \|q(x(s)) - q(x^*(s))\|_X ds \\ &\quad + \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} \|g(s, x_{\rho(s,x_s)}) - g(s, x^*_{\rho(s,x_s^*)})\|_X ds \\ &\quad . + \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} \|f(s, x_{\rho(s,x_s)}, B(x)(s)) - f(s, x^*_{\rho(s,x_s^*)}, B(x^*)(s))\|_X ds \Big\} \\ &\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \|f(s, x_{\rho(s,x_s)}, B(x)(s)) - f(s, x^*_{\rho(s,x_s^*)}, B(x^*)(s))\|_X ds \\ &\le \Big\{ \frac{T^\alpha}{\Gamma(\alpha+1)} L_g + \frac{bT}{a+b} \Big(\frac{T}{b} L_q + \frac{T^{\alpha-1}}{\Gamma(\alpha)} L_g \\ &\quad + \frac{T^{\alpha-1}}{\Gamma(\alpha)} (L_{f1} + L_{f2} B^*) \Big) + \frac{T^\alpha}{\Gamma(\alpha+1)} (L_{f1} + L_{f2} B^*) \Big\} \|x - x^*\|_{PC_T^r} \\ &\le \Delta \|x - x^*\|_{PC_T^r}\end{aligned}$$

As $\Delta < 1$, it implies that P is a contraction map. Thus all the assumptions of the Krasnoselkii's theorem are satisfied. Hence we have that the set PC_T^r has a fixed point which is the solution of system (1.1)–(1.4) on $(-d, T]$. This completes the proof of the theorem. $\square$

4. Example

Consider the following example to demonstrate the application of the results established.

$$
\begin{gathered}
D_t^{\alpha}[x(t)+\int_0^t \frac{1}{47}x(t-\sigma(x)ds]=\frac{e^t x(t-\sigma(x(t)))}{25+x^2(t-\sigma(x(t)))}+\int_0^t \cos(t-s)\frac{xe^s}{4+x}ds,\\
t\in[0,T],\ t\neq t_i,\\
\Delta x(t_i)=\int_{-d}^{t_i}\frac{\gamma_i(t_i-s)x(s)}{25}ds,\quad \Delta x'(t_i)=\int_{-d}^{t_i}\frac{\gamma_i(t_i-s)x(s)}{9}ds,\\
x(t)=\phi(t),t\in(-d,0],\quad x'(0)+x'(T)=\int_0^T \sin(\frac{1}{4}x(s))ds,
\end{gathered}
$$

where $\gamma_i\in C([0,\infty),X)$, $\sigma\in C(X,[0,\infty))$, $0<t_1<t_2<\dots<t_n<T$. Set $\gamma>0$, and choose PC^{γ} as

$$
PC^{\gamma}=\{\phi\in PC((0,\infty],X):\lim_{t\to -d}e^{\gamma t}\phi(t)\,\text{exist}\}
$$

with the norm $\|\phi\|_{\gamma}=\sup_{t\in(0,\infty]}e^{\gamma t}|\phi(t)|$, $\phi\in PC^{\gamma}$. We set

$$
\begin{gathered}
\rho(t,\varphi)=t-\sigma(\varphi(0)),\quad (t,\varphi)\in J\times PC^{\gamma},\\
f(t,\varphi)=\frac{e^t(\varphi)}{25+(\varphi)^2},\quad (t,\varphi)\in J\times PC^{\gamma},\\
g(t,\varphi)=\frac{\varphi}{47}ds,\quad \varphi\in PC^{\gamma},\\
B(x)(t)=\int_0^t \cos(t-s)\frac{xe^s}{(4+x)}ds,\quad (t,x)\in I\times PC^{\gamma},\\
Q_k(x(t_k))=\int_{-d}^{t_i}\frac{\gamma_i(t_i-s)x(s)}{25}ds,\\
I_k(x(t_k))=\int_{-d}^{t_i}\frac{\gamma_i(t_i-s)x(s)}{9}ds.
\end{gathered}
$$

We can see that all the assumptions of Theorem 3.1 are satisfied with

$$
\begin{gathered}
|f(t,\varphi)-f(t,\chi)|\le e^t\frac{\|\varphi-\chi\|}{25}\quad \forall t\in J,\varphi,\chi\in PC^{\gamma},\\
|B(x)-B(y)|\le e^t\frac{\|x-y\|}{4}\quad \forall t\in J,\ x,y\in PC^{\gamma},\\
|g(t,\varphi)-g(t,\chi)|\le \frac{1}{47}\|\varphi-\chi\|,\quad \forall t\in J,\ \varphi,\chi\in PC^{\gamma},\\
|Q_k(x(t_k))-Q_k(y(t_k))|\le \gamma^*\frac{1}{25}\|x-y\|,\quad x,y\in X,\\
|I_k(x(t_k))-I_k(y(t_k))|\le \gamma^*\frac{1}{9}\|x-y\|,\quad x,y\in X,\\
|q(x)-q(y)|\le \frac{1}{4}\|x-y\|,\quad x,y\in X.
\end{gathered}
$$

Further, we observe that

$$
\Big\{mL_I+mTL_Q+\frac{T^{\alpha}}{\Gamma(\alpha+1)}L_g+\frac{bT}{a+b}\Big(\frac{T}{b}L_q+mL_Q+\frac{T^{\alpha-1}}{\Gamma(\alpha)}L_g
$$

$$+\frac{T^{\alpha-1}}{\Gamma(\alpha)}(L_{f1}+L_{f2}B^*)\Big)+\frac{T^{\alpha}}{\Gamma(\alpha+1)}(L_{f1}+L_{f2}B^*)\Big\}$$
$$\approx 0.513\gamma^*+0.534<1.$$

We fix $\gamma^*=\int_{-d}^{t}\gamma_i(t_i-s)ds<0$, $0<t_1<t_2<t_3<1$, $\alpha=3/2$, $T=1$. This implies that there exists a unique solution of the considered problem in this section.

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